Week 2: More Induction and Counting

CSC 236:Introduction to the Theory of Computation

Summer 2024

Instructor: Lily

Announcements

- Volunteer notetaker for Accessibility Services
- Registered study groups
- Solution to unmarked Q2 updated
- New Office Hours (still in BA 2272):
  - Tuesdays: 4-5pm **(NEW! – starts May 21)**
  - Fridays: 1-2pm **(MODIFIED!)**
- Assignment Office Hours (BA 2270):
  - Thursday, 1-2pm
  - Friday 2-3pm

Announcements

- Waitlisted students: **email us!**
  - Q1 and A1 gets average of other quizzes (best 7) and assignments (best 3)

Subject [Waitlist] First Name, Last Name, Student Number, UTORID

- Bonus points: **email us!**
  - 4+ endorsed answers or corrections on piazza (send us list of links, your name, and student ID) --- processed near the end of the semester

Subject [Bonus] First Name, Last Name, Student Number, UTORID

Question 1

1 pts

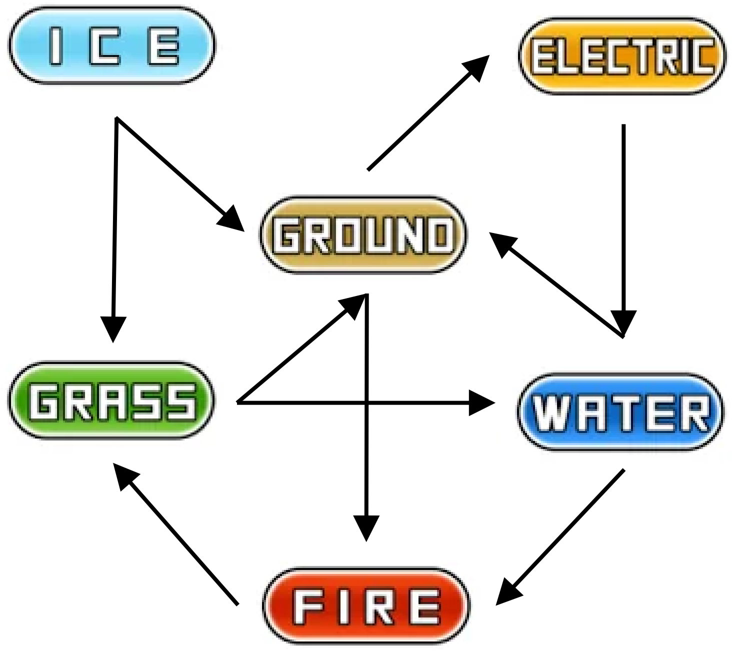
Consider the following proof that all horses have the same color. We induction on n, the number of horses. Choose every line which is ***incorrect***.

1. In the base case, if there are no horses then the statement is vacuously true .If there is one horse it is the same color as itself.
2. For a fixed k, assume that $P(k)$ is true. Show that $P(k+1)$ is true.
3. Take a group of $k+1$ horses and label them $1, \ldots, k+1$. By IH, horses $1, \ldots, k$ are equal.
4. Similarly, by IH, horses $2, \ldots, k+1$ are the same color.
5. Note that the middle horses $2, \ldots, k$ must all be the same color which is the color of horse 1 and horse $k+1$.
6. Thus all horses are the same color.

- [ ] 4
- [ ] 5
- [ ] 1
- [ ] 3
- [ ] 6
- [ ] 2

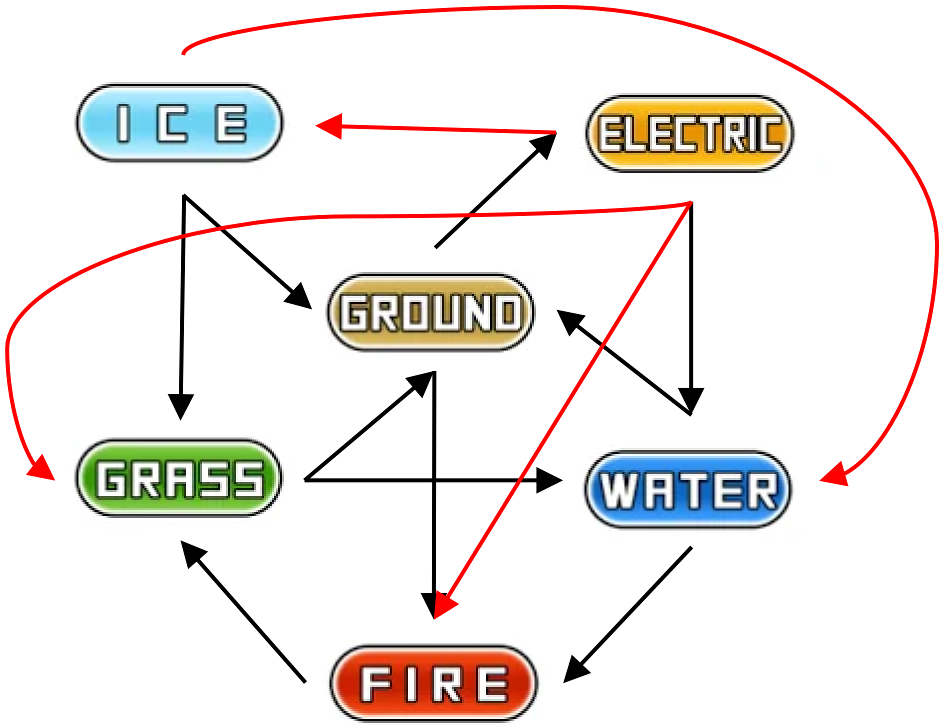
Relations

Ordering:



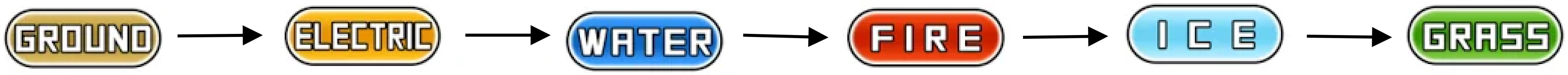
Relations

Total Ordering:



Relations

Well-Ordering:



All other edges implied

Well-Ordering:

A – on a set S is a *total ordering* such that every *non-empty* subset of S has a least element

Well-Ordering Theorem:

Every *nonempty* set S has a relation $<$ on S which is a well ordering.

Well-Ordering Principle:

Every *nonempty* set of $\mathbb{N}$ has a *smallest* element.

Induction implies Well-Ordering Principle

Well-Ordering Principle (WOP):

Every *nonempty* subset of $\mathbb{N}$ has a *smallest* element.

Well-Ordering

For all $n \in \mathbb{N}$ where $n > 1$, n has a prime factorization.

Fundamentals of Counting

- Applications: telephone numbers, IP addresses, password security, biology (e.g. sequencing DNA)

- Putting items (n) into different slots (m)
- Factorials: $n! = n \cdot (n-1) \cdots 2 \cdot 1$
- Binomial Coefficient: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Permutation: Order Matters

How many ways can n children stand in a line for a picture?

Combinations: Order *Does Not* Matter

How many ways can we draw a hand of m cards from a deck of n?
Assume $m \leq n$.

Permutation: Order Matters

How many ways can n children stand in a line for a picture?

Combinations: Order *Does Not* Matter

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

Examples

1. How many permutations of the letters ABCDEFG contains the string "ABC"?
2. How many possibilities are there for the first, second, and third positions in a horse race with 12 horses?
3. A group contains n men and n women. How many ways are there to arrange theses people in a row if the men and women alternate?

Now You Try!

1. How many binary strings contain 3 ones and 5 zeros?
2. Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ for all positive integer n.
3. Suppose there are $2n$ people. How many ways can we form a committee of n people?
4. Prove that $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$ for all positive integer n.

Q1. How many binary strings contain 3 ones and 5 zeros?

Q2. Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ for all positive integer n.

Q3. Suppose there are $2n$ people. How many ways can we form a committee of n people?

Q4. Prove that $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$ for all positive integer n.

Permutation with Replacement

There are n children and m chores. How many ways are there to assign the children to the chores if each child can do any number of chores?

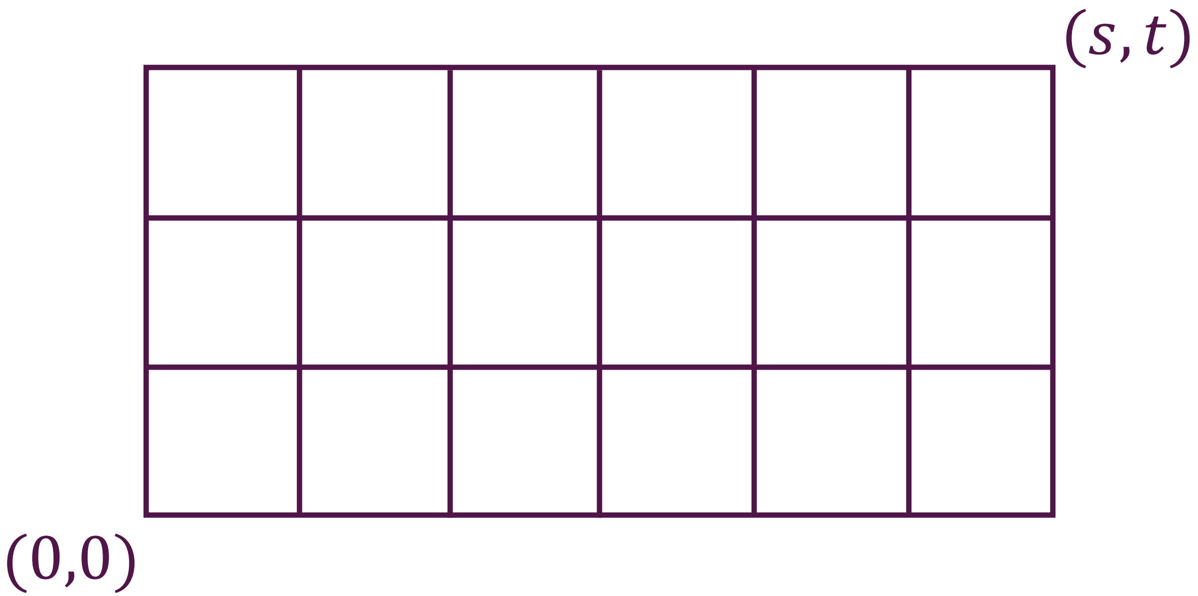
Combination with Replacement

There are m flavors of ice-cream. How many ways are there to make an ice-cream cone with n balls of ice-cream (order does not matter)?

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 = 11$ where $x_1, x_2, x_3 \in \mathbb{N}$.

	Items (n)		
Type		No Replacement	Replacement
	Permutation		
	Combination		

Example. Start at $(0,0)$ and end at (s, t). Valid steps: one step up or one step right. How many different paths?



Recap

- Order relations, total ordering, well-ordering
- Induction and the Well-Ordering Principle are equivalent (*we only saw induction implies well-ordering principle*)
- Permutation (with/without replacement)
- Combination (with/without replacement)
- Lots of Examples

Next time… Discrete Probability, Pigeonhole Principle and Introduction to Graphs.