The Caterer's Problem

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The problem is from `https://www.cs.cmu.edu/puzzle/`. Below is the problem description.

Problem You are organizing a conference in social choice theory, with a festive dinner on the first day. The catering service has 1024 different dinner choices they know how to make, out of which you need to choose 10 to be in the dinner menu (each participant will choose one of these during the dinner). You send an email to the 6875 participants of the conference, with the list of all 1024 choices, asking them to rank the choices in linear order from their favorite to their unfavorite.

You want to find a list L of 10 choices, such that for any dinner choice d not in the list L, if we run a vote of d against L, at least half of people will favor one of the choices in L over d (it may be different dish for different people).

Is it always possible to produce such a list?

Solution Yes, it is always possible to produce such a list.

We first give an algorithm to pick the list L of 10 dishes, then prove a slightly stronger statement on the selected list: for any dish $d \notin L$, we can always find a dish $v^* \in L$ where more than half of people will prefer v^* over d.

First, we construct a directed graph $G = (V, E)$ with 1024 vertices, where each vertex represents a dish. For each pair of vertices/dishes u and v, we look at everyone's preference. If more than half of people prefer u over v, we add a directed edge (v, u) from v to u (i.e. pointing to the preferred dish). Otherwise we add a directed edge (u, v).

As a result, there will be an edge between every pair of vertices, i.e., this is a complete graph with exactly $\binom{1024}{2}$ directed edges.

Now we run the following algorithm:

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Algorithm 1: Select Top 10 Dishes

```
L ← ∅;
while |V| > 1 do
    For every vertex v, compute its in-degree deg⁻(v);
    Find the vertex v* where deg⁻(v*) = max_{v∈V} deg⁻(v);
    L ← L ∪ {v*};
    For every vertex u where (u, v*) ∈ E, remove u along with all edges
      connected to u;
end
if |L| < 10 then
    Fill up the list L with any dishes not selected;
end
return L;
```

Lemma 1 In any iteration of the while-loop, for any vertex u removed from the graph, the dish v^* added to L is preferred over u by more than half of people.

Proof. In that iteration, let v^* be the vertex with maximum in-degree, and let U be the set of removed vertices. For any vertex $u \in U$, it is removed because there is a directed edge $(u, v^*) \in E$. By definition of that directed edge, more than half of people prefer v^* over u. $\square$

Lemma 2 When the while-loop exits, all dishes not in L have been removed from the graph.

Proof. Initially, the graph G is a complete graph. In each iteration, for every $u \in U$ we remove, we also remove all edges connected to u. Thus, the updated graph G is also a complete graph at the end of every iteration.

Now let's consider how many vertices are removed in each iteration. Let n_k be the number of vertices in the graph when we enter the k-th iteration, and v^* is the vertex with the max in-degree in that iteration. We have $deg^-(v^*) \geq \frac{n_k-1}{2}$. This is because in directed complete graphs, we have $\sum_{v\in V} deg^-(v) = |E| = \frac{n_k(n_k-1)}{2}$. Suppose $deg^-(v^*) < \frac{n_k-1}{2}$, then we have $\sum_{v\in V} deg^-(v) \leq n_k \cdot deg^-(v^*) < \frac{n_k(n_k-1)}{2}$. Contradiction! Therefore, in every iteration, we remove at least $\lceil \frac{n_k-1}{2} \rceil$ of the vertices.

We start with $n_1 = 1024 = 2^{10}$ vertices. After the first iteration, we have at most $1024 - \lceil \frac{1024-1}{2} \rceil = 512 = 2^9$ vertices left. After the second iteration, we have at most $256 = 2^8$ vertices left. After k iterations, we have at most 2^{10-k} vertices left. After 9 iterations, we have at most $2^{10-9} = 2$ vertices left. When we enter the tenth iteration, there are 2 vertices left, so there is only 1 directed edge connecting them. We select the preferred dish v^* and add it to L, and remove the other dish u from the graph. The while-loop exits as there is only one vertex v^* left after the tenth iteration. Thus, when the while-loop exits, there is no dish outside of L left in the graph. $\square$

Therefore, by Lemma 2, for any dish d outside of L, it must be removed during some iteration of the while-loop. We just need to find in which iteration d was removed, and then find the corresponding $v^* \in L$ in that iteration, and by Lemma 1 more than half of people will prefer v^* over d.