CSC 310: Information Theory **Winter 2026**

Tutorial 6: Shearer's Lemma and its Combinatorial Applications

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6.1 Shearer's Lemma

Recall that we use $[n]$ to denote the set $\{1, \ldots, n\}$. All the logs are base 2 in this tutorial.

Lemma 6.1 (Shearer's Lemma) *Let $X_1, X_2, \ldots, X_n$ be discrete random variables. Let $S_1, S_2, \ldots, S_m \subseteq [n]$ be subsets of indices such that every index $i \in [n]$ appears in at least k of the subsets S_j. Then we have*

$$k \cdot H(X_1, X_2, \ldots, X_n) \leq \sum_{j=1}^{m} H(X_{S_j}),$$

where X_{S_j} denotes the collection of random variables with indices in S_j.

This inequality was introduced by James Shearer in the mid-1980s as part of combinatorial work using entropy, and it first appeared in paper *Some intersection theorems for ordered sets and graphs* by Fan Chung, Ronald Graham, Peter Frankl, and James Shearer in 1986 in *Journal of Combinatorial Theory, Series A* (a.k.a. JCTA). In this tutorial, we will state and prove a special case of Shearer's Lemma (where $n = 3, k = 2$).

Lemma 6.2 (Shearer's Lemma (special case)) *For discrete random variables X, Y, Z, we have*

$$H(X, Y, Z) \leq \frac{1}{2} \left(H(X, Y) + H(Y, Z) + H(Z, X) \right).$$

Note that this looks like sub-additivity of entropy $H(X, Y, Z) \leq H(X) + H(Y) + H(Z)$ but the bound is tighter.

Proof: A key observation is that conditioning can only reduce entropy. Thus, we have $H(Y|X) \leq H(Y)$ and $H(Z|X, Y) \leq H(Z|X)$ and $H(Z|X, Y) \leq H(Z|Y)$. By chain rule, we have

$$\begin{aligned}
2H(X, Y, Z) &= 2(H(X) + H(Y|X) + H(Z|X, Y)) \\
&= (H(X) + H(Y|X)) + (H(Y|X) + H(Z|X, Y)) + (H(X) + H(Z|X, Y)) \\
&\leq (H(X) + H(Y|X)) + (H(Y) + H(Z|Y)) + (H(X) + H(Z|X)) \\
&= H(X, Y) + H(Y, Z) + H(Z, X).
\end{aligned}$$

This implies $H(X, Y, Z) \leq \frac{1}{2}(H(X, Y) + H(Y, Z) + H(Z, X))$ as wanted. ■

6.2 Combinatorial Applications

Using Shearer's Lemma, we can study combinatorial problems.

Example 1. Suppose S is a set of N points in 3 dimensional space $\mathbb{R}^3$. Then either the projection onto the X, Y plane, or the Y, Z plane, or the Z, X plane has size at least $N^{2/3}$.

Proof: Let (X, Y, Z) be a uniformly random point drawn from S. Note that (X, Y, Z) is uniformly distributed over N points. Therefore, we have $H(X, Y, Z) = N \cdot \frac{1}{N} \cdot \log N = \log N$. By Shearer's lemma, we have $2 \log N = 2H(X, Y, Z) \leq H(X, Y) + H(Y, Z) + H(Z, X)$. By an averaging argument, either $H(X, Y)$ or $H(Y, Z)$ or $H(Z, X)$ is at least $\frac{1}{3} \cdot 2 \log N = \log N^{2/3}$. Since entropy is at most log of the support size, we have $H(X, Y) \leq \log(\mathsf{supp}(X, Y))$, $H(X, Z) \leq \log(\mathsf{supp}(X, Z))$, and $H(Y, Z) \leq \log(\mathsf{supp}(Y, Z))$. Therefore, we have

$$\max\left(\log(\mathsf{supp}(X, Y)),\ \log(\mathsf{supp}(X, Z)),\ \log(\mathsf{supp}(Y, Z))\right) \geq \log N^{2/3}.$$

Taking power of 2 on both sides of inequality, we obtain

$$\max\left(\mathsf{supp}(X, Y),\ \mathsf{supp}(X, Z),\ \mathsf{supp}(Y, Z)\right) \geq N^{2/3}.$$

This means that at least one of the projection has size at least $N^{2/3}$. ■

Example 2. A simple graph is a graph with no multi-edges or self-loops. A triangle is an ordered tuple of 3 vertices such that every pair has an edge. An edge is is an ordered tuple of 2 vertices. Show that if you have an undirected simple graph G with T triangles, then it must have at least $T^{2/3}$ edges.

Proof: Let (X, Y, Z) be a uniformly random triangle in G, where X, Y, Z denote the three vertices of the triangle. Then (X, Y), (Y, Z) and (Z, X) are distributions over edges. Since (X, Y, Z) is uniformly distributed, we have $H(X, Y, Z) = \log T$. Applying Shearer's Lemma, we obtain $2 \log T = 2H(X, Y, Z) \leq H(X, Y) + H(Y, Z) + H(Z, X)$. By an averaging argument, one of $H(X, Y)$ and $H(Y, Z)$ and $H(Z, X)$ is at least $\frac{1}{3} \cdot 2 \log T = \log T^{2/3}$. Since entropy is at most log of the support size, we have $H(X, Y) \leq \log(\mathsf{supp}(X, Y))$, $H(X, Z) \leq \log(\mathsf{supp}(X, Z))$, and $H(Y, Z) \leq \log(\mathsf{supp}(Y, Z))$. Therefore, we have

$$\max\left(\log(\mathsf{supp}(X, Y)),\ \log(\mathsf{supp}(X, Z)),\ \log(\mathsf{supp}(Y, Z))\right) \geq \log T^{2/3}.$$

Taking power of 2 on both sides of inequality, we obtain

$$\max\left(\mathsf{supp}(X, Y),\ \mathsf{supp}(X, Z),\ \mathsf{supp}(Y, Z)\right) \geq T^{2/3}.$$

This means that at least one of the projection, which is a set of (ordered) edges, has size at least $T^{2/3}$. Hence, graph G must have at least $T^{2/3}$ (ordered) edges. ■