Assignment 2

Unmarked Questions

Remember to comment out this section when submitting your assignment.

Here are a few warm-up problems. Make sure you understand their solutions before proceeding to the marked questions. The solutions are provided in the comment blocks (simply move them outside of the comment blocks to read).

Q1. Below are a collection of counting problems.

1. How many ways are there to permute the letters of "BANANA" (same letters are treated as identical).

   Solution. There are six letters so the number of permutations *if all the letters are distinct* is 6!. However, there are three identical "A"s and two identical "N"s which we over-count by 3! and 2! respectively. It follows that the number of ways to permute the letters is $\frac{6!}{2!3!}$.

2. How many ways can k identical balls be distributed among 5 different people?

   Solution. This is a combination-with-replacement problem. We note that there are five categories (people) in which we are distributing k indistinguishable items (balls). There are $\binom{k+4}{k}$ ways to do this.

3. How many ways are there for a horse race with four horses to finish if ties are possible? (any number of horses can tie)

   Solution. It is important to first determine how many ties there are. If all the horses tie, then they must all be in first place. There is 1 way in which this occurs. If three horses tie for first, the remaining horse comes in second. There are 4 ways in which this occurs by picking the second place horse. If there are two horses tied for first, then either the remaining two are tied for second or one comes second and the other comes third. There are 6 and 12 ways in which each occurs respectively. If one horse comes first, then the remaining three can tie for second, two can tie for second while the remaining comes in third, one comes in second while the remaining two tie for third, or there is a unique second, third, and final position. There are 4, 12, 12, 24 ways in which these occur respectively. In total, there are 75 ways.

Q2. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games in total. Show there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Hint: Let $g_1, ..., g_{30}$ be the number of games the team play on each of the thirty days. Let $p_1, ..., p_{30}$ be partial sums of the g_i where $p_1 = g_1$, $p_2 = g_1 + g_2$, and generally $p_k = \sum_{i=1}^{k} g_i$. Also define $q_1, ..., q_{30}$ where $q_i = p_i + 14$.

Solution. Note that because $g_i > 0$ for all i, *all p_is are distinct.* Note that because the p_is are distinct, so are the q_is. Since there were at most 45 games played in total, $q_{30} \leq 59$. However, there are 60 numbers from among the p_is and q_is so by PHP, there must at least two identical numbers.

Since all the p_is are distinct and all the q_is are distinct, we must have $p_i = q_j = p_j + 14$. This means $g_{j+1} + \cdots + g_i = 14$.

Marked Questions

Q1. [10 Points] Counting Labeled Trees *(maximum 2 pages)*

In this question we will walk you through the proof of the following claim for $n \geq 2$.

Claim 1. *There are n^{n-2} different labeled trees on n vertices.*

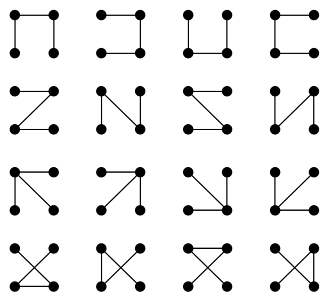


Figure 1: All 16 trees on four vertices labeled by position.

As an example, Figure 1 shows labeled trees on four vertices.

Unmarked. Prove that a connected graph with n vertices is a tree if and only if it has $n-1$ edges.

See Lecture 4 notes.

a. Label the vertices from 1 to n and perform the following procedure to construct two lists A and B:

   i. Take the leaf with the smallest label add its label to B and the label of its neighbour to A.
   ii. Remove this leaf.
   iii. Repeat until only one edge remains.

   For the graph in Figure 2, $A = [2, 2, 1, 1, 7, 1, 10, 10]$ and $B = [3, 4, 2, 5, 6, 7, 1, 8]$.

   Given a sequence $A = [a_1, ..., a_{n-2}]$ we can reverse the procedure as follows: Let $a_{n-1} = n$. For $i = 1, 2, ..., n-1$, let b_i be the vertex with the smallest label *not* in the set

$$\{a_i, a_{i+1}, ..., a_{n-1}\} \cup \{b_1, b_2, ..., b_{i-1}\}.$$

   Prove that every tree has a leaf and that $\{(a_i, b_i) : i = 1, ..., n-1\}$ form the edges of a tree on the vertices $1, ..., n$. *Remember to cite external resources if you use them.*

   Solution.

b. Complete the rest of the argument, why does this procedure prove Claim 1?

   Solution.

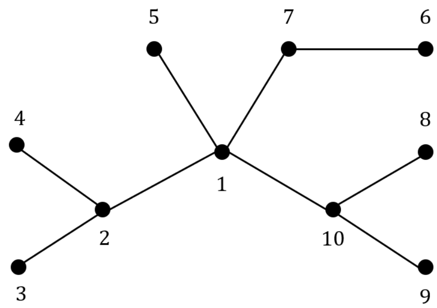


Figure 2: Example tree.

c. Pick one of the follow and answer it (tell us which one you are solving).

(a) For a tree on n nodes with $n \geq 5$ how many contains exactly three leaves?

(b) For a tree on n nodes with $n \geq 5$ how many contains exactly $n - 3$ leaves?

Solution.

Q2. [10 Points] Balatro *(maximum 2 pages)*

Balatro is a solitaire game based on poker hands. *You do not need to buy the game in order to solve the problem*[1]. The game has a bunch of jokers and card modifiers which augment the score, but *I will not be using those.*

Considering the set of cards shown in Figure 3a. I will pick five cards from the set to play and they will score according to the base values shown in Figure 3b. Scores have two components: chips (blue number shown on the left e.g. full houses are worth 40 chips) and multiplier (red number shown on the right e.g. full houses are worth 4 multiplier). Better poker hands result in more chips and (possibly) more multiplier. In addition to the base score for a hand, I also get chips for each card in the hand, added to base chip value *before* multiplying by the multiplier. Number cards are worth their face value (e.g. a nine is worth 9 chips) except aces which are worth 11 chips[2].

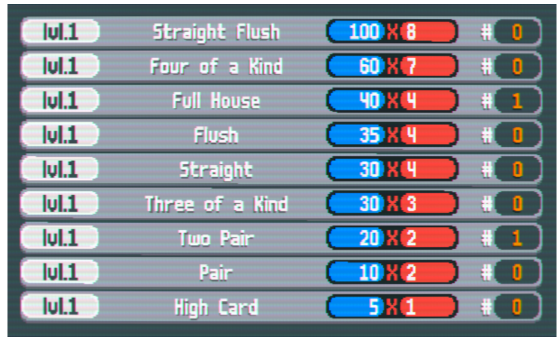
Let's look at an example. From the set of cards shown at the bottom of Figure 3a, I can make a full house consisting of three aces and two sixs. The base score of a full house is 40 chips with a $4\times$ multiplier. The value of the cards in the hand is 11 chips for each ace and 6 for each 6. This means that the total chip value of my hand is $40 + 3(11) + 2(6) = 85$. Together with the multiplier, the score of my hand is $85 \times 4 = 340$. Unfortunately, the *blind* (the target score) is 350.

I have one *discard* left where I can pick $n \leq 5$ cards from the set, remove them, and replace them with n cards drawn uniformly from the ones remaining in my deck (shown at the top of Figure 3a). ***The goal is to determine which cards to discard in order to have the best chance of hitting the blind.*** We do so by considering the types of hands to play. Define "*successfully scoring*" as obtaining a hand worth more than 350 chips. *In the following, no proofs are required, but you need to consider all the cases. You may check your work programmatically, but make sure to state the cases and explain your calculation in your submission.*

[1]It might be helpful to watch a video of someone playing to get a sense of the game (look online for these).

[2]Face cards are worth 10 chips each, but there are no face cards so don't worry about them.

(a) My hand (bottom): ace of spades, clubs, diamonds, 10 of spades, 6 of spades, 6 of clubs, and 3 of clubs. What's in my deck (top): 6 spades, 6 hearts, 6 clubs, and 8 hearts.



(b) Base value of the different poker hands (before adding chip value of played cards): straight flush give 100 chips and 8 multiplier, four of a kind gives 60 chips and 7 multiplier, and so on.

Figure 3: My hand (left) and base values of poker hands (right).

a. What is the probability that after one discard I *successfully score* with a full house?

b. What is the probability that after one discard I *successfully score* with a four of a kind?

c. What is the probability that after one discard I *successfully score* with a straight flush?

Additional Questions

Remember to comment out this section when submitting your assignment.

If you would like more exercises consider trying the following problems from your primary and supplementary textbooks. *We will not be providing solutions to these questions* though you are free to find the solution online and discuss them with your peers.

1. *Concrete Mathematics* Chapter 5: Exercises 1, 2, 3
2. *Discrete Mathematics and Its Applications* Chapter 6.1, 6.2, 6.3, 6.4, 6.5