Etiology of the Major and Minor Scales

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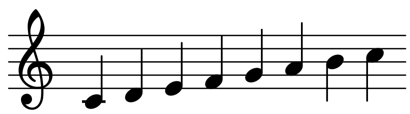
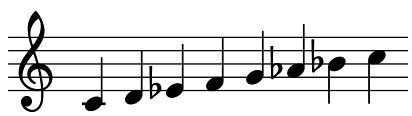
In this short note we propose to explain how the major and minor scales could arise by mathematical design.
Here is the C major scale:



And here is the C minor scale:



1 Twelve Semitones

The infrastructure of the scales is the division of the octave into 12 semitones. This number is chosen so that a relatively good approximation of the interval of (perfect) fifth (3/2) is obtained. This interval is important since two tones in the interval of fifth share the maximal number of overtones over all possible intervals smaller than the octave. Indeed, among all rational fractions strictly between 1 and 2, the fraction 3/2 clearly has the smallest possible parts.

Suppose we divide the octave into n equal semitones (we comment that historically, tunings were not equally tempered). The achievable intervals are of the form $2^{k/n}$. We want to find some small n such that $2^{k/n} \approx 3/2$. Hence, we are looking for rational approximations of $\log_2 3/2$. The best rational approximations are obtained from the continued fraction:

$$\log_2 \frac{3}{2} = [0; 1, 1, 2, 2, 3, 1, 5, 2, 23 \ldots].$$

The corresponding convergents are

$$0, 1, \frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \ldots$$

The convergent 7/12 corresponds to $[0; 1, 1, 2, 2]$. The next term in the continued fraction is 3, which is relatively large (with respect to the former entries). Thus $2^{7/12}$ should be a reasonable approximation to 3/2. This prompts division of the octave into 12 semitones. The interval of fifth is approximated by 7 semitones.

2 Seven Notes

Why is the number of notes in the scale 7? One natural requirement of a scale is that for each note in the scale, the fifth above it is also in the scale. Unfortunately, since $(7, 12) = 1$, this requirement is only fulfilled if we take all 12 possible notes. We can relax our requirements by allowing some errors. A natural way to accomplish that is to take m notes for an m such that $7m \pmod{12}$ is small. That way, we will have only one mistake, which will also be small.

If $7m \pmod{12} = 0$ then $m = 12$, as we've already mentioned. The next best choice is $7m \pmod{12} = \pm 1$, and here we have two solutions: $m = 5$ with $7m \pmod{12} = -1$, and $m = 7$ with $7m \pmod{12} = +1$. Since $2^{6/12}$ is slightly closer to $3/2$ than $2^{8/12}$, the bad fifth of a heptatonic scale is slightly better than the bad fifth of a pentatonic scale.

3 Major mode up to shift

We have seen that our scale should support all fifths. Since it contains 7 notes, the distances between adjacent notes, measured in semitones, are 2 for 5 of the notes, and 1 for 2 of the notes. We can describe the situation by assigning 1/2 values for the 7 distances C, D, E, F, G, A, B. The only combination of values giving exactly 7 (the interval of a fifth) is $2 + 2 + 2 + 1$ (since we have only two 1s). Thus, if our scale is going to have all fifths but one (say, the one starting on B), then we should have

$$\begin{aligned} C + D + E + F &= 7 \\ D + E + F + G &= 7 \\ E + F + G + A &= 7 \\ F + G + A + B &= 7 \\ G + A + B + C &= 7 \\ A + B + C + D &= 7 \end{aligned}$$

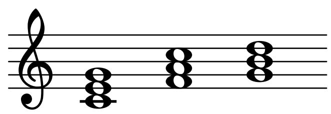
We can eliminate adjacent pairs of equations to get $C = G = D = A$, $F = C$, $E = B$. Thus our scale is of the form

$$CCBCCCB$$

Thus $C = 2$ and $B = 1$, and we get (up to shift) both the major scale and the (harmonic) minor scale, as well as other modes. The exact mode we get depends on our choice of bad fifth.

4 Major and minor modes

The actual construction of the major and minor modes relies on triads. A triad is build from a note by taking its third and fifth (the third is a natural complement to the fifth, because the second and fourth are dissonant). A reasonable requirement of a scale is that the triad built on the tonic be the same as the ones built on the dominant (fifth above tonic) and subdominant (fifth below tonic, and so forth above tonic). In pictures:



This gives us the following equations

$$C + D = F + G = G + A$$
$$E + F = A + B = B + C$$

These reduce to $A = C = F$ and so $B = E$, $D = G$. Thus our scale is of the form

$$ABADBAD$$

Since A appears three times, necessarily $A = 2$:

$$2B2DB2D$$

There are two solutions:

$$2221221$$
$$2122122$$

The first is the major mode, the second the minor mode.