Discrepancy and Related Problems

by

Xinyuan Li

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy

Department of Computer Science
University of Toronto

Discrepancy and Related Problems

Xinyuan Li
Doctor of Philosophy

Department of Computer Science
University of Toronto
2024

Abstract

A set system is a collection of sets on an underlying universe of elements. It can be modeled by a $\{0,1\}$ incidence matrix. The rows of the matrix represent sets, while the columns represent elements. The discrepancy of a set system is the least balanced that we can make the most unbalanced set in the set system by coloring elements of the universe ± 1. Since its inception, extensions and limitations have necessitated the introduction of many variants of discrepancy. In this thesis we consider linear and hereditary discrepancy, first introduced by Lovász, Spencer, and Vesztergombi [LSV86], in detail.

Linear discrepancy captures a common problem in applied mathematics: round a real-valued vector $\mathbf{w}$ with entries in the interval $[0,1]$ to a binary vector $\mathbf{x}$. For a given matrix $\mathbf{A}$, we want $\mathbf{Ax}$ to be as close to $\mathbf{Aw}$ as possible under a fixed metric, often ℓ_∞. In LP rounding, for example, the real-valued solution of a LP approximates the solution of NP-hard optimization problem. The worst-case error over all choices of $\mathbf{w}$ incurred by the best possible rounding is the linear discrepancy of $\mathbf{A}$. We study of the computational complexity of linear discrepancy by proving hardness results and presenting new algorithms to evaluate the linear discrepancy of a special family of matrices.

Hereditary discrepancy is a generalization of discrepancy that maximizes the unbalance over all submatrices of the incidence matrix. It has applications to data structures, differential privacy, and spectral graph theory. A general way to prove lower bounds on the hereditary discrepancy of a matrix is via its determinant lower bound. Building upon the works of Matousek [Mat13] and Jiang and Reis [JR22], we show that for a matrix of dimension $m \times n$, the bound of $O\left(\sqrt{\log m \log n}\right)$ on the ratio of its hereditary discrepancy to its determinant lower bound is tight for nearly all ranges of m.

Finally, we consider a graph partitioning problem studied in many fields with many different names. Given a graph G, the goal is to partition the vertices of G into two or more equally sized parts so that every vertex has more neighbours in its own part than the others. We show constructive results from the social choice perspective using discrepancy theory. From a graph theory perspective, existential results for Erdös-Rényi random graphs generalized to digraphs.

To those who believe improvement is possible.

Acknowledgements

This work would not be possible — not to mention far less enjoyable — without the support of my advisor, committee members, collaborators, friends, and family.

I would like to sincerely thank my advisor, Sasho Nikolov, with whom I collaborated on all of my projects and who was a caring and helpful mentor. Sasho was always patient with me when I got confused by unfamiliar concepts, providing explanations, examples, and references. He was generous with his time, going above and beyond weekly meetings by agreeing to be my sounding board as I puzzled through a probability textbook. Working with him made me a better writer. He is able to highlight points of confusion which we then rework and refine them until the message becomes clear. Sasho is a great inspiration to me and the lessons he has taught me has shaped the way I approach problems.

Many thanks to my committee members Mike Molloy and Sushant Sachdeva. Throughout my time in graduate school, our yearly meetings were steady beacons in the murky waters of academia, guiding me along the way. They not only offered technical feedback on my research proposals and thesis, but also provided thoughtful suggestions about my trajectory during and after graduate school.

Further, I would like to thank my external examiner Thomas Rothvoß and my internal external Swastik Kopparty for attending my defense on such short notice, asking insightful questions during the presentation, and providing feedback on my thesis. They gave me a different perspective to analyze my work and for that I am thankful.

I would like to thank my collaborators Evi Micha and Nisarg Shah. It was fascinating to dive deep into a problem in another field. Without this exposure, I would be missing half of a thesis and also miss out on a wonderful collaborative experience.

I would like to thank the up-and-coming theory students Ben, Devansh, Harry, Haohua, Lawrence, Yibin, Ziyang and others for revitalizing the sociable atmosphere at the office post COVID. Board games, TSS, and theory seminars (and the subsequent lunches) were enjoyable due to their enthusiastic engagement. I have no doubt that the group will be a fun and welcoming place for the new students joining this year and in the future.

Many thanks to my friends who have graduated from U of T or will do so shortly. Hamoon is my best running partner and intrepid friend. He is no doubt deep in the exploration of quantum mysteries and his local environs. Alex is a generous and thoughtful person with a like-minded community. With the right company, I am sure his skills will be greatly appreciated. Kevan is also job hunting at the time of writing, but I am sure that it will conclude with a positive outcome. Hopefully he will find a fulfilling career working with compassionate people. Greg M. is a skilled coder with a variety of interests. Whatever he decides to do in the future, I hope he sees it through to the end. Suhail is a puzzle master and the most famous person I know. He inspired me to be better at mathematical exposition and I hope he will return to it one day. Ian is witty and charismatic, responding "a normal quartet" to "What's a Gaussian ensemble?" without missing a beat. Even though Coventry is nothing like a Japanese metropolis, I hope he enjoys his time there all the same. Deepanshu is almost done and has to choose among a plethora of tantalizing options. Whichever he picks, I hope it sparks joy as I am sure he will excel. Soroush is exceptional at both the theoretical and technical sides of computer science. I hope his internship will lead to many opportunities so that both he and

Parand will have their pick of places to live. Noah has returned to the academic path. I always thought it suited him well. Now there is plenty of time for him to pursue his eclectic interests.

I would like to thank Deeksha, Greg R., and Morgan who were in my cohort and are now all soon-to-be-professors. I thoroughly enjoyed our time together during lunches, walking around the city, and attending AI talks for all the right reasons. Morgan has an encyclopedic knowledge of esoteric complexity classes and is my go-to person for alcohol recommendations. We will be in the same geographic region soon and I hope we get to hangout. I do not think Greg will object if I said that his skills in cooking and household-chores is inversely proportional to his skill as a TCS researcher. He is, after all, a great researcher. He is also an innovator in pizza topping and neologisms such as the "statistical tie". Deeksha was my office mate for the majority of my time here at U of T and a great one at that. We started out in a cold empty office on the south side of SF, moved to SF4306, and then moved down to the third floor together. Whenever we did not feel like working we would talk or go for a walk and this made research feel a lot less lonely. I wish them all the best.

Finally, to my mom and dad: even though they were initially skeptical of graduate school, I appreciate their support in the end.

Thank you.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

A set system, denoted $(X, \mathcal{S})$, consists of an underlying universe X of size n and a family $\mathcal{S}$ of size m containing subsets $S \subseteq X$. For simplicity, let the sets of $\mathcal{S}$ be $S_1, ..., S_m$ and the elements of X be $[n] \coloneqq \{1, ..., n\}$. A *coloring* χ assigns ± 1 to every element of X. The value of χ restricted to $S \subseteq X$ is $\chi(S) = \sum_{x \in S} \chi(x)$. If $\chi(S)$ is small, then χ is "balanced" on S, i.e., there are an approximately equal number of elements in S colored $+1$ and -1. Conversely, if $\chi(S)$ is large, then χ is "unbalanced" on S, i.e., a large proportion of elements in S are assigned the same color. The combinatorial discrepancy of $\mathcal{S}$ is a measure of minimum achievable balance over all possible colorings χ. For a fixed χ, the discrepancy of $\mathcal{S}$ on χ is defined as $\operatorname{disc}(\mathcal{S}, \chi) = \max_{S \in \mathcal{S}} |\chi(S)|$. The *discrepancy* of $(X, \mathcal{S})$ is

$$\operatorname{disc}(\mathcal{S}) = \min_{\chi: X \to \{-1,+1\}} \max_{S \in \mathcal{S}} |\chi(S)|.$$

Discrepancy and its variants are well-studied topics in combinatorics and computer science [BS96; Cha01; Mat09]. We highlight two variants, namely linear and hereditary discrepancy, and their applications here and will define them formally in Section 1.2.

Many questions in mathematics and computer science are captured by the following rounding problem: for $m \times n$ matrix $\mathbf{A}$ and $\mathbf{w} \in [0,1]^n$, find an boolean vector $\mathbf{x} \in \{0,1\}^n$ so that $\mathbf{Ax}$ is as close as possible to $\mathbf{Aw}$ in a specified metric. For example, the following integer linear program,

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{such that} \quad & \mathbf{Ax} \geq \mathbf{b} \\ & \mathbf{x} \in \{0,1\}^n, \end{aligned}$$

models many NP-hard optimizations problems. By relaxing $\mathbf{x}$ to real-valued variables $\mathbf{w} \in [0,1]^n$, then rounding $\mathbf{w}$ to an integer feasible solution $\mathbf{x}$ where $\mathbf{Ax} \geq \mathbf{b}$, we obtain a solution whose objective value $\mathbf{c}^\top \mathbf{x}$ is not much greater than $\mathbf{c}^\top \mathbf{w}$. As an intermediate step, it is often helpful to guarantee that $\mathbf{x}$ is approximately optimal, i.e., showing that the coordinates of $\mathbf{b} - \mathbf{Ax}$ are bounded above. In particular, we can ensure the existence of $\mathbf{x}$ which achieves bounds that are a function of the linear discrepancy of $\mathbf{A}$. This approach was used in the works of Rothvoss [Rot13] and Hoberg and Rothvoss [HR17] to develop additive approximate algorithms for bin-packing, has applications to scheduling [BKN14] by the works of Bansal, Krishnaswamy, and Nagaragan, and designing point sets well distributed with respect to arbitrary Borel measures [ABN18; Nik17]. A better understanding of linear discrepancy has allowed tighter integrality gaps for special families of linear programming relaxations [BDM23] and may allow for tighter results more generally for these important optimization problems.

Discrepancy also has many connections to problems in other fields. It was shown to be intimately connected to range searching in group models where results from discrepancy have been used to obtain better query time and update time tradeoffs for various range search problems [Lar14]. Such connections, when applied to a more robust version of discrepancy known as hereditary discrepancy and have been used to construct differentially private algorithms [MN12]. Further, the approaches developed to compute discrepancy have been used to construct better spectral sparsifiers [JRT24].

This thesis contains results from four papers. Chapter 2 covers material from *On the Gap Between Hereditary Discrepancy and Determinant Lower Bound* [LN24] which is a joint work with Aleksandar Nikolov published in SIDMA 2024. Chapter 3 covers material from *On the Computational Complexity of Linear Discrepancy* [LN20] which is another joint work with Aleksandar Nikolov published in ESA 2020. Chapter 4 covers part of *Partitioning Friends Fairly* [LMNS23] which is a joint work with Evi Micha, Aleksandar Nikolov, and Nisarg Shah published in AAAI 2023. The full paper considers two notions of fairness: envy-freeness and the existence of a core, but only the portion pertaining to envy-freeness — a concept we will define and motivate in Section 1.3.1 — is relevant here. Chapter 5 covers material from *Balanced Friendly Partitions of Random Digraphs*, another joint work with Aleksandar Nikolov, which is currently in submission.

1.1 Notation

We summarize the commonly used mathematical notation found throughout this thesis.

Variables. For $n \in \mathbb{N}$, let $[n] = \{1, ..., n\}$. For a set S, let $|S|$ be the number of elements in S. For any $\alpha \in \mathbb{R}$, let $\lceil \alpha \rfloor$ be α rounded to a closest integer. For two sets S_1 and S_2, let $S_1 \sqcup S_2$ be the disjoint union of the two sets.

Lower-case letters (e.g. u) represents scalars while lower-case bold letters (e.g. $\mathbf{u}$) denote vectors. The subscript of an element denotes its position in the vector (e.g. u_i is the i^{th} element in vector $\mathbf{u}$). Upper-case letters (e.g. A) represents sets while upper-case bold letter (e.g. $\mathbf{A}$) denote matrices. Rows or columns of matrices are denoted by their lower-case bold counterparts (e.g. $\mathbf{a}_i$ for the i^{th} column) depending on the context. Entries are sub-scripted scalars (e.g. $a_{i,j}$). Universal constants will typically be represented by the letter c and are unrelated in different theorems even though they may be represented by the same symbol.

Relations. We adopt the short-hand notation $a \lesssim b$ to denote $a \leq c \cdot b$ for some constant c independent of a and b. Similarly $a \gtrsim b$ denotes $a \geq c \cdot b$. If $a \lesssim b$ and $a \gtrsim b$, then $a \asymp b$. When an inequality has a subscript, i.e. $a \lesssim_u b$, then the constant in the inequality depends on u.

In addition to standard asymptotic notation, we use $\widetilde{O}$, e.g., $f(n) = \widetilde{O}\left(g(n)\right)$, to represent $f(n) = O\left(g(n) \log^k g(n)\right)$ for some constant k.

Discrepancy. Let $\mathbf{A} \in \{0,1\}^{m \times n}$ be the incidence matrix of the set system $(X, \mathcal{S})$ where $|\mathcal{S}| = m$ with $\mathcal{S} = \{S_1, ..., S_m\}$ and $X = [n]$ unless stated otherwise. Row i of $\mathbf{A}$ is the indicator vector of set $S_i \in \mathcal{S}$ and column j of $\mathbf{A}$ is the indicator vector for the element $j \in X$. We will often consider colorings $\chi : X \to \mathcal{D}$ for some domain $\mathcal{D}$. Observe that χ can be interpreted as a vector in $\mathcal{D}^X$ and we will often use these interpretations interchangeably. When indexing, χ_i corresponds to the color of χ on $i \in X$ and $\chi(S) = \sum_{i \in S} \chi(i)$ for set $S \in \mathcal{S}$.

Linear Algebra. For vector $\mathbf{v}$, $\|\mathbf{v}\|_p$ is the ℓ_p-norm of $\mathbf{v}$. If $\mathbf{v} \in \mathbb{R}^n$, then $\text{Diag}(\mathbf{v})$ is the $n \times n$ matrix with the entries of $\mathbf{v}$ along the diagonal. $\mathbb{1}$ is the all ones vector of the appropriate dimension. $\mathbb{I}_n$ is the $n \times n$ identity matrix and $\mathbb{J}_n$ is the $n \times n$ all ones matrix. Whenever $\mathbf{A}$ is multiplied or divided by a scalar, *every entry* of $\mathbf{A}$ is multiplied or divided by the same scalar. For two square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimension, $\mathbf{A} \succ \mathbf{B}$ means $\mathbf{A} - \mathbf{B} \succ \mathbf{0}$ or that the matrix $\mathbf{A} - \mathbf{B}$ is positive definite (PD). Similarly, $\mathbf{A} \succeq \mathbf{B}$ means $\mathbf{A} - \mathbf{B} \succeq \mathbf{0}$ or that the matrix $\mathbf{A} - \mathbf{B}$ is positive semi-definite (PSD).

Probability. ϕ and Φ are the probability density and cumulative density functions of the standard Gaussian respectively. Φ^{-1} denotes the quantile function[1] for the standard Gaussian. Capitalized letters X, Y, and Z typically denote random variables and, in particular, Z — and any of its subscripted versions — denotes a standard Gaussian random variable unless otherwise specified. For a body D in $\mathbb{R}^n$, let $\gamma_n(D)$ be its standard Gaussian measure, i.e.

$$\gamma_n(D) = \int_D \frac{\exp\left(-\frac{\mathbf{x}^\top \mathbf{x}}{2}\right)}{(2\pi)^{k/2}} d\lambda^n(\mathbf{x})$$

where λ^n is the n-dimensional Lebesgue measure.

1.2 Discrepancy

We define and motivate the definition of two variants of discrepancy: hereditary and linear discrepancy. First, though the set system definition of combinatorial discrepance above is intuitive, it can be cumbersome to use. Instead, let $\mathbf{A} \in \{0,1\}^{m \times n}$ be the *incidence matrix* of the set system $(X, \mathcal{S})$ and redefine $\text{disc}(\mathcal{S})$ in terms of $\mathbf{A}$. Each row of $\mathbf{A}$ represents a set in $\mathcal{S}$, while each column of $\mathbf{A}$ represents an element in X. The entry in row i, column j of $\mathbf{A}$ is equal to one if and only if $j \in S_i$. Thus, the discrepancy of $(X, \mathcal{S})$ is equivalent to

$$\text{disc}(\mathbf{A}) = \min_{\chi \in \{-1,+1\}^n} \|\mathbf{A}\chi\|_\infty.$$

We can further define variants of discrepancy based on other norms. Just as $\text{disc}(\mathbf{A})$ was defined in terms of L_∞, for L_1, we can define

$$\text{disc}_1(\mathbf{A}) \coloneqq \min_{\mathbf{x} \in \{\pm 1\}^n} \frac{\|\mathbf{A}\mathbf{x}\|_1}{m}$$

[1]The quantile function of a random variable X with CDF F_X is equal to $F_X^{-1}(p) = \min\{x : F_X(x) = p\}$.

and generally for L_p, we can define

$$\operatorname{disc}_p(\mathbf{A}) := \min_{\mathbf{x}\in\{\pm1\}^n} \left(\frac{\|\mathbf{A}\mathbf{x}\|_p^p}{m}\right)^{1/p}.$$

Note that $\operatorname{disc}_p(\mathbf{A}) \leq \operatorname{disc}_q(\mathbf{A})$ when $p \leq q$.

Sometimes $\operatorname{disc}(\mathbf{A})$ can be small "by accident". For any matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$, we can define $\mathbf{A} \in \mathbb{R}^{m\times 2n}$ as $\mathbf{A} = [\mathbf{B}, \mathbf{B}]$ (i.e., the concatenation of two copies of $\mathbf{B}$ side by side). Regardless of the discrepancy of $\mathbf{B}$, $\operatorname{disc}(\mathbf{A}) = 0$ since there exists $\mathbf{x} \in \{-1, 1\}^n$ such that $\|\mathbf{A}\mathbf{x}\|_\infty = 0$, namely

$$\mathbf{x}^\mathsf{T} = [\underbrace{-1, ..., -1}_{n}, \underbrace{1, ..., 1}_{n}].$$

Instead of an inherent property of the matrix — $\mathbf{A}$ contains $\mathbf{B}$, so we would hope that $\operatorname{disc}(\mathbf{A})$ and $\operatorname{disc}(\mathbf{B})$ are related and that $\operatorname{disc}(\mathbf{A}) \geq \operatorname{disc}(\mathbf{B})$ —, we find that discrepancy is too brittle and unable to capture the underlying structure present in the matrix.

That is not to say similar matrices necessarily have approximately the same discrepancy. Consider this next example where the universe X is composed of two disjoint sets $A \sqcup B$ where $|A| = |B| = n$ for n a multiple of four. Let $(X, \mathcal{S})$ and $(\mathcal{T}, X)$ be two different set-systems on the same universe X. $\mathcal{S}$ is the set-system consisting of all sets of size $n/2$ with $n/4$ elements from A and $n/4$ from B. $\mathcal{T}$ is the set-system consisting of the set A. Observe that $\operatorname{disc}(\mathcal{S}) = 0$; the coloring χ where $\chi(a) \mapsto 1$ for $a \in A$ and $\chi(b) \mapsto -1$ for $b \in B$ achieves zero discrepancy. Similarly, $\operatorname{disc}(\mathcal{T}) = 0$: assign half the elements of A to 1 and the other to -1. Consider the set-system $(\mathcal{S} \cup \mathcal{T}, X)$. Note that $\operatorname{disc}(\mathcal{S} \cup \mathcal{T}) = n/2$. For any coloring $\chi : X \to \{\pm1\}$, let $\chi^{-1}(1)$ and $\chi^{-1}(-1)$ be the set of vertices colored 1 and -1, respectively, by χ. Without loss of generality, suppose that the majority of the elements in B are colored 1, i.e., $|\chi^{-1}(1) \cap B| \geq |\chi^{-1}(-1) \cap B|$. If there are $n/4$ or fewer elements colored 1 in A (i.e., $|\chi^{-1}(1) \cap A| \leq n/4$), then $\left|\sum_{a\in A} \chi(a)\right| \geq n/2$. Conversely, if there are $n/4$ or more elements colored 1 in A (i.e., $|\chi^{-1}(1) \cap A| \geq n/4$), then for any subset $A' \subset A$ such that $|A'| = n/4$ and $A' \in \chi^{-1}(1)$ as well as $B' \subset B$ such that $|B'| = n/4$ and $B' \in \chi^{-1}(1)$, $\chi(A' \cap B') = n/2$. Even though the two set systems separately have zero discrepancy, their union interacts in such a way as to increase the discrepancy substantially.

From these examples, we see that discrepancy can change dramatically with seemingly trivial changes to the set system (e.g., duplicating elements or adding an additional set). Thus, it is useful to define a more robust variant of discrepancy. The *hereditary discrepancy*, initially introduced by Lovász, Spencer, and Vesztergombi [LSV86], is one such variant. For a matrix $\mathbf{A}$, it is the maximum discrepancy over all sub-matrices, i.e.,

$$\operatorname{herdisc}(\mathbf{A}) = \max_{\mathbf{B}} \operatorname{disc}(\mathbf{B}), \tag{1.1}$$

where $\mathbf{B}$ ranges over submatrices of $\mathbf{A}$. Since adding rows can never decrease the discrepancy of a matrix, it suffices to consider only sub-matrices $\mathbf{B}$ whose columns are a subset of the columns of $\mathbf{A}$.

Discrepancy can also be interpreted as a rounding problem. Applying the linear transformation $x \mapsto \frac{1-x}{2}$, we have that $\operatorname{disc}(\mathbf{A}) = 2\min_{\chi\in\{0,1\}^n} \|\mathbf{A}(\mathbf{w} - \chi)\|_\infty$ where $\mathbf{w} = \frac{1}{2}\mathbb{1}$. This is the smallest possible error when rounding a zero-one combination of the columns of $\mathbf{A}$ to the vector $\mathbf{A}\mathbf{w}$ in

ℓ_∞-norm. If, instead of rounding to a fixed $\mathbf{w}$, we let $\mathbf{w}$ be an arbitrary vector in $[0,1]^n$, then we have defined the linear discrepancy of $\mathbf{A}$ with respect to $\mathbf{w}$, i.e.,

$$\operatorname{lindisc}(\mathbf{A},\mathbf{w}) = \min_{\mathbf{x}\in\{0,1\}^n} \|\mathbf{A}\,(\mathbf{w}-\mathbf{x})\|_\infty.$$

More generally, the *linear discrepancy* of $\mathbf{A}$ is the worst-case linear discrepancy over all $\mathbf{w} \in [0,1]^n$.

$$\operatorname{lindisc}(\mathbf{A}) = \max_{\mathbf{w}\in[0,1]^n} \operatorname{lindisc}(\mathbf{A},\mathbf{w}). \tag{1.2}$$

There is an illuminating geometric interpretation of discrepancy, hereditary discrepancy, and linear discrepancy. In order to compare these three functions, we must ensure that they have the same domain. Thus, we will use the definition of linear discrepancy where $\mathbf{w} \in [-1,1]^n$ and $\mathbf{x} \in \{-1,1\}^n$, i.e., *before* applying the map $x \mapsto \frac{1-x}{2}$. Denote by $\mathbf{a}_i^\top$ the i^{th} row of $\mathbf{A}$. For discrepancy, we want to find the smallest r such that $|\langle \mathbf{a}_i, \chi\rangle| \le r$ for all $i \in [m]$. To see this, note that $|\langle \mathbf{a}_i, \chi\rangle| \le r$ represents a pair of linear constraints $-r \le \langle \mathbf{a}_i, \chi\rangle \le r$. Over all rows, we obtain the linear system $-r\mathbb{1} \le \mathbf{A}\chi \le r\mathbb{1}$. Let $\mathcal{P}_\mathbf{A} = \{\mathbf{x} \in [-1,1]^n : \|\mathbf{A}\mathbf{x}\|_\infty \le 1\}$ be the *fundamental parallelepiped* of $\mathbf{A}$. The colorings χ are exactly the corners of the $[-1,1]^n$ hypercube. If a constant r satisfies $-r\mathbb{1} \le \mathbf{A}\chi \le r\mathbb{1}$, then we have a certificate of a coloring that achieves $\operatorname{disc}(\mathbf{A},\chi) \le r$. Thus, $\operatorname{disc}(\mathbf{A})$ is the *smallest* value of r such that $r \cdot \mathcal{P}_\mathbf{A}$ centered at the origin contains *some* corner of the $[-1,1]^n$ hypercube. Equivalently, $\operatorname{disc}(\mathbf{A})$ is the smallest scaling r of 2^n copies of $r \cdot \mathcal{P}_\mathbf{A}$ placed at the corners of $[-1,1]^n$ which contains the origin. Using the same geometric view, hereditary discrepancy is the smallest scaling s so that copies of $s \cdot \mathcal{P}_\mathbf{A}$ placed at the corners of $[-1,1]^n$ containing a face of $[-1,1]^n$ cover the center of the same face, e.g., since the hypercube $[-1,1]^n$ is the full-dimensional face, we must be able to cover the origin with a copy of $\mathcal{P}_\mathbf{A}$ scaled by s and centered at a corner of $[-1,1]^n$. Similarly, linear discrepancy is the smallest scaling t so that the 2^n copies of $t \cdot \mathcal{P}_\mathbf{A}$ placed at the corners of $[-1,1]^n$ cover every point of $[-1,1]^n$. See Figure 1.1.

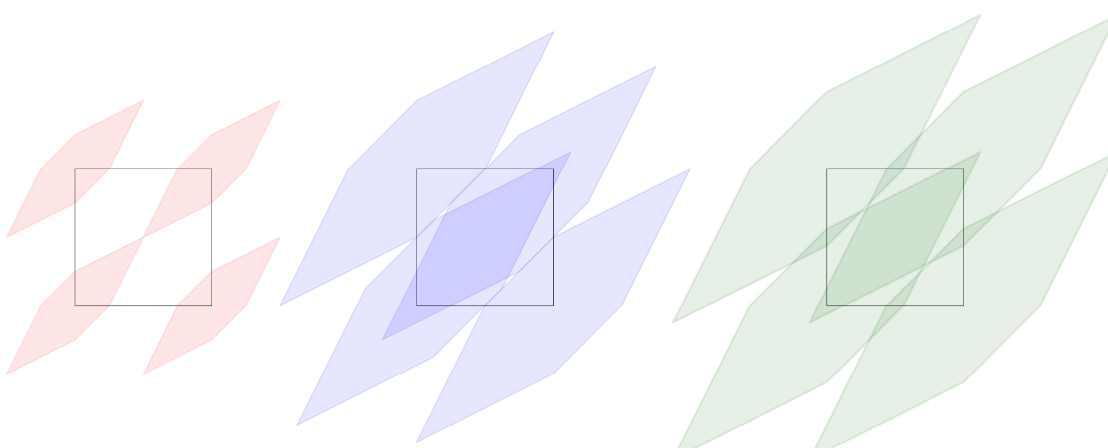


Figure 1.1: The scaling of the fundamental parallelopiped represents discrepancy (left), hereditary discrepancy (center), and linear discrepancy (right).

Note that the relationship between hereditary and linear discrepancy is nuanced. Even though hereditary discrepancy only requires covering the centers of all the faces of $[-1,1]^n$, while linear discrepancy requires covering every point of $[-1,1]^n$, the parallelepipdeds of hereditary discrepancy *must be centered at the corners surrounding the face.* Further, if we consider the one-row matrix $\mathbf{A} = [1, 2, ..., 2^{n-1}]$, we have $\text{herdisc}(\mathbf{A}) = 2^{n-1}$ while $\text{lindisc}(\mathbf{A}) = 1$ (with domain $\{-1,1\}^n$) so, at least in this case, $\text{lindisc}(\mathbf{A}) \leq \text{herdisc}(\mathbf{A})$.

A fundamental result of Lovász, Spencer, and Vesztergombi shows that linear discrepancy can generally be bounded above by twice the hereditary discrepancy.

Theorem 1. [LSV86, Corollary 1]. $\text{lindisc}(\mathbf{A}) \leq 2 \cdot \text{herdisc}(\mathbf{A})$.

The determinant is another inherent property of a matrix $\mathbf{A}$. Let the *determinant lower bound* of $\mathbf{A}$ be defined as

$$\text{detlb}(\mathbf{A}) \coloneqq \max_{k \in \min(m,n)} \max_{\mathbf{B}} |\det(\mathbf{B})|^{1/k}$$

where $\mathbf{B}$ is a $k \times k$ submatrix of $\mathbf{A}$. The work of Lovász, Spencer, and Vestergombi also highlights a connection between the hereditary discrepancy of a matrix $\mathbf{A}$ and its determinant lower bound.

Theorem 2. [LSV86, Lemma 2]. *For any matrix* $\mathbf{A} \in \mathbb{R}^{m \times n}$,

$$\text{herdisc}(\mathbf{A}) \geq \frac{1}{2} \max_{k \in [\min(m,n)]} \max_{\mathbf{B}} |\det(\mathbf{B})|^{1/k}$$

where $\mathbf{B}$ *ranges over all* $k \times k$ *submatrices of* $\mathbf{A}$.

To get a rough upper-bound for the discrepancy of a matrix $\mathbf{A}$, consider the random coloring. By the Chernoff bound, the probability that the discrepancy of any row exceeds $c\sqrt{n \log m}$ for some constant c occurs with probability at most m^{-c}. Taking a union bound over all m rows, with positive probability, the coloring achieves discrepancy at most $O(\sqrt{n \log m})$. Conversely, suppose that $\mathbf{A}$ is a random incidence matrix. Then, the probability that any row achieves discrepancy $\Omega(\sqrt{n})$ is a constant. Using the probabilistic method, we have that

$$\text{disc}(\mathbf{A}) = \Omega\left(\sqrt{n}\right). \tag{1.3}$$

Thus, the gap between the upper and lower bounds is $\sqrt{\log m}$.

1.2.1 Discrepancy Minimization

Non-constructive Results

Spencer [Spe85], in his celebrated 1985 work[2], showed that the upper bound on the discrepancy of $\mathbf{A}$ can be improved to $O\left(\sqrt{n \log \frac{2m}{n}}\right)$. When $m = n$, his paper achieves $\text{disc}(\mathbf{A}) \leq 6\sqrt{n}$, giving it a striking name: *Six Standard Deviations Suffice*. In combination with the discrepancy lower bound for the random matrix in Equation (1.3), this result is tight.

The following is a short summary of Spencer's approach. See Matoušek's textbook [Mat09] or Spencer's exposition [Spe94] for more details. In order to obtain the $O\left(\sqrt{n \log \frac{2m}{n}}\right)$ bound on $\text{disc}(\mathbf{A})$, Spencer's proof uses Beck's partial coloring method. The goal is to construct a coloring

[2] Similar results were shown in a geometric setting by Gluskin [Glu89].

vector $\mathbf{u}$ incrementally. Begin with $\mathbf{u} = \mathbf{0}$. At every iteration, maintain the condition that the entries of $\mathbf{u}$ are in $\{-1, 0, +1\}$ and that the product of $\mathbf{A}$ and $\mathbf{u}$ achieves low discrepancy. For variables $i \in [n]$, i is *floating* or *fixed* depending on whether $u_i = 0$ or $u_i \in \{-1, +1\}$ respectively. If $\mathbf{u}$ has a floating variable, then it is a *partial coloring*. When all variables are fixed, $\mathbf{u}$ is a *full coloring*. At each step we update $\mathbf{u}$ so that at least half of the floating variables become fixed. This process of constructing a full coloring from a series of partial colorings is called the *partial coloring method*. There is a related notion of the *fractional coloring method* which allows coordinates of $\mathbf{u}$ to take values in $[-1, 1]$ throughout the coloring process.

The key idea of Spencer's proof is to show the existence of two full colorings which assign many coordinates the different color and have similar discrepancies on all sets in the set-system. Then, it suffices to only color those elements which were assigned the same color by the two full colorings to obtain a partial coloring. In order to show the existence of these full colorings, Spencer used the Pigeon-Hole Principle. Despite the existence of a coloring χ that achieves $\text{disc}(\mathbf{A}, \chi) \leq 6\sqrt{n}$, Spencer's result is non-constructive due to the central role of the Pigeon Hole Principle in the proof. Hence, Spencer posed as an open problem: Does there exist a polynomial time algorithm which finds a coloring achieving $O(\sqrt{n})$ discrepancy? After more than two decades, Bansal was able to answer in the affirmative [Ban10].

Constructive Results

In 2010, Bansal [Ban10] resolved Spencer's conjecture by showing that it *was* possible to construct a coloring which achieves the discrepancy bounds of Spencer's theorem using the solution of an SDP. Subsequent to Bansal's work, others have sought to simplify the proof, making it truly constructive and more general [LM15; Rot17].

The original proof of Bansal is a randomized poly-time algorithm which finds a coloring of $\mathbf{A}$ achieving discrepancy $O\left(\sqrt{n}\log(2m/n)\right)$. Note that this is slightly weaker than Spencer's full result by a factor of $O\left(\sqrt{\log(2m/n)}\right)$, but when $m = n$, he recovers Spencer's bound of $O\left(\sqrt{n}\right)$. Roughly, his algorithm constructs a feasible SDP using constraints on the sets of the set-system given by the current fractional coloring. He takes the solution of the SDP and updates the fractional coloring.

Inspired by Bansal's work, Lovett and Meka [LM15] came up with a constructive algorithm using random walks instead of SDPs. Unlike Bansal who used Spencer's result, this work was "truely constructive" as it did not rely on prior existential results. Their result also eliminates the $O\left(\sqrt{\log(2m/n)}\right)$ factor gap between the general non-constructive result of Spencer and the constructive result of Bansal. In particular, their main theorem states: there exists a randomized algorithm which computes in time $\widetilde{O}\left((n+m)^3\right)$, a coloring for $\mathbf{A}$ which achieves discrepancy at most $O\left(\sqrt{n\log(2m/n)}\right)$ with constant probability. The key to this main theorem is the following partial coloring theorem.

Theorem 3. (Lovett-Meka partial coloring [LM15, Theorem 4].) *Let* $\mathbf{a}_1, ..., \mathbf{a}_m \in \mathbb{R}^n$ *and* $\mathbf{u}_0 \in [-1, 1]^n$ *be the initial starting position. Further let* $\Delta_1, ..., \Delta_m \geq 0$ *be thresholds such that*

$$\sum_{j=1}^{m} \exp\left(-\Delta_j^2/16\right) \leq n/16.$$

For any small error parameter $\epsilon > 0$, there exists an efficient randomized algorithm which, with constant probability, say $1/10$, finds a point $\mathbf{u} \in [-1,1]^n$ satisfying

1. *Discrepancy Constraint: for all $j \in [m]$,*

$$|\langle \mathbf{u} - \mathbf{u}_0, \mathbf{a}_j \rangle| \leq \Delta_j \|\mathbf{a}_j\|_2.$$

2. *Variable Constraint: $|u_i| \geq 1 - \epsilon$ for at least $n/2$ indices $i \in [n]$.*

Their algorithm produces a full coloring $\mathbf{u}$ by repeatedly applying Theorem 3.

Rothvoß [Rot17] also has a constructive discrepancy result which, instead of finding a random rounding based on the solutions of an SDP as in Bansal or random walks in the space orthogonal to the tight discrepancy and integrality constraints as in Lovett-Meka, projects the fractional coloring vector in a random direction. This result is shown in Theorem 4.

Theorem 4. (Rothvoß Partial Coloring [Rot17, Theorem 3.1]). *Let $P \subset \mathbb{R}^n$ be a symmetric convex set with Gaussian measure at least $e^{-n/500}$ and suppose that we have a polynomial time separation oracle for P. Then there exists a randomized polynomial time algorithm which finds a point $\mathbf{u} \in P \cap [-1,1]^n$ with $u_i \in \{-1,1\}$ for at least $n/9000$ many coordinates.*

Again, the algorithm produces a fractional coloring $\mathbf{u}$ by repeatedly applying Theorem 4.

Further, one can ask: what is the expected discrepancy of a random matrix? Hoberg and Rothvoß and others [FS20; HR19; TMR20] consider matrices $\mathbf{A} \in \{0,1\}^{m \times n}$ drawn from a Bernoulli ensemble where each entry is Bernoulli with probability $p \geq \log n/m$ with high probability. When $n = \Omega(m^2 \log m)$ they show that $\mathrm{disc}(\mathbf{A}) = O(1)$. Later work by Altschuler and Niles-Weed[AN21], and independently MacRury et al. [MMPP23], improved this result and showed that for a Bernoulli ensemble with parameter $p \coloneqq p(n)$, there exists a universal constant $C > 0$ such that if $n \geq Cm \log m$, then $\mathrm{disc}(\mathbf{A}) \leq 1$ with high probability. In addition, Altschuler and Niles-Weed [AN21] showed that for a Poisson ensemble with parameter λ for every entry, when $n \geq m$, $\mathrm{disc}(\mathbf{A}) = O\left(2^{-n/m}\sqrt{n\lambda} + 1\right)$ with high probability. When $m/np \to 0$, every row of $\mathbf{A}$ has many non-zero entries in expectation so the techniques are similar to those from prior works: second moment and local limit theorems. When $m/np \not\to 0$, the rows of $\mathbf{A}$ may have very few non-zero entries and such techniques, whose error is on the order of $O(1/w)$ where w is the number of non-zero entries, are too coarse, so the authors require more sophisticated probabilistic tools.

1.2.2 Determinant Lower Bound and Hereditary Discrepancy

From the result of Lovász, Spencer, and Vestergombi Theorem 2, recall that $\mathrm{herdisc}(\mathbf{A}) \geq \frac{1}{2}\mathrm{detlb}(\mathbf{A})$ for matrices $\mathbf{A} \in \{0,1\}^{m \times n}$. Matoušek [Mat13] provided an upper-bound for hereditary discrepancy in-terms of the determinant lower-bound.

Theorem 5 (Matoušek [Mat13], Theorem 2)**.** *For any matrix $\mathbf{A} \in \{0,1\}^{m \times n}$,*

$$\mathrm{herdisc}(\mathbf{A}) \leq O\left(\mathrm{detlb}(\mathbf{A}) \log(mn) \sqrt{\log n}\right). \tag{1.4}$$

His proof required another variant of discrepancy, vector discrepancy, denoted $\mathrm{vecdisc}(\mathbf{A})$. This quantity is similar to discrepancy with elements of the universe "colored" by vectors rather than by ± 1. It is the smallest $\lambda \geq 0$ for which there exists unit vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ satisfing the SDP:

$$\|\mathbf{V}\mathbf{a}_i\|_2 \leq \lambda \text{ for each set } j \in [m].$$

where $\mathbf{V}$ is the matrix with columns $\mathbf{v}_1, ..., \mathbf{v}_n$ and each $\mathbf{a}_j$ is the j_{th} row of $\mathbf{A}$. Vector discrepancy is a lower bound on discrepancy since a coloring $\chi : X \to \{\pm 1\}$ can be interpreted as a set of vectors where all vectors are parallel to each other. The *hereditary vector discrepancy of* $\mathbf{A}$, denoted hervecdisc, is the maximum vector discrepancy of any subset of the columns of $\mathbf{A}$.

We have that $\mathrm{vecdisc}(\mathbf{A}) \leq \mathrm{disc}(\mathbf{A}) \leq \mathrm{herdisc}(\mathbf{A})$ and

$$\mathrm{hervecdisc}(\mathbf{A}) \leq \mathrm{herdisc}(\mathbf{A}) \leq \mathrm{hervecdisc}(\mathbf{A}) \log mn$$

where the second inequality follows from a theorem of [Ban10]. Matoušek [Mat13] showed that

$$\mathrm{vecdisc}(\mathbf{A}) \leq \mathrm{detlb}(\mathbf{A})\sqrt{\log n} \leq \mathrm{herdisc}(\mathbf{A})\sqrt{\log n}.$$

To obtain the inequality of Equation 1.4, combine Bansal's theorem (second inequality from the second chain) with the first inequality by replacing $\mathbf{A}$ with a sub-matrix $\mathbf{B}$ with largest vector discrepancy among all sub-matrices. To show the inequality of Bansal's paper, apply SDP duality to the definition of vector discrepancy and consider a subset of roughly equal positive coordinates in the solution of the dual.

In a sense, this definition generalizes total unimodularity as TUM matrices are *exactly* the set of hereditary discrepancy one matrices with entries in $\{-1, 0, 1\}$. From Theorem 1 of Lovász, Spencer, and Vesztergombi [LSV86], the determinant lower bound of $\mathbf{A}$ satisfies $2 \cdot \mathrm{herdisc}(\mathbf{A}) \geq \mathrm{detlb}(\mathbf{A})$ so for TUMs, both quantities are equal to one.

Matoušek's bound was not believed to be tight as the largest known value of $\frac{\mathrm{herdisc}(\mathbf{A})}{\mathrm{detlb}(\mathbf{A})}$ is on the order of $\log n$. Both the three permutations family of Newman, Neiman, and Nikolov [NNN12] (see also [Fra21]) and the construction of Pálvölgyi [Pál10] that we described above achieve this gap.

Jiang and Reis [JR22] used a similar approach but instead of using hereditary vector discrepancy like Matoušek [Mat13], used hereditary partial vector discrepancy, denoted $\mathrm{herpvdisc}(\mathbf{A})$ for a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. $\mathrm{herpvdisc}(\mathbf{A})$ is defined to be the smallest $\lambda \geq 0$ satisfying: For every subset $S \subseteq [n]$, there exist unit vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{R}^n$ satisfying the following SDP constraints:

$$\begin{aligned}
\|\sum_{j \in S} a_{i,j}\mathbf{v}_j\|_2^2 &\leq \lambda^2 \qquad \forall i \in [m] \\
\sum_{j \in [n]} \|\mathbf{v}_j\|_2^2 &\geq \frac{|S|}{2} \\
\|\mathbf{v}_j\|_2^2 &\leq 1 \qquad \forall j \in S, \\
\|\mathbf{v}_j\|_2^2 &= 0 \qquad \forall j \in [n] \backslash S.
\end{aligned}$$

They were able to show that

$$\begin{aligned} \mathrm{herdisc}(\mathbf{A}) &\le O\left(\sqrt{\log(m)\log(n)} \cdot \mathrm{herpvdisc}(\mathbf{A})\right) \\ \mathrm{herpvdisc}(\mathbf{A}) &\le O\left(\mathrm{detlb}(\mathbf{A})\right) \end{aligned}$$

which together achieves $\frac{\mathrm{herdisc}(\mathbf{A})}{\mathrm{detlb}(\mathbf{A})} \le O\left(\sqrt{\log(m)\log(n)}\right)$.

1.2.3 Main Results

Our thesis contributes results relating to both hereditary and linear discrepancy.

Determinant Lower Bound and Hereditary Discrepancy

Jiang and Reis [JR22] showed that $\frac{\mathrm{herdisc}(\mathbf{A})}{\mathrm{detLB}(\mathbf{A})} \le O\left(\sqrt{\log(m)\log(n)}\right)$. This bound is tight when $m = O(n^c)$ for constant c by a set system of Pálvölgyi (see Section 2.1.2), but we wanted to know if the factor of $\sqrt{\log m}$ is necessary when $m \gg n$. We will show in Chapter 2 that it is, by considering a particular matrix $\mathbf{A} = \mathbf{P}_N \otimes \mathbf{A}_k$ where $\mathbf{P}_N$ is the $2^N \times N$ power matrix whose rows are all the k-ary binary strings and $\mathbf{A}_k$ is the Haar Basis defined in Section 2.1.3. Proving Theorem 6 requires the use of discrepancy amplification and a lemma from Matoušek's original result [Mat13].

Theorem 6 (Hereditary Discrepancy and Detlb Lower Bound)**.** *For any real number $\varepsilon \in (0,1)$, any integers $n \ge 2$ and $m \in \left[n, 2^{n^{1-\varepsilon}}\right]$, there exists a matrix $\mathbf{A} \in \{0,1\}^{m \times n}$ such that*

$$\frac{\mathrm{herdisc}(\mathbf{A})}{\mathrm{detlb}(\mathbf{A})} \gtrsim \sqrt{\log m \log n}. \tag{1.5}$$

Note that the lower bound in Theorem 6 only holds for $m \le 2^{n^{1-\varepsilon}}$ where ε is an arbitrarily small but fixed constant. This leaves open whether such a lower bound holds all the way to $m = 2^n$. The next theorem gives a new upper bound on herdisc($\mathbf{A}$) in terms of detlb($\mathbf{A}$), which implies that Theorem 6 cannot be extended to $m = 2^{\omega(n/\log n)}$.

Theorem 7 (Hereditary Discrepancy and Detlb Upper Bound)**.** *For all positive integers m and n, and all matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$, we have*

$$\frac{\mathrm{herdisc}(\mathbf{A})}{\mathrm{detlb}(\mathbf{A})} \lesssim \sqrt{n}.$$

This upper bound is based on the relationship between the volume lower bound on discrepancy studied in [DNTT18], and the determinant lower bound. In particular, the volume lower bound is bounded by a constant multiple of $\sqrt{n} \cdot \mathrm{detlb}(\mathbf{A})$ and, using a result of [DNTT18], it is possible to characterize the hereditary discrepancy of partial colorings in terms of the volume lower bound.

Linear Discrepancy

Let us begin with a simple observation: When $\mathbf{A}$ is a single row matrix, deciding $\mathrm{lindisc}(\mathbf{A}, t\mathbb{1}) = 0$ is equivalent to the NP-hard Subset Sum problem with target sum $t\sum_{j=1}^{n} A_{1,j}$. This *does not show* that computing $\mathrm{lindisc}(\mathbf{A})$ is NP-hard, but suggests that linear discrepancy is closely related to hard problems. In this work we show a hardness result for linear discrepancy in Theorem 8 but suspect that the problem is actually Π_2-Hard. Later, Manurangsi [Man21] showed that linear discrepancy *is indeed* Π_2-Hard to approximate up to $\frac{9}{8} - \epsilon$ for $\epsilon > 0$.

Theorem 8 (Linear Discrepancy Hardness)**.** *Given an $m \times n$ matrix $\mathbf{A}$ with rational entries, and a rational number t, deciding whether $\mathrm{lindisc}(\mathbf{A}) \leq t$, is NP-hard and is contained in the class Π_2.*

We present algorithms for computing linear discrepancy exactly when the matrix $\mathbf{A}$ has a constant number of rows. Beginning with a matrix consisting of a single row, we have,

Theorem 9 (Linear Discrepancy for One Row Matrix)**.** *For any matrix $\mathbf{A} \in \mathbb{R}^{1\times n}$, $\mathrm{lindisc}(\mathbf{A})$ can be computed in time $O(n \log n)$.*

Contrast Theorem 9 with the observation at the start of this section where computing $\mathrm{lindisc}(\mathbf{A}, \mathbf{w})$ *for a fixed* $\mathbf{w}$ is hard even for a single-row matrix $\mathbf{A}$. This suggests that even though the structure of $\|\mathbf{A}\mathbf{w}\|_\infty$ maybe hard to predict, the structure of the fundamental parallelepiped is more regular — at least in low dimensions.

We also give a rounding algorithm, showing that, for single row matrices, any $\mathbf{w} \in \mathbb{Q}^n$ can be efficiently rounded to a $\{0, 1\}^n$ vector with error bounded above by the linear discrepancy of the one-row matrix.

Theorem 10 (Linear Discrepancy for One Row Matrix Approximation)**.** *For any matrix $\mathbf{A} \in \mathbb{Q}^{1\times n}$ and any $\mathbf{w} \in ([0, 1] \cap \mathbb{Q})^n$, we can find an $\mathbf{x} \in \{0, 1\}^n$ such that $\|\mathbf{A}(\mathbf{w} - \mathbf{x})\|_\infty \leq \mathrm{lindisc}(\mathbf{A})$ in time $O(n \log n)$.*

We can extend Theorem 9 to the case of matrices with a bounded number of rows as shown in Theorem 11, with the additional assumption that the entries of $\mathbf{A}$ are bounded. We leave open the task of removing this assumption. As before, a corresponding approximation algorithm appears in Theorem 12.

Theorem 11 (Linear Discrepancy for Matrices with Constantly Many Rows)**.** *For any matrix $\mathbf{A} \in \mathbb{Z}^{d\times n}$ where d is some fixed constant and $\max_{i,j} |A_{i,j}| \leq \delta$, $\mathrm{lindisc}(\mathbf{A})$ can be computed in time $O\left(d(n\delta)^{d^2+d}\right)$.*

Theorem 12 (Approximate Linear Discrepancy for Matrices)**.** *For any matrix $\mathbf{A} \in \mathbb{Q}^{m\times n}$, the linear discrepancy of $\mathbf{A}$ can be approximated in polynomial time within a factor of 2^{n+1}.*

1.3 Graph Partition

A college dean needs to assign students to one of two equal-size dorms. Each student has a list of preferred dormmates. Every person on their preferred list assigned to the same dorm contributes

$+1$ to the student's satisfaction and -1 if they are assigned to the other dorm. In particular, a student is *satisfied* if they are in a dorm with at least half of the people on their preferred list. The standard Dean's Problem seeks to find a partition where every student is satisfied, but many other variants are possible. The real-valued variant sees students assigning real values to the other students allowing for more precise representation of the relationships. The symmetric preferences variant requires preference lists to be symmetric, i.e., if student i is in student j's list then student j must be in student i's list as well. The k-dorms variant finds the dean assigning students to k equal-size dorms while maintaining one of several generalized definitions of satisfaction which we will describe in detail in Chapter 5.

We can express the *Dean's Problem* using directed graphs $G = (V, E)$ where the vertices represent the students and the edges represent preferences, i.e., edge $(u, v) \in E$ if and only if student v is the preference list of student u. If whenever a student u is in the preference list of a student v, then v must also be in preference lists of u we can model the Dean's Problem using an undirected graph.

Let $d_S(v)$, for any $S \subseteq V$, be the number of neighbours of v in S and $d(v) \coloneqq d_G(v)$ be the number of neighbours of v in the full graph. A partition (P_1, P_2) of the V into two *non-trivial*[3] parts is a *γ-friendly partition* if for every vertex $v \in P_1$, $d_{P_1}(v) \geq d_{P_2}(v) + \gamma$ and for every vertex $u \in P_2$, $d_{P_2}(u) \geq d_{P_1}(u) + \gamma$. We call a k-partition $\pi = (P_1, ..., P_k)$ *balanced* if $||P_i| - |P_j|| \leq 1$ for all $i, j \in [k]$, i.e., the size of any two parts differs by at most one. In particular, we call a balanced 2-partition a *balanced bisection.* Using this terminology, the Dean's Problem seeks to find a 0-friendly bisection. We call these *satisfactory bisections.*

The notion of a desirable partition of the students into groups is captured by two closely related terms: envy-free partitions and friendly partitions.

1.3.1 Envy-Free Partitions

Envy-free partitions is a well-studied notion in the Social Choice literature. In the standard case, we distribute a set of indivisible items $\mathcal{R}$ among a set of agents $\mathcal{A}$. For each agent a_i, let $v_i : \mathcal{P}(\mathcal{R}) \to \mathbb{R}$ be the utility function which determines the value a_i places on each subset of items. The goal is to find a partition $\pi : \mathcal{R} \to \mathcal{A}$ of items into bundles assigned to each agent satisfying certain properties. We say that π is *envy-free up to one* (EF-1)[4] if for every agent a_i with bundle b_i, after removing the most valuable item from the bundle of some agent a_j to obtain bundle b'_j, $v_i(b_i) \geq v_i(b'_j)$. For weakly additive utilities[5] the round-robin protocol[6] finds an EF-1 partition in linear time.

We consider a variant of the problem where $\mathcal{A}$ and $\mathcal{R}$ are the same set, i.e., agents define utility functions over other agents. In the Social Choice literature the setting where agents have preferences over all subsets of other agents, is known as hedonic games. When pairs of agents feel the same way about one another, i.e. $v_i(a_j) = v_j(a_i)$, the games are *symmetric.* When agent utilities are boolean, then the games are *boolean.*

A *k-partition* $X = (X_0, \ldots, X_{k-1})$ is a collection of subsets of V where all the subsets are disjoint

[3] A partition is *non-trivial* if no part is empty.

[4] There is a related notion of EF-x which removes the *least valuable* item of the other agent's bundle.

[5] A utility function is weakly additive if, for four bundles A, B, C, D with $A \cap C = \emptyset$ and $B \cap D = \emptyset$, A is preferable to B and C is preferable to D then $A \sqcup C$ is preferable to $B \sqcup D$.

[6] In the *round-robin* protocol agents take turns picking their most prefered item.

(i.e., $X_j \cap X_{j'} = \emptyset$ for all distinct $j, j' \in [k]$), non-empty (i.e., $X_j \neq \emptyset$ for all $j \in [k]$), and cover all of V (i.e., $\cup_{j\in[k]} X_j = V$). Each X_j is a *part* of X. We denote by $X(i)$ the part to which agent i belongs. Assume that $n \geq k$, so a k-partition exists. A k-partition is *balanced* if $\lfloor n/k \rfloor \leq |X_j| \leq \lceil n/k \rceil$ for all $j \in [k]$, and is *imbalanced* otherwise.

In this work, we focus on envy-free balanced k-partitions in boolean symmetric hedonic games formally defined in Definition 13. Note that the utility of agent i for $S \subseteq V$ is denoted by $u_i(S)$ and we assume that utilities are additive, i.e., $u_i(S) = \sum_{i' \in S} u_i(i') = |S \cap N(i)|$.

Definition 13. *For $r \geq 0$, a k-partition X is* envy-free up to r, *denoted EF-r, if, for every pair of agents $i, i' \in V$, $u_i(X(i)) \geq u_i(X(i') \cup \{i\} \setminus \{i'\}) - r$. When $r = 0$, we simply refer to this as envy-freeness (EF).*

Much like the Dean's Problem, we can model the envy-free balanced partition problem with graphs in the natural way: $V = [n]$ is a set of agents who are members of a social network. The network is represented by an undirected graph $G = (V, E)$, where agents are nodes and an edge $(i, i') \in E$ indicates friendship between agents i and i'. This induces a utility function for agent i, denoted $u_i : V \to \{0, 1\}$, where $u_i(i') = 1$ if $(i, i') \in E$ and 0 otherwise. Let $N_G(i)$ denote the set of neighbors of agent i in G, i.e., $N_G(i) = \{i' \in V : (i, i') \in E\}$. $d_G(i) = |N_G(i)|$ is the degree of agent i and we omit G to write $d(i)$ when the graph is clear from context.

Suppose that the adjacency matrix $\mathbf{A}$ of the graph $G = (V, E)$, has discrepancy $\text{disc}(\mathbf{A})$. Then there exists a partition (A, B) of the vertices such that for every vertex $a_i \in A$,

$$\left| \sum_{a_j \in A} u_i(a_j) - \sum_{a_k \in B} u_i(a_k) \right| \leq \text{disc}(\mathbf{A})$$

and similarly for every vertex in B. This means that the partition is envy-free up to $\text{disc}(\mathbf{A})$ i.e. no vertex has too many friends in the other part and no vertex has too many non-friends in the other part either. Since only the first property is necessary for envy-free partitions, we can define a variant of discrepancy to better model this problem.

Looking back at the definition of discrepancy, we note that in this setting every vertex corresponds to both a set in the set-system and an element in the universe of $\mathbf{A}$. Thus, for set S_i corresponding to element i, we can maximize $\chi_i \cdot \chi(S)$ instead of the absolute value of $\chi(S)$. Let this variant of discrepancy be called one-sided discrepancy

$$\text{sdisc}(\mathcal{S}, X) := \min_{\chi: X \to \{-1,1\}} \max_{S_i \in \mathcal{S}} \chi_i \cdot \chi(S_i).$$

In terms of the adjacency matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$,

$$\text{sdisc}(\mathbf{A}) = \min_t \exists \chi \in \{-1, 1\}^n : \chi_i \cdot \langle \mathbf{a}_i, \chi \rangle \leq t \text{ for all } i \in [m]$$

where $\mathbf{a}_1^\top, ..., \mathbf{a}_m^\top$ are the rows of $\mathbf{A}$.

1.3.2 Friendly Partitions

Friendly partitions are similar to envy-free partitions and have been studied in various fields under various names. A non-exhaustive list includes: *satisfactory* [BTV10] or *internal* [BL16] partitions in graph theory, *assortative partitions* [FKNSS22; BAKZ22] or *offensive alliances* [FFGHHKLS22] in statistical physics, and *local max-cut* [CGVYZ20] in optimization among many others.

Indeed, local min-cut is a useful way to think about friendly partitions. If there exists a min-cut where both parts contain at least two vertices, then this partition would already be 0-friendly; Every vertex must have as many neighbors in its own part as the other or else moving the vertex will decrease the number of edges crossing the cut. The difficulty lies with those graphs whose min-cut has a part containing only one node.

In the simplest case, a γ-friendly partition (P_1, P_2) of the vertices into two *non-empty* parts satisfies the following condition: for every vertex $v \in P_1$, there is a lower bound on the number of neighbours of v in P_1. In particular, $d_{P_1}(v) \geq d_{P_2}(v) + \gamma$. A similar requirement exists for all vertices $u \in P_2$, namely $d_{P_2}(u) \geq d_{P_1}(u) + \gamma$. If (P_1, P_2) satisfies these inequalities for all vertices in the graph when $\gamma = 0$, then the partition is *satisfactory*.

It is known that every graph has a (-2)-friendly partition by a non-constructive result of Stiebitz [Sti96]. The actual statement of Stiebitz' result is more general. For any two functions $a, b : V \to \mathbb{Z}$, an (a, b)-partition is a partition (A, B) such that $d_A(v) \geq a(v)$ and $d_B(v) \geq b(v)$. Stiebitz showed that if $d(v) \geq a(v) + b(v) + 1$ for every $v \in V$, then there exists an (a, b)-partition. By plugging in $\lfloor d(v)/2 \rfloor - 1$ for both $a(v)$ and $b(v)$, $d(v) \geq a(v) + b(v) + 1$ is true for every $v \in V$, so an (a, b)-partition exists and we can recover the existence of a (-2)-friendly partition.

Since then, Stiebitz' result was made constructive by Bazgan, Tuza, and Vanderpooten [BTV07], and extended constructively by Ban and Linial [BL16] to weighted graphs. For a *weighted graph* G, $d_S(v)$ is defined to be $\sum_{(v,u) \in E, u \in S} w_{v,u}$ where $w_{v,u}$ is the weight of edge (v, u).

We cannot control the size of the parts in the partition resulting from the above constructive algorithms. However, in many applications, including the Dean's Problem described at the beginning, we want the parts to be balanced. We encountered the following deceptively simple conjecture in a prior work of Bollobás and Scott [BS02]:

Conjecture 14. *Every graph has a* (-2)*-friendly bisection.*

We leave the conjecture open and instead consider a variant of the problem pertaining to Erdös-Rényi random graph $G \sim \mathcal{G}_{n,1/2}$. This follows a recent line of work by Ferber et al. [FKNSS22] who showed constructively that, with high probability, $G \sim \mathcal{G}(n, 1/2)$ has a bisection where all but $o(n)$ vertices are 0-friendly, answering a problem of Füredi [Gre, Problem 91]. Even more recently, a result of Minzer, Sah, and Sawhney [MSS23] showed non-constructively that, with high probability, *even an* $\Omega(\sqrt{n})$*-friendly bisection exists*, and determined the exact constant coefficient of the leading term.

In parallel, Dandi, Gamarnik, and Zdeborová [DGZ23], extending techniques of Gamarnik and Li [GL18], showed a similar result for random sparse graphs, where up to $o(n)$ vertices are allowed to buck the $\Omega(\sqrt{n})$-friendliness requirement. In a prior related work, Behrens et al. [BAKZ22] showed the γ-friendliness of random d-regular graphs for a range of γ values using non-rigorous, but standard, tools from statistical physics. Similar questions about locally energy minimizing states of Hamiltonians have been studied in statistical physics [ADLO19].

The above works focus on undirected graphs. In this work, we consider the γ-friendly bisection and balanced k-partitions problem for random *directed* graphs (digraphs) drawn from the directed Erdös-Rényi random graph model. Formally, our random graph model is described in Definition 15.

Definition 15 (Erdös-Rényi Digraph Model). *$\mathcal{G}_B(n)$ is a distribution on random digraphs with n vertices. To construct $G \sim \mathcal{G}_B(n)$, take the complete graph on n vertices and for each of the $n(n-1)$ directed edges, add the edge to G independently with probability* $\mathrm{Bern}(1/2)$*. Note that G does not have self loops.*

The friendliness problem on random digraphs has received much less attention than random graphs. Only a recent result of Anastos et al. [ACKK23], which we state later in Theorem 20, considered anything similar.

In addition to γ-friendly bisection results, we are also the first to consider γ-friendly balanced k-partitions for $k \geq 3$ in random digraphs. Note that Bazgan, Tuza, and Vanderpooten [BTV06, Section 5] describe three different ways of generalizing γ-friendly 2-partitions to k-parts.

Definition 16 (γ-Friendly k-Partitions). *Given a graph $G = (V, E)$ and a partition $\pi = (P_1, \ldots, P_k)$ of V into non-empty parts, we say that π is an*

- average γ-friendly k-partition *if, for every $v \in P_i$, $d_{P_i}(v) \geq \frac{1}{k-1} \sum_{j \neq i} d_{P_j}(v) + \gamma$;*
- max γ-friendly k-partition *if, for every $v \in P_i$, $d_{P_i}(v) \geq \max_{j \neq i} d_{P_j}(v) + \gamma$.*
- sum γ-friendly k-partition *if, for every $v \in P_i$, $d_{P_i}(v) \geq \sum_{j \neq i} d_{P_j}(v) + \gamma$.*

If $k = 2$, these three definitions are equivalent, so we say that $v \in P_i$ is a γ-friendly vertex if $d_{P_i}(v) \geq d_{P_j}(v) + \gamma$ for $j \neq i$.

Unfortunately deciding if there is an average, maximum, or sum γ-friendly k-partition is NP-complete [BTV06]. Fortunately, if we do not require the partitions to be balanced, as a consequence of Stiebitz, we have that average satisfactory k-partition *up to two* can be computed in polynomial time, i.e., there exists a k-partition where every vertex has $\lfloor d(v)/k \rfloor - 1$ neighbours in its own part. The hardness of the other two problems up to k is yet unknown.

In the k-part case we mostly consider average γ-friendliness, and sometimes omit "average" if there is no danger of confusion. We show that, with high probability, a graph $G \sim \mathcal{G}_B(n)$ has a $\Omega(\sqrt{n})$-friendly balanced k-partition when $k \geq 3$ which we discuss in more detail in the next section.

Again we want to know if there exists a γ-friendly balanced k-partition under each setting for directed graphs. Under certain numerical assumption, and again using the second moment method, we will show that *average* balanced friendly k-partitions exist for $\gamma = \Omega(\sqrt{|V|})$ with high probability.

1.3.3 Main Results

Envy-Free Balanced Paritititions

Since finding an EF-1 partition has been resolved constructively by Steibitz, Bazgan, Tuza, and Vanderpooten, we want to use discrepancy to finding the best possible envy-free bisection in a graph, i.e., an algorithm which finds a EF-γ partition for the largest value of γ. Here, for graph $G = (V, E)$, in addition to requiring that the partition (A, B) of V be *envy-free up to* k, we also require that the partition has parts which are approximately of equal size i.e. $||A| - |B|| \leq 1$.

To this end, we were able to show Theorem 17 using a result from multicolor discrepancy theory [HS14, Theorem 61].

Theorem 17 (Constructive Envy-Free Partitions)**.** *For any $k \geq 2$, a k-partition that is EF-$O(\sqrt{\frac{n}{k}\ln k})$ is guaranteed to exist and can be computed in polynomial time.*

Further, we present an algorithm which computes the EF-1 partition for trees which can be generalized to arbitrarily many parts as well as an algorithm which runs in polynomial time and distinguishes if a tree is EF-0 or EF-1 when dividing the vertices into two equal parts.

Theorem 18 (Compute EF-1 Balanced k-partition in Trees)**.** *For all $k \geq 2$ and every tree, we can find a balanced EF-1 k-partition in polynomial time.*

Theorem 19 (Distinguishing EF-0 and EF-1 in Trees)**.** *For $k = 2$ and every tree, we can distinguish whether it has an EF-1 partition or not in polynomial time.*

Friendly Blanced Partitions in Random Graphs

Our first result pertains to the existence of a γ-friendly bisection in random digraphs. It is similar to a result of Anastos et al. [ACKK23, Theorem 1.7] stated below in Theorem 20. In Anastos et al. the distribution $\mathbb{D}(n, p)$ is defined on binomial random directed graphs with n vertices and each of whose $n(n-1)$ possible directed edges are present independently with probability p. Comparing this with the distribution shown in Definition 15, we restrict p to be 1/2, but our results apply for any constant p after a slight modification to the proofs which we will describe after the overview of the proof procedure.

Theorem 20 (Theorem 1.7, Anastos et al.)**.** *Let p_n such that $np_n(1 - p_n) \to \infty$. Then with high probability $D \sim \mathbb{D}(n, p_n)$ has an $o(1)$-almost-majority bisection.*

Note that an $o(1)$-*almost-majority bisection* is a balanced 2-partition where all but $o(n)$ nodes of an n-vertex graph have at least half of their neighbours in the opposite part, i.e., for partition (P_1, P_2) all $v \in P_1$ satisfies $d_{P_2}(v) \geq d_{P_1}(v)$ and $u \in P_2$ satisfies $d_{P_1}(u) \geq d_{P_2}(u)$. These *external*, *majority*,*disassortative* partitions, or *local max-cuts* also appear frequently in the literature [ABPW17; BDP20; CGVYZ20].

For a graph G and its complement graph $\overline{G}$, we note that if G contains a γ-friendly bisection, then $\overline{G}$ contains a $(\gamma - 1)$-unfriendly bisection — corresponding to the majority bisections mentioned above. Let (A, B) be a majority bisection of $V(G)$. It follows that, in G, for every vertex $a \in A$, $d_A(v) \leq d_B(v)$ and similarly for the vertices of B. Since the parts are balanced, the number of vertices in the two parts can differ by at most one. If $\overline{d}_A(v)$ is the number of neighbours of v in $\overline{G}$,

$$\overline{d}_A(v) = |A| - d_A(v) \geq |A| - d_B(v) \geq |B| - 1 - d_B(v) = \overline{d}_B(v) - 1.$$

Thus the result of Anastos et al. implies that for $G \sim \mathbb{D}(n, p_n)$ where $|V(G)| = n$, with high probability, there exists a bisection where $n(1 - o(1))$ nodes are -1-friendly. Compare this with our result stated in Theorem 21. Unlike Theorem 1.7 of Anastos et al., Theorem 21 does not exclude $o(n)$ vertices from the (-1)-friendly requirement and is able to ensure that all vertices are (-1)-friendly. Unfortunately, as a trade-off, our result only holds with uniformly positive probability rather than with high probability though we believe the latter is possible with stronger tools.

Theorem 21 (γ-Friendly Bisections in Erdös-Rényi Random Digraphs)**.** *Let $G \sim \mathcal{G}_B(2n)$ as in Definition 15. Then, for all constant integers $\gamma \leq -1$, G has a γ-friendly bisection with uniform positive probability. Further, with high probability, G does not have a balanced γ-friendly for integer $\gamma \geq 0$.*

Similarly to Anastos et al., we use the second moment method, but require more delicate estimates of the second moment in order to avoid excluding vertices. In particular, we needed to formulate and prove a new variant of the Laplace approximation lemma. It is fairly typical that the second moment method does not yield high probability results by itself (see e.g. [APZ19]), and strengthening Theorem 21 to a high probability result is an interesting open problem.

Contrast Theorem 21 with Theorem 1.3 of Minzer, Sah, and Sawhney [MSS23] stated below.

Theorem 22 (Theorem 1.3, Minzer, Sah, and Sawhney)**.** *Fix $\epsilon > 0$ and let $G \sim \mathbb{G}(n, 1/2)$. Given Assumption 23, with high probability, G has a $(\gamma_{\text{crit}}/\sqrt{2} - \epsilon)\sqrt{n}$-friendly equipartition. Furthermore, with high probability G does not have a $(\gamma_{\text{crit}} + \epsilon)\sqrt{n}$-friendly equipartition.*

Assumption 23 (Assumption 1.5, Minzer, Sah, and Sawhney)**.** *For $\gamma \in \mathbb{R}$, define*

$$\begin{aligned}
F_1(\alpha) &:= \log 2 - \alpha^2 + \log\left(\mathbb{P}_{Z\sim\mathcal{N}(0,1)}\left[z \geq (\gamma + \alpha)\sqrt{2}\right]\right), \\
f(\beta, \alpha) &:= \mathbb{P}\left[\sqrt{\frac{\beta}{2}}Z_1 + \sqrt{\frac{1-\beta}{2}}Z_2 \geq \gamma + \alpha \wedge \sqrt{\frac{\beta}{2}}Z_1 - \sqrt{\frac{1-\beta}{2}}Z_2 \geq \gamma + \alpha\right], \\
F_2(\beta, \alpha_1, \alpha_2) &:= 2\log 2 - 2\beta\log\beta - 2(1-\beta)\log(1-\beta) - 2\alpha_1^2 - 2\alpha_2^2 \\
&\quad + 2\beta\log f(\beta, \alpha_1) + 2(1-\beta)\log f(1-\beta, \alpha_2).
\end{aligned}$$

Fix $\epsilon = 10^{-25}$ and $\gamma \in [\gamma_{\text{crit}} - \epsilon, \gamma_{\text{crit}} + \epsilon]$. Then we have

$$\sup_{\beta\in[0,0.001],\alpha_1,\alpha_2\in\mathbb{R}} F_2(\beta, \alpha_1, \alpha_2) = 2\sup_{\alpha\in\mathbb{R}} F_1(\alpha)$$

and

$$\sup_{\beta\in[0.001,0.999],\alpha_1,\alpha_2\in\mathbb{R}} F_2(\beta, \alpha_1, \alpha_2) = 4\sup_{\alpha\in\mathbb{R}} F_1(\alpha).$$

In the above notation $\mathbb{G}(n, 1/2)$ is the Erdös-Rényi graph with probability $1/2$. Minzer, Sah, and Sawhney showed that, with high probability a graph drawn from $\mathbb{G}(n, 1/2)$ has a $\Omega(\sqrt{n})$-friendly bisection. This suggests that random digraphs and random graphs have fundamental structural differences which are particularly relevant when considering friendly bisections. In the undirected case, the dependence among the vertices induced by the edges is enough to ensure that *most* graphs have a friendly bisection where, for every vertex, many more of its neighbours are in its part as opposed to the other one. In undirected graphs, the edges going out of different vertices are independent, and, as a result, the best achievable friendliness is negative with high probability.

Our other contribution pertains to the three generalizations of γ-friendly bisection stated in Definition 16. For the average γ-friendly balanced k-partition problem, our result requires a numerical assumption about the concavity of the probability of two partitions being simultaneously friendly for the same random digraph. This probability arises naturally in the second moment calculation. See Assumption 24. Upon first reading, it is *not* necessary to understand the precise statement of the assumption. The key is to note that the constants β and γ and the functions in Equation (1.6) and thus in Equation (1.7) only depend on k, the number of parts in the balanced partition, and not on n, where kn is the number of vertices in the graph. In Section 5.2, we will show that Equation (1.6) is an upper bound on the probability of two partitions with given overlap being friendly, and apply the Laplace method on $\ln g(\mathbf{A})$. Assumption 24 is needed to make sure that the application of the Laplace method is justified.

It is worth noting that *such assumptions are quite common in the literature*. Many of the prior works that we build upon have similar assumptions. Sometimes these can be verified numerically, see, e.g., Anastos et al. [ACKK23, Lemma 8.6], Minzer, Sah, Sawhney [MSS23, Assumption 1.5], and Dandi, Gamarnik, and Zdeborová [DGZ23]. More generally, making this type of assumptions, without or without numerical evidence, is common in applications of the second moment method, see, e.g., [APZ19, Hypothesis 3], [PX21a, Assumption 1], [DS19, Condition 1.2]. This is because the probabilities which appear in the second moment function are often complicated functions of the overlaps between two variable assignments (or partitions, in our case) and it is difficult to verify their concavity rigorously. Nevertheless, the functions in question are low dimensional, and the concavity can be observed: see Figure 5.4 which depicts g_n. Since g_n can be observed to be concave, $\ln g_n$ is concave as well.

Assumption 24. *For any integer $k > 2$ and constant $\delta > 0$, define the function $f : [0,1] \to [0,1]$ defined over the variable a as*

$$f(a) \coloneqq \mathbb{P}\left[\sigma_1 Z_1 + \sigma_2 Z_2 \geq (c_k - \delta) \wedge \sigma_1 Z_1 - \sigma_2 Z_2 \geq (c_k - \delta)\right], \tag{1.6}$$

where Z_1 and Z_2 are standard Gaussians, $c_k \coloneqq \Phi^{-1}\left(1 - \frac{1}{k}\right)$, $\sigma_1 \coloneqq \sqrt{\frac{(1+a)k-2}{2(k-1)}}$, and $\sigma_2 \coloneqq \sqrt{\frac{(1-a)k}{2(k-1)}}$. Further, define the function g

$$g\left(\mathbf{A}\right) = \prod_{i,j \in [k]} \left(\frac{f(a_{i,j})}{a_{i,j}}\right)^{a_{i,j}} \tag{1.7}$$

defined on the $k \times k$ doubly stochastic matrix $\mathbf{A}$ with entry in row i and column j denoted $a_{i,j}$.

The function $\ln g(\mathbf{A})$ has a unique maximum over the set of doubly stochastic matrices at $\mathbf{A}^* \coloneqq \frac{1}{k}\mathbb{J}$.

In order to provide some intuition for why Assumption 24 is true, we show that $\ln g$ has a *local maximum* at $\mathbb{J}/k$ in Claim 81. Further, it is worth noting that such assumptions are often made in the literature, for example in the work on the capacity of the Ising perceptron [DS19; APZ19; PX21b].

For γ-friendly balanced k-partitions we are able to show the following theorems.

Theorem 25 (Average γ-Friendly k-Partition)**.** *For any constant integer $k > 2$, let $G \sim \mathcal{G}_B(kn)$, $c_k := \Phi^{-1}(1 - 1/k)$ where Φ is the CDF of the standard normal distribution, and $\sigma = \sqrt{\frac{nk}{4(k-1)}}$. If Assumption 24 holds, then for any $\delta > 0$, with high probability G has an average $\sigma(c_k - \delta)$-friendly balanced k-partition.*

Conversely, with high probability, G will not have an average $\sigma \cdot c_k$-friendly balanced k-partition.

Theorem 26 (Maximum γ-Friendly k-Partition)**.** *For constant $k > 2$, let $G \sim \mathcal{G}_B(kn)$. Then, if $\gamma \geq 0$, with high probability G does not have any max γ-friendly balanced k-partitions.*

Theorem 27 (Sum γ-Friendly k-Partition)**.** *For constant $k > 2$, let $G \sim \mathcal{G}_B(kn)$. Then,* even with *$\gamma \lesssim_k -n$, with high probability G does not have any sum γ-friendly balanced k-partitions.*

Note that Theorem 25 shows that, for any $k > 2$, a random digraph has, with high probability, a balanced k-partition that is average γ-friendly for γ on the order of $\sqrt{n}$. Thus we observe a transition at $k = 2$ for random digraphs: positive friendliness is impossible for $k = 2$, but quite large positive friendliness is possible as soon as $k = 3$. The intuitive reason for this behaviour is that there are many more balanced 3-partitions than balanced bisections. On the other hand, Theorems 26 and 27 show that no positive friendliness is possible with respect to the more stringent max and sum definitions. We prove our positive result in Theorem 25 using the second moment method, while the three negative results follow from the first moment. There are several lemmas associated with these theorems, but we will only state those in Chapter 5.

Chapter 2

Hereditary Discrepancy

This chapter covers a joint work with Aleksandar Nikolov [LN24].

2.1 Special Set Systems

Throughout the history of combinatorial discrepancy, several set systems played important roles in proving or disproving certain conjectures and exhibiting tight bounds. We begin the chapter by highlighting some of these set systems, their associated bounds and conjectures, and other properties that they exhibit. In particular, we cover: the k-permutation matrix, the Hoffman set system, the closely related Pálvölgi set system, and the Haar Basis matrix. In Section 2.1.1 we explain the use of the k-permutation matrix in disproving a conjecture of Beck. In Section 2.1.2 we show how the Hoffman and Pálvölgi set systems were used to answer a question of Sós. In this discussion, we present a new interpretation of Pálvölgi's set system so that proving a tight bound on its discrepancy becomes nearly as simple as for the Hoffman set system. Finally, in Section 2.1.3, we state and prove many properties of the Haar basis. These are used in our lower bound result, Theorem 6, and discussed in detail in Section 2.2. We were not the first to use the Haar basis for discrepancy. That honor goes to Kunisky [Kun23].

2.1.1 Discrepancy of k-Permutations

A k-permutation matrix is an incidence matrix with dimension $kn \times n$. For each of k different permutations $\sigma_1, ..., \sigma_k$ on $[n]$, we define one row for each $i \in [n]$ with non-zero entries $\sigma_j(1), ..., \sigma_j(i)$ where $j \in [k]$. We often denote such matrices by $\mathbf{A}(\sigma_1, ..., \sigma_k)$. There is a simple proof[1] that the discrepancy of any two permutation matrix is at most 1. Beck conjectured that for all constant k,

[1] The proof is as follows. We construct a graph on the elements $[n]$ in the universe. We assume that n is even — add a dummy variable if this is not the case. Without loss of generality we can assume σ_1 is the identity permutation. For each permutation, each pair of adjacent indices form an edge e.g. for σ_1, we get edges $(1, 2), (3, 4), (5, 6), ...$. Note that the degree of each node in the graph is two as it receives an incident edge from σ_1 and another from σ_2. Further there are no odd cycles: the edges alternate between σ_1 and σ_2. It follows that we can decompose the graph into even cycles. If we color the nodes on the cycles with alternating colors, then, regardless of the prefix of either permutation, we can maintain a discrepancy of at most one.

the discrepancy of a k-permutation matrix is also $O(1)$, but Newman and Nikolov showed that the discrepancy of a 3-permutation matrix is actually $\Omega(\log n)$ [NNN12].

Their original proof used induction to show that a particular set of three permutations $\sigma_1, \sigma_2, \sigma_3$ on $[n]$ achieved $\operatorname{disc}(\mathbf{A}(\sigma_1, \sigma_2, \sigma_3)) = \Omega(\log n)$ via induction. The proof was later simplified by Franks [Fra21] who showed that the high discrepancy sets of permutations were subsets of the symmetric group on three elements and had a rigid hierarchical structure.

For upper-bounds, Bohus [Boh90] showed that every k-permutation has discrepancy $O(k \log n)$ by defining constraints on blocks of m variables to have zero discrepancy. Then he found a sequence of $O(\log_{2k/m} n)$ fractional colorings such that every set *eventually* has discrepancy at most $O(m \log_{2k/m} n)$. It suffices to take $m = \Theta(k)$ to get the upper-bound. Later, Spencer [SST01] was able to improve the bound to $O(\sqrt{k} \log n)$.

2.1.2 Hoffman's and Pálvölgi's Set System

Hoffman's set system is contructed from the nodes of a perfect k-ary tree of depth k, denoted T_k. Let elements of the universe consist of the vertices V of T_k. There are two different families of sets $\mathcal{P}$ and $\mathcal{C}$. For every root-to-leaf path in T_k, $P \in \mathcal{P}$ consists of the vertices on that path. For every internal node u in T_k, $C \in \mathcal{C}$ consists of the nodes which are the children of u. Our analysis of the discrepancy of Hoffman's set system follows the exposition of Matoušek [Mat09, Proposition 4.11]. We show that either there exists a monochromatic sibling set $C \in \mathcal{C}$ or a monochromatic root-to-leaf path $P \in \mathcal{P}$. If some $C \in \mathcal{C}$ is monochromatic, then we are done, so assume that none are. Suppose the root is colored $+1$. Since the set of all children of the root is *not* monochromatic, there exists a child of the root which is also colored $+1$. Take the edge from the root to this child. Repeat this process starting from the child until we have reached a leaf. This is possible since *every* sibling set $C \in \mathcal{C}$ has a node colored $+1$ and another colored -1.

Sós [LSV86] asked: for two set systems $\mathcal{S}_1$ and $\mathcal{S}_2$ on the same universe, is it possible to bound $\operatorname{disc}(\mathcal{S}_1 \cup \mathcal{S}_2)$ by some function of $\operatorname{disc}(\mathcal{S}_1)$ and $\operatorname{disc}(\mathcal{S}_2)$? With Hoffman's set system, it becomes apparent that the answer is no. Both $\mathcal{P}$ and $\mathcal{C}$ have discrepancy ≤ 1: For $\mathcal{P}$, it suffices to color a node based on the parity of its height (i.e., even and odd parity gets colors $+1$ and -1 respectively) and for $\mathcal{C}$ it suffices to index the nodes of T_k based on a BFS traversal and color the nodes based on the parity of the index (i.e., even and odd parity indices gets colors $+1$ and -1 respectively). The discrepancy of their union, however, is $\operatorname{disc}(\mathcal{P} \cup \mathcal{C}) = k$. Note that T_k has k^k nodes. The discrepancy of Hoffman's set system with respect to the number of nodes n is $\operatorname{disc}(\mathcal{P} \cup \mathcal{C}) = \Omega(\log n / \log \log n)$.

Pálvölgyi's's set system also resolves Sós' question in the negative but does so with two sets $\mathcal{F}$ and $\mathcal{T}$ each of whose discrepancies are ≤ 1 and whose union achieves $\operatorname{disc}(\mathcal{F} \cup \mathcal{T}) = \Omega(\log n)$ when the underlying universe has size n. The set system, as it originally appears [Pál10], is geometric in nature. From the exposition of Matoušek [Mat13] which aims to simplify the construction in order to compute its discrepancy: The construction is inductive, and requires parameters k and ℓ. For every $k, \ell \geq 1$, and ground set V of $n = \binom{k+\ell}{k} - 1$ elements, there exists set system $\mathcal{F}_1 = \mathcal{F}_1(V, k, \ell)$, $\mathcal{F}_2 = \mathcal{F}_2(V, k, \ell)$ on V such that $\mathcal{F}_1$ consists of k-tuples, $\mathcal{F}_2$ consists of ℓ-tuples and there exists a monochromatic $+1$ coloring of a set in $\mathcal{F}_1$ or a monochromatic -1 coloring of a set in $\mathcal{F}_2$. We omit the inductive definition here as it is quite involved.

Instead, we show that Pálvölgyi's set system is simply a "more compact" variant of Hoffman's set system by exploring the structure of its underlying sets. Instead of encoding all sets in the set system as root-to-leaf paths or sibling sets as was the case in Hoffman's Example, the sets of Pálvölgyi's' set system are encoded as root-to-leaf paths and *sibling-like* sets (including piblings[2], great piblings, and so on). We define two set systems $\mathcal{F}(k,\ell)$ and $\mathcal{T}(k,\ell)$ on the universe V which are the nodes in a tree $T_{k,\ell}$ of level k and degree ℓ at the root node. $\mathcal{T}(k,\ell)$ contains the root-to-leaf paths in $T_{k,\ell}$ and $\mathcal{F}(k,\ell)$ the pibling sets of $T_{k,\ell}$.

$T_{k,\ell}$ is define recursively. The root node has ℓ children. For each internal node with c non-leaf children, its children have degrees $c, c-1, ..., 1$. Note that there are $\binom{k+\ell-1}{\ell}$ nodes in $T_{k,\ell}$ since $1+\sum_{i=1}^{\ell}\binom{k-2+i}{i} = \binom{k+\ell-1}{\ell}$ by the Hockey-stick Identity; The LHS term is the sum of the subtrees rooted at each one of the children of the root with $+1$ representing the root. We label the nodes of $T_{k,\ell}$ in a postfix manner. See Figure 2.1. Define a pibling set in $T_{k,\ell}$ as follows. Begin with any sibling set S of an internal node p, i.e., all children of a common parent p. Walk back up the tree from p until we hit the root. Add to S any siblings of p and its ancestors with label smaller than those of the nodes originally in S. Observe that each pibling set consists of exactly ℓ nodes; a child of the root has degree $\ell - c + 1$ exactly when it is the c^{th} child with $c-1$ siblings occurring before it. $\mathcal{F}(k,\ell)$ is the set of all pibling sets of $T_{k,\ell}$.

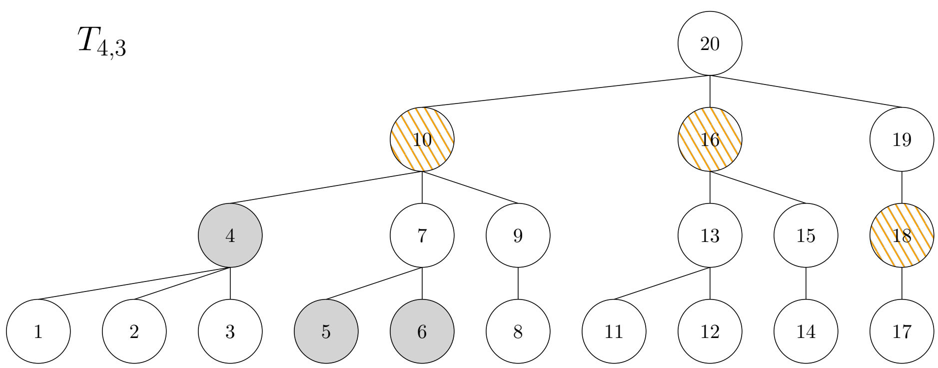


Figure 2.1: Example of a labelled $T_{3,3}$. Note that two different pibling sets are colored: $\{4,5,6\}$ in solid gray and $\{10,16,18\}$ in lined orange.

Proposition 28. (Pálvölgyi's Set System) *Let $\mathcal{P}(k,\ell) = \mathcal{F}(k,\ell) \cup \mathcal{T}(k,\ell)$ where $\mathcal{F}$ and $\mathcal{T}$ are the set system described above. Then $\mathcal{P}(k,\ell)$ has a monochromatic $\mathcal{F}(k,\ell)$ set or a monochromatic $\mathcal{T}(k,\ell)$, i.e., monochromatic pibling set or a monochromatic root-to-leaf path in $T_{k,\ell}$.*

Proof. We consider red-blue colorings of $T_{k,\ell}$. Suppose there does not exist a monochromatic pibling set, i.e., none of sets in $\mathcal{F}(k,\ell)$ are monochromatic. We will find a monochromatic root-to-leaf path in $\mathcal{T}(k,\ell)$, i.e., some set in $\mathcal{T}(k,\ell)$ which is monochromatic. In particular, we claim that for every node, if the node has maximum degree then it will have two children c_1 and c_2 such that there is a monochromatic c_1-to-leaf path and a monochromatic c_2-to-leaf path of *different* colors. Further, for all other nodes u, either

1. both red and blue appear as the colors of u's piblings with smaller index (to the left), or

[2] Piblings are siblings of parents. Great piblings are siblings of grandparents and so on.

2. if u's piblings with smaller index are monochromatic, say red, either u is also red or there exists a monochromatic blue u-to-leaf path.

Since there does not exist a monochromatic pibling set and the root has maximum degree, there must exists a red monchromatic path from one child of the root down to a leaf and a blue monochromatic path from another child of the root down to a leaf. Thus, regardless of the root's color, there will be a monochromatic root-to-leaf path. We prove the claim using induction on the node labels in $T_{k,\ell}$.

In the base case with the node labeled one, and generally for a leaf, any single node constitutes a monochromatic path. Consider any internal node labeled i in $T_{k,\ell}$. Since we assign labels in postfix order (i.e., label all the nodes in my subtree before labeling myself) we can apply the induction hypothesis to all decendants of i as well as all piblings with smaller label — these appear to the left of i. Either i has maximum degree or it does not. When it *does not*, consider the children of i along with its piblings of smaller index. These form a pibling set. If two piblings of i have different colors then we are done. Otherwise all of the piblings of i with smaller label have the same color, say red. Then there must exist a child of i colored blue as no pibling set can be monochromatic (the children of i and i's piblings with smaller label form a pibling set). Let c be the child of i satisfying this property with *smallest index* (left-most node). Since all of c's siblings to the left are colored red, by the induction hypothesis, there must be a monochromatic blue path from c to a root. Node i is blue or red. If i is blue, then there exists a monochromatic blue i-to-leaf path though c. If i is red, then it has the same color as all its piblings to the left.

When the internal i has maximum degree, let its children be $c_1, ..., c_k$. Without loss of generality, suppose that c_1 is colored red and let c_2 be the first child of i colored blue; Since no pibling set is monochromatic, such an c_j must exist. There is a monochromatic red c_1-to-leaf path. By the inductive hypothesis, we know that there is also a monochromatic blue c_2-to-leaf path: c_2 does not have maximum degree and all its siblings of smaller index have the same color. □

Corollary 29. *Let $\mathcal{P}(k,k)$ be Pálvölgyi's Set System on a universe V of size n, then* $\operatorname{disc}(\mathcal{P}(k,k)) \geq k$ *where $n = \binom{2k+1}{k}$ or $k = \Theta(\log n)$.*

2.1.3 Haar Basis

The $2^k \times 2^k$ discrete Haar basis matrix, $\mathbf{A}_k$, has $\{-1, 0, +1\}$ entries and is defined recursively with $\mathbf{A}_0 = [1]$ and

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{A}_{k-1} & \mathbf{I}_{2^{k-1}} \\ \mathbf{A}_{k-1} & -\mathbf{I}_{2^{k-1}} \end{bmatrix}, \tag{2.1}$$

where $\mathbf{I}_{2^{k-1}}$ is the $2^{k-1} \times 2^{k-1}$ identity matrix. This matrix arises from the following tree structure: Construct a depth k perfect binary tree, and let r be an additional node. We make the root of the perfect binary tree the left child of r, and let r be the root of our tree. Every non-leaf node represents a column in the matrix while every root-to-leaf path corresponds to a row in the matrix. Whenever the path proceeds down the left child from some node i, entry i of the corresponding row will have value $+1$. If instead the path proceeds down the right child of i, entry i of the corresponding row will have value -1. Thus every row will have k non-zero entries. It is also not hard to show that, for any ± 1 coloring of the columns, there is a row whose nonzero entries are equal to the corresponding column colors, and, therefore, $\mathrm{disc}(\mathbf{A}_k) = k$. Kunisky [Kun23] describes this in detail.

In addition, we define the $\{0,1\}^{2^k \times 2^k}$ matrices $\mathbf{A}_k^+$ and $\mathbf{A}_k^-$ to be the indicator matrices of the positive and negatives elements of $\mathbf{A}_k$ respectively. Here an *indicator matrix* will have one in some entry if and only the corresponding entry of $\mathbf{A}_k$ is non-zero and positive, in the case of $\mathbf{A}_k^+$, or negative, in the case of $\mathbf{A}_k^-$. Note that $\mathbf{A}_k = \mathbf{A}_k^+ - \mathbf{A}_k^-$. Finally define,

$$\mathbf{A}_k^{\pm} := \begin{pmatrix} \mathbf{A}_k^+ \\ \mathbf{A}_k^- \end{pmatrix}. \tag{2.2}$$

Lemma 30. $\mathrm{detlb}(\mathbf{A}_k) \leq 2$.

Proof. We show that any $i \times i$ square submatrix $\mathbf{B}$ of $\mathbf{A}_k$ satisfies $|\det(\mathbf{B})| \leq 2^i$. First, define $M_k(i) := \max_{\mathbf{B}} |\det(\mathbf{B})|$ where the maximum is taken over all $i \times i$ submatrices $\mathbf{B}$ of $\mathbf{A}_k$. We compute $M_k(i)$ recursively by considering the forms that all $i \times i$ submatrices of $\mathbf{A}_k$ can take:

1. $\mathbf{B}$ only contain elements from the first 2^{k-1} columns of $\mathbf{A}_k$,
2. $\mathbf{B}$ only contain elements from the second 2^{k-1} columns of $\mathbf{A}_k$, or
3. $\mathbf{B}$ contain elements from both the first and second 2^{k-1} columns of $\mathbf{A}_k$.

We use the recursive formula (2.1) to analyze these cases. In the first case the resulting submatrix is either entirely contained in $\mathbf{A}_{k-1}$ up to rearranging rows, or contains a duplicated row. The magnitude of the determinant of these submatrices can be bounded above by $M_{k-1}(i)$ and 0, respectively. In the second case we note that the submatrix is a totally unimodular matrix [3] have hereditary discrepancy at most one [Sch98]. Further, a result of Ghouila-Houri [Gho62] (TUM). To see this, recall that the second 2^{k-1} columns of $\mathbf{A}_k$ consist of an identity matrix and its negation stacked on top of one another. Any square submatrix is entirely contained in the identity matrix or contains duplicated (and negated) rows. Thus the absolute value of the determinant of this kind of submatrix is at most one. It remains to consider the third case. Let $\mathbf{B}$ be a submatrix of $\mathbf{A}_k$ with some j columns coming from the second 2^{k-1} columns of $\mathbf{A}_k$ for $1 \leq j < i$. For any such column there is either one or two non-zero entries, equal to 1 or -1. If there is only one non-zero entry, then this reduces to computing $M_k(i-1)$ since we can perform a co-factor expansion on this column. If there are two non-zero entries, then we can permute the rows so that they are adjacent. This only changes the sign of the resulting determinant. Notice that the two rows are identical except for the sign of the non-zero entries. When performing a co-factor expansion on these two entries, the $(i-1) \times (i-1)$ submatrix that results when removing either row and the column is identical. Thus $|\det(\mathbf{B})|$ is at most twice the absolute value of the determinant of this $(i-1) \times (i-1)$ submatrix. After removing all the columns and associated rows of $\mathbf{B}$ from the second half of $\mathbf{A}_k$ in this way, we see that $|\det(\mathbf{B})| \leq 2^j M_{k-1}(i-j)$. Since, in the base case, $M_0(1) = 1$, the claim follows. $\square$

Using a similar argument as above, we can show that the matrices $\mathbf{A}_k^+$ and $\mathbf{A}_k^-$ are totally unimodular matrices (TUM). Note that, using (2.1), $\mathbf{A}_k^+$ and $\mathbf{A}_k^-$ can be recursively defined as $\mathbf{A}_0^+ = [1]$,

[3] A matrix $\mathbf{A}$ is TUM if every square submatrix of $\mathbf{A}$ has determinant in $\{-1, 0, 1\}$. A linear systems of the form $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ for TUM $\mathbf{A}$, integral $\mathbf{b}$, and $0 \leq \mathbf{x}$ has an integral polyhedron as its feasible region.

$\mathbf{A}_0^- = [0]$, and

$$\mathbf{A}_k^+ = \begin{bmatrix} \mathbf{A}_{k-1}^+ & \mathbf{I}_{2^{k-1}} \\ \mathbf{A}_{k-1}^+ & \mathbf{0} \end{bmatrix} \qquad \mathbf{A}_k^- = \begin{bmatrix} \mathbf{A}_{k-1}^- & \mathbf{0} \\ \mathbf{A}_{k-1}^- & \mathbf{I}_{2^{k-1}} \end{bmatrix}, \tag{2.3}$$

where $\mathbf{0}$ is the all zeros matrix of appropriate dimension, and $\mathbf{I}_{2^{k-1}}$ is the 2^{k-1} by 2^{k-1} identity matrix.

Corollary 31. *$\mathbf{A}_k^+$ and $\mathbf{A}_k^-$ are TUM matrices where $\mathbf{A}_k^+$ and $\mathbf{A}_k^-$ are indicators of the positive and negatives entries of $\mathbf{A}_k$, respectively.*

Proof. We only consider $\mathbf{A}_k^+$ as the proof that $\mathbf{A}_k^-$ is a TUM matrix is similar. The proof proceeds by induction on k. Consider some $i \times i$ submatrix $\mathbf{B}$ of $\mathbf{A}_k^+$. If $\mathbf{B}$ is entirely contained in first half of the columns of $\mathbf{A}_k^+$, then we are done by the inductive hypothesis; if $\mathbf{B}$ is entirely contained in the second half of the columns of $\mathbf{A}_k^+$, then $\mathbf{B}$ is a submatrix of the identity matrix or has a row of 0's, and the absolute value of its determinant is at most 1. Thus it suffices to consider the case where $\mathbf{B}$ has some columns from the first half of $\mathbf{A}_k^+$ and some columns from the second half of $\mathbf{A}_k^+$. Since any column from the second half has only one non-zero entry, equal to 1, performing co-factor expansions on the columns in the second half shows that the absolute value of the determinant will only be as large as the absolute value of the determinant of some smaller square sub-matrix in $\mathbf{A}_{k-1}^+$. Note that in the base case, $|\det(\mathbf{A}_0^+)| = 1$. ∎

Lemma 32. $\mathrm{disc}_1(\mathbf{A}_k) = \frac{k+1}{2^k}\binom{k}{\lfloor (k+1)/2 \rfloor} \cong \sqrt{k}$.

Proof. The proof for $k = 0$ is trivial, so we focus on the case $k \geq 1$. Let $\tilde{\mathbf{A}}_k$ denote the $2^k \times (2^k - 1)$ matrix equal to $\mathbf{A}_k$ with the first column, all of whose entries are 1, removed. Note that this is equivalent to removing the root node r and keeping only the perfect binary tree of depth k in the tree structure of the Haar basis, as described at the beginning of the section. We have the following key claim.

Claim 33. *For any $\mathbf{x} \in \{\pm 1\}^{2^k - 1}$ there exists a permutation which maps the entries of $\tilde{\mathbf{A}}_k\mathbf{x}$ to those of $\tilde{\mathbf{A}}_k\mathbf{1}$.*

Proof. Our proof is by induction on k. When $k = 1$,we have a root node with two children corresponding to the matrix

$$\tilde{\mathbf{A}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

When $\mathbf{x} = [1]$, $\tilde{\mathbf{A}}_1\mathbf{x} = \tilde{\mathbf{A}}_1\mathbf{1}$ and the identity permutation suffices; when $\mathbf{x} = [-1]$, $\tilde{\mathbf{A}}_1\mathbf{x} = -\tilde{\mathbf{A}}_1\mathbf{1}$ and it suffices to swap the two entries.

Consider some height k perfect binary tree corresponding to $\tilde{\mathbf{A}}_k$. Let u be the root of the tree with left and right children u_+ and u_- respectively. Since every root-to-leaf path must go through u_+ or u_-, this forms a partition of the rows of $\tilde{\mathbf{A}}_k$. In particular, we can rearrange $\tilde{\mathbf{A}}_k$ as

$$\tilde{\mathbf{A}}_k = \begin{bmatrix} \mathbf{1} & \tilde{\mathbf{A}}_{k-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \tilde{\mathbf{A}}_{k-1} \end{bmatrix}.$$

Consider the $\tilde{\mathbf{A}}_{k-1}$ submatrices which appear in $\tilde{\mathbf{A}}_k$. The $\tilde{\mathbf{A}}_{k-1}$ submatrix in the first 2^{k-1} rows has rows which correspond to root-to-leaf paths with leaves in the subtree rooted at u_+. Its columns correspond to nodes in the same subtree. The $\tilde{\mathbf{A}}_{k-1}$ submatrix in the second 2^{k-1} rows is defined similarly on the subtree rooted at u_-. Write the vector $\mathbf{x}$ as $[x_u, \mathbf{x}_+, \mathbf{x}_-]^\top$ where x_u is the color of the node u and $\mathbf{x}_+$ and $\mathbf{x}_-$ are the colors of the nodes in the subtrees rooted at u_+ and u_-, respectively. Consider the value of x_u. If $x_u = 1$, then by the inductive hypothesis, there exists a permutation which takes the entries of $\tilde{\mathbf{A}}_{k-1}\mathbf{x}_+$ to the entries of $\tilde{\mathbf{A}}_{k-1}\mathbf{1}$ and another permutation which takes the entries of $\tilde{\mathbf{A}}_{k-1}\mathbf{x}_-$ to the entries of $\tilde{\mathbf{A}}_{k-1}\mathbf{1}$. These two permutations can be combined to form a permutation which maps the entries of $\tilde{\mathbf{A}}_k\mathbf{x}$ to the entries of $\tilde{\mathbf{A}}_k\mathbf{1}$. Otherwise $x_u = -1$. Again, there exists a permutation π_1 which takes $\tilde{\mathbf{A}}_{k-1}\mathbf{x}_+$ to $\tilde{\mathbf{A}}_{k-1}\mathbf{1}$ and another permutation π_2 which takes $\tilde{\mathbf{A}}_{k-1}\mathbf{x}_-$ to $\tilde{\mathbf{A}}_{k-1}\mathbf{1}$. We can construct a permutation which maps the elements of $\tilde{\mathbf{A}}_k\mathbf{x}$ to those of $\tilde{\mathbf{A}}_k\mathbf{1}$ by by first applying π_1 to the first 2^{k-1} entries of $\tilde{\mathbf{A}}_k\mathbf{x}$, and π_2 to the remaining 2^{k-1} entries, and then swapping the first 2^{k-1} entries with the second 2^{k-1} entries. □

Note that, for any $\mathbf{x} \in \{\pm 1\}^{2^k}$, $\mathrm{disc}_1(\mathbf{A}_k, \mathbf{x}) = \mathrm{disc}_1(\mathbf{A}_k, -\mathbf{x})$. Then, we have

$$\mathrm{disc}_1(\mathbf{A}_k, \mathbf{x}) = \frac{1}{2}(\mathrm{disc}_1(\mathbf{A}_k, \mathbf{x}) + \mathrm{disc}_1(\mathbf{A}_k, -\mathbf{x}))$$

By Claim 33, and, since all entries of the first column of $\mathbf{A}_k$ are equal to 1,

$$\begin{aligned}
\mathrm{disc}_1(\mathbf{A}_k, \mathbf{x}) &= \frac{1}{2^k}\sum_{i=1}^{2^k} \frac{|\tilde{\mathbf{a}}_i^\top \mathbf{1} + 1| + |\tilde{\mathbf{a}}_i^\top \mathbf{1} - 1|}{2} \\
&= \frac{1}{2^{k+1}}\sum_{i=1}^{2^k} |\tilde{\mathbf{a}}_i^\top \mathbf{1} + 1| + \frac{1}{2^{k+1}}\sum_{i=1}^{2^k} |\tilde{\mathbf{a}}_i^\top \mathbf{1} - 1|,
\end{aligned}$$

where $\tilde{\mathbf{a}}_i^\top$ is the i-th row of $\tilde{\mathbf{A}}_k$. Recall that each row of $\tilde{\mathbf{A}}_k$ has exactly k non-zero entries, and every sign pattern for these k entries appears exactly once, as each row in $\tilde{\mathbf{A}}_k$ corresponds to a root-to-leaf path in a depth k perfect binary tree, and the sign pattern corresponds to the sequence of left and right turns made by the path. In particular, there are exactly $\binom{k}{\ell}$ rows $\tilde{\mathbf{a}}_i$ for which $\tilde{\mathbf{a}}_i^\top \mathbf{1}$ equals $k - 2\ell$, since such rows have $k - \ell$ non-zero entries equal to $+1$, and ℓ non-zero entries equal to -1. Substituting above, we have

$$\begin{aligned}
&\frac{1}{2^{k+1}}\sum_{i=1}^{2^k} |\tilde{\mathbf{a}}_i^\top \mathbf{1} + 1| + \frac{1}{2^{k+1}}\sum_{i=1}^{2^k} |\tilde{\mathbf{a}}_i^\top \mathbf{1} - 1| \\
&\qquad = \frac{1}{2^{k+1}}\sum_{\ell=0}^{k} \binom{k}{\ell}|k + 1 - 2\ell| + \frac{1}{2^{k+1}}\sum_{\ell=0}^{k} \binom{k}{\ell}|k - 1 - 2\ell| \\
&\qquad = \frac{1}{2^{k+1}}\sum_{\ell=0}^{k} \binom{k}{\ell}|k + 1 - 2\ell| + \frac{1}{2^{k+1}}\sum_{\ell=1}^{k+1} \binom{k}{\ell - 1}|k + 1 - 2\ell| \\
&\qquad = \frac{1}{2^{k+1}}\sum_{\ell=0}^{k+1} \left(\binom{k}{\ell} + \binom{k}{\ell - 1}\right)|k + 1 - 2\ell| \\
&\qquad = \frac{1}{2^{k+1}}\sum_{\ell=0}^{k+1} \binom{k+1}{\ell}|k + 1 - 2\ell|.
\end{aligned}$$

The second equality above follows by a change of variables in the second sum, the third equality uses the convention $\binom{k}{k+1} = \binom{k}{-1} = 0$, and the last equality follows form Pascal's identity.

Now, by Lemma 34 below, we have that

$$\sum_{\ell=0}^{k+1} \binom{k+1}{\ell} |k+1-2\ell| = 2(k+1)\binom{k}{\lfloor (k+1)/2 \rfloor},$$

which implies that $\mathrm{disc}_1(\tilde{\mathbf{A}}_k) \cong \sqrt{k}$ since $\binom{k}{\lfloor (k+1)/2 \rfloor} \cong 2^k/\sqrt{k}$ by Stirling's approximation. $\square$

There is a probabilistic interpretation for the above proof: For a uniformly random row $\mathbf{a}^\top$ of $\tilde{\mathbf{A}}_k$ and a fixed coloring $\mathbf{x} \in \{\pm 1\}^{2^k-1}$, $\mathbf{a}^\top \mathbf{x}$ is distributed like $X_1 + \cdots + X_k$ where the X_is are independent Rademacher random variables (i.e. random variables uniform in $\{-1, +1\}$). Recall that uniformly choosing a row of $\tilde{\mathbf{A}}_k$ corresponds to uniformly choosing a root-to-leaf path in the depth k perfect binary tree. Further, the non-leaf nodes of the tree correspond to the columns of $\tilde{\mathbf{A}}_k$. Thus we know that exactly k entries of the row will be non-zero. Let the indices of these entries be $U_1, ..., U_k$, and note that $\mathbf{a}^\top \mathbf{x} = x_{U_1} a_{U_1} + \cdots + x_{U_k} a_{U_k}$. The key observation is that, conditional on the values of $U_1, \ldots, U_\ell$, a_{U_ℓ} is equally likely to be -1 or $+1$, since a uniformly random path in the binary tree going through $U_1, \ldots, U_\ell$ is equally likely to visit the left or the right child of U_ℓ. We can then show that $x_{U_1} a_{U_1} + \cdots + x_{U_k} a_{U_k}$ has the same distribution as a sum of k independent Rademacher random variables by induction on k. In the base case, a_{U_1} is uniform in $\{-1, +1\}$, and so is $x_{U_1} a_{U_1}$. Suppose $x_{U_1} a_{U_1} + \cdots + x_{U_{k-1}} a_{U_{k-1}}$ is distributed as the sum of $k-1$ independent Rademacher random variables. Conditional on the choice of $U_1, \ldots, U_k$, a_{U_k}, and, therefore, $x_{U_k} a_{U_k}$ are equally likely to be -1 or $+1$. Taking expectation over the choice of $U_1, \ldots, U_k$ finishes the proof.

The next lemma is a well-known calculation. We include a proof due to Lavrov, for completeness.

Lemma 34 ([Lav18])**.** $\sum_{\ell=0}^{k} \binom{k}{\ell} |k-2\ell| = 2k \cdot \binom{k-1}{\lfloor k/2 \rfloor}$.

Proof. Recall the identity $\binom{k}{\ell}\ell = \binom{k-1}{\ell-1}k$. We write

$$\begin{aligned}
\sum_{\ell=0}^{k} \binom{k}{\ell} |k-2\ell| &= \sum_{\ell<k/2} \binom{k}{\ell}(k-2\ell) - \sum_{\ell>k/2} \binom{k}{\ell}(k-2\ell) \\
&= k\left(\sum_{\ell<k/2} \binom{k}{\ell} - \sum_{\ell>k/2} \binom{k}{\ell}\right) - 2\left(\sum_{\ell<k/2} \binom{k}{\ell}\ell - \sum_{\ell>k/2} \binom{k}{\ell}\ell\right) \\
&= 2k\left(\sum_{\ell>k/2} \binom{k-1}{\ell-1} - \sum_{\ell<k/2} \binom{k-1}{\ell-1}\right) \\
&= 2k\binom{k-1}{\lfloor k/2 \rfloor}.
\end{aligned}$$

Here, the first equality follows since $\binom{k}{\ell}(k-2(k/2)) = 0$ when k is even. The last equality follows by consider the parity of k; when k is even, we obtain a $\binom{k-1}{k/2}$ term after cancellation, and when k is odd, we obtain a $\binom{k-1}{\lfloor k/2 \rfloor}$ term after cancellation. $\square$

Note that when we divide the identity by 2^k, we obtain the expectation of a sum of k independent

Rademacher random variables. The asymptotic version of this identity follows from Khintchine's inequality.

We see that $\mathbf{A}_k$ — with its decomposition into $\mathbf{A}_k^+$ and $\mathbf{A}_k^-$ — is another counter-example of Sós' question that we saw in Section 2.1.2. While it matches the $\Omega(\log n)$ discrepancy lower bound of the Pálvölgyi and Newman-Neiman-Nikolov constructions, it is simpler to analyze.

Claim 35. *With $\mathbf{A}_k^\pm$ as described above Corollary 31,*

$$\operatorname{disc}\left(\mathbf{A}_k^\pm\right) \gtrsim k.$$

Proof. Recall that $\operatorname{disc}(\mathbf{A}_k) = k$. We claim that $\operatorname{disc}(\mathbf{A}_k^\pm) \geq \frac{1}{2}\operatorname{disc}(\mathbf{A}_k)$, and this proves the claim. Indeed, take any coloring $\mathbf{x}$. Let $\mathbf{a}^\top$ be the row of $\mathbf{A}_k$ achieving $|\mathbf{a}^\top\mathbf{x}| = \operatorname{disc}(\mathbf{A}_k, \mathbf{x})$ and let $\mathbf{a}_+^\top$ and $\mathbf{a}_-^\top$ be the corresponding rows in the copy of $\mathbf{A}^+$ and $\mathbf{A}^-$ in $\mathbf{A}^\pm$ respectively. Since $\operatorname{disc}(\mathbf{A}_k) \leq |\mathbf{a}^\top\mathbf{x}| = |\mathbf{a}_+^\top\mathbf{x} - \mathbf{a}_-^\top\mathbf{x}|$, by the triangle inequality we have that either $|\mathbf{a}_+^\top\mathbf{x}| \geq \operatorname{disc}(\mathbf{A}_k)/2$ or $|\mathbf{a}_-^\top\mathbf{x}| \geq \operatorname{disc}(\mathbf{A}_k)/2$. □

Finally, note another property of $\mathbf{A}_k$.

Claim 36. $|\det(\mathbf{A}_k)| = 2^{2^k-1}$.

Proof. $\mathbf{A}_k$ have orthogonal columns so $|\det(\mathbf{A}_k)|$ is equal to the product of the ℓ_2-norms of columns. Since the ith column has magnitude $2^{2^{i-1}}$, $|\det(\mathbf{A}_k)| = 2^{2^{k-1}+2^{k-2}+\cdots+2^0} = 2^{2^k-1}$. □

2.2 Hereditary Discrepancy

In this section we prove Theorem 6 and Theorem 7 restated below.

Theorem 5 (Matoušek [Mat13], Theorem 2)**.** *For any matrix $\mathbf{A} \in \{0,1\}^{m\times n}$,*

$$\operatorname{herdisc}(\mathbf{A}) \leq O\left(\operatorname{detlb}(\mathbf{A})\log(mn)\sqrt{\log n}\right). \tag{1.4}$$

Theorem 6 (Hereditary Discrepancy and Detlb Lower Bound)**.** *For any real number $\varepsilon \in (0,1)$, any integers $n \geq 2$ and $m \in \left[n, 2^{n^{1-\varepsilon}}\right]$, there exists a matrix $\mathbf{A} \in \{0,1\}^{m\times n}$ such that*

$$\frac{\operatorname{herdisc}(\mathbf{A})}{\operatorname{detlb}(\mathbf{A})} \gtrsim \sqrt{\log m \log n}. \tag{1.5}$$

Theorem 7 (Hereditary Discrepancy and Detlb Upper Bound)**.** *For all positive integers m and n, and all matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$, we have*

$$\frac{\operatorname{herdisc}(\mathbf{A})}{\operatorname{detlb}(\mathbf{A})} \lesssim \sqrt{n}.$$

Recall the result of Matoušek [Mat13] mentioned in the introduction and restated here in Theorem 5.

The crux of Matoušek's result were a pair of inequalities:

$$\mathrm{herdisc}(\mathbf{A}) \lesssim \log(2mn) \cdot \mathrm{hervecdisc}(\mathbf{A}) \tag{2.4}$$

$$\mathrm{hervecdisc}(\mathbf{A}) \lesssim \sqrt{\log 2n} \cdot \mathrm{detlb}(\mathbf{A}) \tag{2.5}$$

Recently, Jiang and Reis [JR22] were able to improve Matoušek's result by showing that $\frac{\mathrm{herdisc}(\mathbf{A})}{\mathrm{detlb}(\mathbf{A})} \lesssim \sqrt{\log 2m \log 2n}$. Their work left open whether the $\sqrt{\log m}$ term can be replaced by $\sqrt{\log n}$ for large m. In the present work, we show this is not possible for almost the entire range of m, i.e., the factor $\sqrt{\log m}$ is necessary for all m in the range $n \le m \le 2^{n^{1-\epsilon}}$ for any constant $\epsilon > 0$. Note that when $m > 2^n$, the matrix contains duplicated rows whose removal will not change the value of $\mathrm{herdisc}(\mathbf{A})$ nor $\mathrm{detlb}(\mathbf{A})$. Thus, our lower bound covers nearly all values for m.

The proof of Theorem 6 crucially relies upon the $2^k \times 2^k$ Haar wavelet basis, denoted $\mathbf{A}_k$, which we discussed in Section 2.1.3. For the lower bound of Theorem 7, we provide the proof in Section 2.2.3. We also give a simpler proof for the special case of $\mathbf{A} \in \{0,1\}^{m \times n}$ in Section 2.2.4, using the theory of VC dimension.

For the Haar Basis, we show that $\mathbf{A}_k$ is tight for Equation (2.5) in Section 2.2.2.

Theorem 37. *With $n = 2^k$ for an integer $k \ge 1$,* $\mathrm{hervecdisc}(\mathbf{A}_k) \gtrsim \sqrt{\log n} \cdot \mathrm{detlb}(\mathbf{A}_k)$.

It is yet unknown whether Equation (2.4) is tight as Jiang and Reis improved upon Matoušek's bound by circumventing the inequality altogether. The resolution of this problem via an efficient algorithm would imply new and old constructive bounds, for example, the constructive version of Banaszczyk's upper bound for the Beck-Fiala problem [Ban98; BDG16], and a constructive version of Nikolov's upper bound for Tuśnady's problem [Nik17].

Our use of the matrix $\mathbf{A}_k$ is inspired by work of Kunisky [Kun23], who first used this matrix in the context of proving discrepancy lower bounds.

2.2.1 Proof Structure

In order to prove Theorem 6 we will find a family of matrices which satisfy equation (1.5). Our candidates will have the form $\mathbf{P}_N \otimes \mathbf{A}$ where $\mathbf{P}_N$ is the $2^N \times N$ incidence matrix of the power set, and $\mathbf{A}$ is some $p \times p$ matrix with a gap between $\mathrm{detlb}(\mathbf{A})$ and $\mathrm{disc}_1(\mathbf{A})$. In particular, we let $\mathbf{A}$ be the Haar basis matrix used in the work of Kunisky [Kun23], and prove some properties of $\mathbf{A}$ in-order to obtain the present result.

We bound $\mathrm{detlb}(\mathbf{P}_N \otimes \mathbf{A})$ from above by showing that $\mathrm{detlb}(\mathbf{P}_N \otimes \mathbf{A}) \lesssim \sqrt{N} \cdot \mathrm{detlb}(\mathbf{A})$ using standard linear algebra and Lemma 4 from [Mat13]. See Lemma 38. For our choice of $\mathbf{A}$, we will show that $\mathrm{detlb}(\mathbf{A}) \lesssim 1$. We also bound $\mathrm{disc}(\mathbf{P}_N \otimes \mathbf{A}) \gtrsim N \cdot \mathrm{disc}_1(\mathbf{A})$ using a discrepancy amplification argument. See Lemma 39. By finding a tight lower bound on $\mathrm{disc}_1(\mathbf{A})$, we obtain the lower bound $\mathrm{disc}(\mathbf{P}_N \otimes \mathbf{A}) \gtrsim N \cdot \sqrt{\log p}$. Taken together, these bounds gives us a gap on the order of $\sqrt{N} \cdot \sqrt{\log p}$ between $\mathrm{detlb}(\mathbf{P}_N \otimes \mathbf{A})$ and $\mathrm{disc}(\mathbf{P}_N \otimes \mathbf{A})$.

Lemma 38. *For the power matrix $\mathbf{P}_N$, and any real matrix $\mathbf{A}$,*

$$\mathrm{detlb}\left(\mathbf{P}_N \otimes \mathbf{A}\right) \le \sqrt{eN} \cdot \mathrm{detlb}(\mathbf{A}).$$

Proof. Let $\mathbf{u}_1, ..., \mathbf{u}_N$ be the columns of $\mathbf{P}_N$ and $\mathbf{A} \in \mathbb{R}^{p\times p}$. Divide the columns of $\mathbf{P}_N \otimes \mathbf{A}$ into N contiguous blocks of size $(2^N p) \times p$ each representing $\mathbf{u}_\ell \otimes \mathbf{A}$. Note that $\mathbf{u}_\ell \otimes \mathbf{A}$ consists of 2^N blocks of $\mathbf{A}$ or $\mathbf{0}$ stacked on top of one another. We claim that $\operatorname{detlb}(\mathbf{u}_\ell \otimes \mathbf{A}) \le \operatorname{detlb}(\mathbf{A})$. Consider an $s \times s$ sub-matrix $\mathbf{B}$ of $\mathbf{u}_\ell \otimes \mathbf{A}$ with the rows indexed by I and columns indexed by J. Note that if any row of $\mathbf{B}$ is zero, then $\det(\mathbf{B}) = 0$ so in order for the determinant to be non-zero, the rows of $\mathbf{B}$ must be parts of rows of $\mathbf{A}$ with columns indexed by J. If there are multiple copies of the same row of $\mathbf{A}$, then again $\det(\mathbf{B}) = 0$. Thus $\mathbf{B}$ must come from distinct rows of $\mathbf{A}$ with columns indexed by J. It follows that $\mathbf{B}$ is actually a sub-matrix of $\mathbf{A}$ up to rearrangement of the rows, so $|\det(\mathbf{B})|^{1/s} \le \operatorname{detlb}(\mathbf{A})$. Since this is true for all choices of the submatrix $\mathbf{B}$, we have $\operatorname{detlb}(\mathbf{u}_\ell \otimes \mathbf{A}) \le \operatorname{detlb}(\mathbf{A})$.

Recall from [Mat13] Lemma 4, that for real matrices $\mathbf{B}_1, ..., \mathbf{B}_t$ each with the same number of columns and $D \coloneqq \max_{i=1,2,...t} \operatorname{detlb}(\mathbf{B}_i)$, any matrix $\mathbf{B}$ whose rows are copies of the rows of the matrices $\mathbf{B}_i$ satisfies $\operatorname{detlb}(\mathbf{B}) \le D\sqrt{et}$. By applying Lemma 4 to $(\mathbf{P}_N \otimes \mathbf{A})^\top$ with $\mathbf{B}_i = (\mathbf{u}_i \otimes \mathbf{A})^\top$, and we have that

$$\operatorname{detlb}(\mathbf{P}_N \otimes \mathbf{A}) \le \sqrt{eN} \cdot \max_{\ell\in[N]} \operatorname{detlb}(\mathbf{u}_\ell \otimes \mathbf{A}) \le \sqrt{eN} \cdot \operatorname{detlb}(\mathbf{A}). \qquad \square$$

Lemma 39. (Discrepancy Amplification). *For the power matrix $\mathbf{P}_N$ and any real matrix $\mathbf{A}$,*

$$\operatorname{disc}(\mathbf{P}_N \otimes \mathbf{A}) \ge \frac{N \cdot \operatorname{disc}_1(\mathbf{A})}{2}.$$

Proof. Let $\mathbf{A} \in \mathbb{R}^{p\times q}$ and $t \coloneqq \operatorname{disc}_1(\mathbf{A})$. Consider some vector $\mathbf{x} \in \{\pm 1\}^{qN}$ composed of vectors $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(N)}$ stacked on top of each other containing p entries each. We compute $\|(\mathbf{P}_N \otimes \mathbf{A})\mathbf{x}\|_\infty$. Note that

$$\|(\mathbf{P}_N \otimes \mathbf{A})\mathbf{x}\|_\infty = \max_{S\subseteq[N]} \left\| \sum_{i\in S} \mathbf{A}\mathbf{x}^{(i)} \right\|_\infty = \max_{S\subseteq[N]} \max_{j\in[p]} \left| \sum_{i\in S} \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right|.$$

From the assumption, we have $\frac{1}{p}\|\mathbf{A}\mathbf{x}^{(i)}\|_1 \ge t$ for every $i \in [N]$. Taking an average over all choices of i,

$$t \le \frac{1}{pN} \sum_{i=1}^{N} \sum_{j=1}^{p} \left| \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right| = \frac{1}{pN} \sum_{j=1}^{p} \sum_{i=1}^{N} \left| \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right| \implies \sum_{i=1}^{N} \left| \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right| \ge Nt$$

for some $j \in [p]$. With $S^+ = \{i : (\mathbf{A}\mathbf{x}^{(i)})_j > 0\}$ and $S^- = \{i : (\mathbf{A}\mathbf{x}^{(i)})_j < 0\}$,

$$\begin{aligned} Nt \le \sum_{i=1}^{N} \left| \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right| &= \sum_{i\in S^+} \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j - \sum_{i\in S^-} \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \\ &= \left| \sum_{i\in S^+} \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right| + \left| \sum_{i\in S^-} \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right| \\ &\implies \max\left\{ \left| \sum_{i\in S^+} \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right|, \left| \sum_{i\in S^-} \left(\mathbf{A}\mathbf{x}^{(i)}\right)_j \right| \right\} \ge \frac{Nt}{2}. \end{aligned}$$

$\square$

Lemmas 38 and 39 together imply that

$$\frac{\text{herdisc}(\mathbf{P}_N \otimes \mathbf{A})}{\text{detlb}(\mathbf{P}_N \otimes \mathbf{A})} \geq \frac{\sqrt{N}}{2\sqrt{e}} \cdot \frac{\text{disc}_1(\mathbf{A})}{\text{detlb}(\mathbf{A})}.$$

Note that, if $\mathbf{A} \in \mathbb{R}^{p \times q}$, then $\mathbf{P}_N \otimes \mathbf{A}$ is an $(2^N p) \times (Nq)$ matrix, so $\sqrt{N}$ is roughly $\sqrt{\log m}$ for small enough p, where $m \coloneqq 2^N p$ is the number of rows of $\mathbf{P}_N \otimes \mathbf{A}$. To prove Theorem 6, we need to find a matrix $\mathbf{A}$ that exhibits a large gap between $\text{disc}_1(\mathbf{A})$ and $\text{detlb}(\mathbf{A})$. In the next section, we show that a matrix whose columns are the discrete Haar basis vectors has this property. We bound the hereditary discrepancy to determinant lower bound ratio for both $\mathbf{P}_N \otimes \mathbf{A}_k$ and $\mathbf{P}_N \otimes \mathbf{A}_k^{\pm}$.

Theorem 40. *For the power matrix $\mathbf{P}_N$, the discrete Haar basis $\mathbf{A}_k$, and the stacked indicator matrix $\mathbf{A}_k^{\pm}$ as defined in equation (2.2),*

$$\frac{\text{herdisc}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right)}{\text{detlb}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right)} \gtrsim \sqrt{N \cdot k}, \tag{2.6}$$

$$\frac{\text{herdisc}\left(\mathbf{P}_N \otimes \mathbf{A}_k^{\pm}\right)}{\text{detlb}\left(\mathbf{P}_N \otimes \mathbf{A}_k^{\pm}\right)} \gtrsim \sqrt{N \cdot k}. \tag{2.7}$$

Proof. First we apply the proof structure described in the previous section to $\mathbf{A}_k$. In particular, we show that $\text{detlb}(\mathbf{A}_k) = O(1)$ in Lemma 30 and that $\text{disc}_1(\mathbf{A}_k) \gtrsim \sqrt{k}$ in Lemma 32. Applying Lemma 38 to the first result and Lemma 39 to the second, we have that $\text{detlb}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right) \lesssim \sqrt{N}$ and $\text{disc}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right) \gtrsim N \cdot \sqrt{k}$. It follows that

$$\frac{\text{herdisc}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right)}{\text{detlb}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right)} \geq \frac{\text{disc}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right)}{\text{detlb}\left(\mathbf{P}_N \otimes \mathbf{A}_k\right)} \gtrsim \sqrt{N \cdot k}.$$

The process for $\mathbf{A}_k^{\pm}$ is similar. To show an upper bound on $\text{detlb}(\mathbf{A}_k^{\pm})$, use Corollary 31 where $\mathbf{A}_k^{+}$ and $\mathbf{A}_k^{-}$ are shown to be TUM. Since the determinant of any square submatrix of either matrix is at most one in absolute value, we can apply Lemma 4 of [Mat13] to $\mathbf{A}_k^{+}$ and $\mathbf{A}_k^{-}$ to obtain $\text{detlb}(\mathbf{A}_k^{\pm}) = O(1)$. To obtain the lower bound on $\text{disc}_1(\mathbf{A}_k^{\pm})$, we will recall that $\text{disc}_1(\mathbf{A}_k) \gtrsim \sqrt{k}$ from Lemma 32. Note that, for any $\mathbf{x} \in \{-1, +1\}^{2^k}$, by the triangle inequality

$$\frac{1}{2^k}\|\mathbf{A}_k \mathbf{x}\|_1 = \frac{1}{2^k}\|(\mathbf{A}_k^{+} - \mathbf{A}_k^{-})\mathbf{x}\|_1 \leq 2\left(\frac{1}{2^{k+1}}\|\mathbf{A}_k^{+}\mathbf{x}\|_1 + \frac{1}{2^{k+1}}\|\mathbf{A}_k^{-}\mathbf{x}\|_1\right).$$

Therefore, $\text{disc}_1(\mathbf{A}_k^{\pm}) \gtrsim \sqrt{k}$ as well. Apply Lemma 38 and Lemma 39 to $\text{detlb}(\mathbf{A}_k^{\pm}) = O(1)$ and $\text{disc}_1(\mathbf{A}_k^{\pm 1}) \gtrsim \sqrt{k}$ respectively to obtain equation (2.7). □

2.2.2 Proof

Next we prove Theorem 37, showing that $\mathbf{A}_k$ serves as a tight example of Equation (2.5) in [Mat13].

Proof of Theorem 37. To see this, it suffices to show that $\text{vecdisc}\left(\mathbf{A}_k\right)^2 = \Omega(k)$. Let $\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_q$ be the vector colors assigned to the 2^k columns of $\mathbf{A}_k$. Recall that $\mathbf{A}_k$ corresponds to a tree with root node r, where r has no right child, and the left child is the root of a perfect binary tree of depth k. The root to leaf paths of this tree represent rows in $\mathbf{A}_k$. For any path $r, t_1, ..., t_i$ from r to a node t_i, let $\overline{\mathbf{v}}_{t_i} = \mathbf{v}_r + \sum_{j=1}^{i-1} a_{t_j}\mathbf{v}_{t_j}$ where a_{t_j} is 1 if t_{j+1} is the left child of t_j, and -1 otherwise. We will show that there exists a root-to-leaf t_k path $t_1, ..., t_k$ such that $\|\overline{\mathbf{v}}_{t_k}\|_2^2 \geq k$. In particular we show that at every internal node t, with children t_+ and t_-, must have $\|\overline{\mathbf{v}}_{t_+}\|^2 \geq 1 + \|\overline{\mathbf{v}}_t\|^2$ or $\|\overline{\mathbf{v}}_{t_-}\|^2 \geq 1 + \|\overline{\mathbf{v}}_t\|^2$. To see this, note that

$$\|\overline{\mathbf{v}}_{t_+}\|^2 = \|\overline{\mathbf{v}}_t + \mathbf{v}_t\|^2 = \|\overline{\mathbf{v}}_t\|^2 + \|\mathbf{v}_t\|^2 + 2\langle \overline{\mathbf{v}}_t, \mathbf{v}_t\rangle = \|\overline{\mathbf{v}}_t\|^2 + 1 + 2\langle \overline{\mathbf{v}}_t, \mathbf{v}_t\rangle.$$

Similarly, we have that $\|\overline{\mathbf{v}}_{t_-}\|^2 = \|\overline{\mathbf{v}}_t\|^2 + 1 - 2\langle \overline{\mathbf{v}}_t, \mathbf{v}_t\rangle$. The claim then follows since either $\langle \overline{\mathbf{v}}_t, \mathbf{v}_t\rangle \geq 0$ or $-\langle \overline{\mathbf{v}}_t, \mathbf{v}_t\rangle \geq 0$. The theorem then follows from the tree interpretation of $\mathbf{A}_k$. □

As mentioned in Section 2.1.2 Sós which asks if the hereditary discrepancy of a union of two sets systems is bounded above by the discrepancy of each individual set system. The Hoffman set system (Section 2.1.2), the example of Pálvölgyi [Pál10], and the three permutations family of Newman, Neiman, and Nikolov [NNN12] (Section 2.1.1) showed instances of constantly many set systems where no such bounds exist.

Proof of Theorem 6. Consider the range of m in $[n, n^2]$ and $\left(n^2, 2^{n^{(1-\epsilon)}}\right]$ separately. In the first interval, we let $\mathbf{A}$ be the matrix $\mathbf{A}_k$ padded with $m-n$ rows of zeros. Here, $\frac{\text{herdisc}(\mathbf{A})}{\text{detlb}(\mathbf{A})} \cong \log n \cong \sqrt{\log m \cdot \log n}$. When $m \in \left(n^2, 2^{n^{(1-\epsilon)}}\right]$, we consider the matrix $\mathbf{P}_N \otimes \mathbf{A}_k$ where $N = \lfloor \log_2(m/n) \rfloor$ and $k = \lfloor \log_2 n^\epsilon \rfloor$. Observe that $\mathbf{P}_N \otimes \mathbf{A}_k$ is an $m' \times n'$ matrix where $m' = 2^{N+k} \leq m/n^{1-\epsilon} < m$ and $n' = N \cdot 2^k \leq \log_2(m/n) \cdot n^\epsilon \leq n - n^\epsilon \log n$ since $m \leq 2^{n^{(1-\epsilon)}}$. We obtain $\mathbf{A}$ by padding $\mathbf{P}_N \otimes \mathbf{A}_k$ with zero vectors so that it has exactly m rows and n columns. Note that $\log m \cong N$ and $\log n \cong k$. By Theorem 40, $\frac{\text{herdisc}(\mathbf{A})}{\text{detlb}(\mathbf{A})} \gtrsim \sqrt{Nk} \gtrsim \sqrt{\log m \log n}$, as required.

The reader might object that the matrix $\mathbf{A}_k$ has negative entries which would not occur for incidence matrices of a set system. We remedy this by considering $\mathbf{A}_k^{\pm}$ as defined in Theorem 40 instead. □

2.2.3 Upper Bound on Hereditary Discrepancy

In this section we prove Theorem 7. To this end we introduce the volume lower bound on hereditary discrepancy, introduced by Lovász, Spencer, and Vesztergombi [LSV86], and, in a more general setting, by Banaszczyk [Ban93], and studied by Dadush, Nikolov, Talwar, and Tomczak-Jaegermann [DNTT18].

Let $\mathbf{A}$ be an $m \times n$ real matrix, and define the symmetric convex set $K_{\mathbf{A}} := \{x \in \mathbb{R}^n : \|\mathbf{A}\mathbf{x}\|_\infty \leq 1\}$. Let us define the *volume lower bound of A*, denoted volLB(A), by

$$\text{volLB}(A) = \max_{k \in [n]} \max_{S \subseteq [n], |S| = k} \frac{1}{\text{vol}_k\left(K_{\mathbf{A}} \cap W_S\right)^{1/k}},$$

where W_S is the canonical subspace in the dimensions indexed by S (i.e. $W_S = \text{span}\{\mathbf{e}_i,\, i \in S\}$) and vol_k is the k-dimensional volume within W_S, i.e., the Lebesgue measure restricted to this subspace. We also define a dual volume lower bound by

$$\text{volLB}^*(A) = \max_{k \in [n]} \max_{S \subseteq [n], |S| = k} \frac{\text{vol}_k\left(\text{conv}(\pm \mathbf{\Pi}_S \mathbf{a}_1, \ldots \pm \mathbf{\Pi}_S \mathbf{a}_m)\right)^{1/k}}{c_k^{2/k}},$$

where $\mathbf{\Pi}_S$ is the orthogonal projection onto W_S, $\mathbf{a_i}^\top$ is the i-th row of $\mathbf{A}$, and $c_k = \frac{\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$ is the volume of the k-dimensional unit Euclidean ball.

We also need the concept of a polar set of a set $K \subseteq \mathbb{R}^n$, defined as

$$K^\circ \coloneqq \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y}^\top \mathbf{x} \le 1 \; \forall \mathbf{x} \in K\}.$$

It is a consequence of the hyperplane separator theorem that for any closed convex K containing 0, $K^{\circ\circ} = K$ [Roc70, Section 14]. Moreover, it is clear from the definition that $K \subseteq L$ implies $L^\circ \subseteq K^\circ$.

We have the following relationship between $\mathrm{volLB}(\mathbf{A})$ and $\mathrm{volLB}^*(\mathbf{A})$.

Claim 41. *For any matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathrm{volLB}(\mathbf{A}) \asymp \mathrm{volLB}^*(\mathbf{A})$.

Proof. Let $K_\mathbf{A}$ be defined as above, and let $L_\mathbf{A} \coloneqq \mathrm{conv}(\pm\mathbf{a}_1, \dots \pm \mathbf{a}_m)$. We claim that, for any set $S \subseteq [n]$, $(K_\mathbf{A} \cap W_S)^\circ = \mathbf{\Pi}_S L_\mathbf{A}$, where the polar $(K_\mathbf{A} \cap W_S)^\circ$ is taken within the subspace W_S. It is sufficient to show this for $S = [n]$, since we can always replace $\mathbf{A}$ with its submatrix consisting of the columns indexed by S. In the case $S = [n]$, we just need to show $K_\mathbf{A}^\circ = L_\mathbf{A}$. Notice that

$$\begin{aligned} K_\mathbf{A} &= \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{A}\mathbf{x}\|_\infty \le 1\} \\ &= \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle \le 1 \text{ for all } \|\mathbf{y}\|_1 \le 1\} \\ &= \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{A}^\top\mathbf{y}\rangle \le 1 \text{ for all } \|\mathbf{y}\|_1 \le 1\}. \end{aligned}$$

By the definition of polar, we see that $K_\mathbf{A} = L_\mathbf{A}^\circ$ as

$$L_\mathbf{A} = \{\mathbf{A}^\top \mathbf{y} : \mathbf{y} \in \mathbb{R}^m \text{ where } \|\mathbf{y}\|_1 \le 1\}.$$

Thus $K_\mathbf{A}^\circ = L_\mathbf{A}^{\circ\circ} = L_\mathbf{A}$ as required.

Once we have established that $(K_\mathbf{A} \cap W_S)^\circ = \mathbf{\Pi}_S L_\mathbf{A}$, we have, by the Santaló-Blaschke and the reverse Santaló inequalities (see Chapters 1 and 8 of [AGM15]),

$$\mathrm{vol}_k(K_\mathbf{A} \cap W_S)^{1/k} \mathrm{vol}_k((K_\mathbf{A} \cap W_S)^\circ)^{1/k} \asymp c_k^{2/k}.$$

This completes the proof. ∎

The next lemma shows a relationship between $\mathrm{volLB}(\mathbf{A})$ and $\mathrm{detlb}(\mathbf{A})$ that, as far as we are aware, has not been observed before.

Lemma 42. *For any matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathrm{volLB}^*(\mathbf{A}) \lesssim \sqrt{n} \cdot \mathrm{detlb}(\mathbf{A})$.

In the proof of Lemma 42 we use the following result of Nikolov [Nik15, Theorem 10]. Closely related results were shown earlier by Dvoretzky and Rogers [DR50, Theorem 5B] and Ball [Bal89, Proposition 7].

Lemma 43. ([Nik15, Theorem 10]). *Let $m \ge n$ and $E \subseteq \mathbb{R}^n$ be a minimum volume ellipsoid containing the points $\pm\mathbf{a}_1, \dots, \pm\mathbf{a}_m \in \mathbb{R}^n$. Then there exists a set $T \subseteq [m]$ of size n such that*

$$|\det((\mathbf{a}_i)_{i\in T})| \ge \sqrt{\frac{n!}{n^n}} \frac{\mathrm{vol}_n(E)}{c_n} \asymp n^{1/4} e^{-n/2} \frac{\mathrm{vol}_n(E)}{c_n},$$

where $(\mathbf{a}_i)_{i\in T}$ *is the matrix with columns* $\mathbf{a}_i$ *for* $i \in T$*, and* vol_n *is the* n*-dimensional Lebesgue measure.*

Note that originally in [Nik15] the theorem shows that there is a distribution on random multisets T for which $\mathbb{E}|\det((\mathbf{a}_i)_{i\in T})|^2 = \frac{n!}{n^n}\frac{\mathrm{vol}_n(E)^2}{c_n^2}$. Since the determinant is zero unless T is a set, this implies Lemma 43.

Proof of Lemma 42. Take some $S \subseteq [n]$ of size k such that

$$\mathrm{volLB}^*(A) = \frac{\mathrm{vol}_k(\mathbf{\Pi}_S L_{\mathbf{A}})^{1/k}}{c_k^{2/k}},$$

where $\mathbf{a}_1^\top, \ldots, \mathbf{a}_m^\top$ are the rows of $\mathbf{A}$, and $L_{\mathbf{A}} \coloneqq \mathrm{conv}(\pm\mathbf{a}_1, \ldots \pm \mathbf{a}_m)$. Applying Lemma 43 to $\pm\mathbf{\Pi}_S\mathbf{a}_1, \ldots, \pm\mathbf{\Pi}_S\mathbf{a}_m$, we have that, taking $E \subseteq W_S$ to be the smallest volume ellipsoid containing $\pm\mathbf{\Pi}_S\mathbf{a}_1, \ldots, \pm\mathbf{\Pi}_S\mathbf{a}_m$, there exists a set $T \subseteq [m]$ of size k for which

$$|\det((\mathbf{\Pi}_S\mathbf{a}_i)_{i\in T})| \gtrsim k^{1/4}e^{-k/2}\frac{\mathrm{vol}_k(E)}{c_k} \geq k^{1/4}e^{-k/2}\frac{\mathrm{vol}_k(\mathbf{\Pi}_S L_{\mathbf{A}})}{c_k}.$$

The last inequality follows because $L_{\mathbf{A}} \subseteq E$. Re-arranging and raising to the power $1/k$, this gives us that

$$\mathrm{volLB}^*(A) = \frac{\mathrm{vol}_k(\mathbf{\Pi}_S L_{\mathbf{A}})^{1/k}}{c_k^{2/k}} \lesssim \frac{|\det((\mathbf{\Pi}_S\mathbf{a}_i)_{i\in T})|^{1/k}}{c_k^{1/k}} \lesssim \sqrt{k}\cdot \mathrm{detlb}(A),$$

where, in the final inequality, we used that $(\mathbf{\Pi}_S\mathbf{a}_i)_{i\in T}$ is the transpose of a k by k submatrix of $\mathbf{A}$, and we also used the estimate $c_k^{-1/k} \lesssim \sqrt{k}$, which follows from Stirling's approximation. Since $k \leq n$, the result follows. $\square$

We remark in passing that the trivial inequality $\mathrm{vol}_k(E) \geq \mathrm{vol}_k(\mathbf{\Pi}_S L_{\mathbf{A}})$ for a k-dimensional symmetric convex polytope with $2m$ vertices $\mathbf{\Pi}_S L_{\mathbf{A}}$ and an ellipsoid E containing it can be improved to $\mathrm{vol}_k(E) \geq \sqrt{\frac{k}{\log(2m)}}\mathrm{vol}_k(\mathbf{\Pi}_S L_{\mathbf{A}})$ when m is small, using, e.g., results of Gluskin [Glu89]. Substituting this inequality in the proof of Lemma 42 gives the bound $\mathrm{volLB}^*(\mathbf{A}) \lesssim \sqrt{\log 2m}\cdot \mathrm{detlb}(\mathbf{A})$.

The final ingredient we need for the proof of Theorem 7 is an upper bound on the hereditary discrepancy of partial colorings in terms of the volume lower bound, due to Dadush, Nikolov, Tomczak-Jaegermann, and Talwar.

Lemma 44. [DNTT18, Lemma 8]). *There exist universal constants* $c \geq 1$ *and* $\epsilon_0 \in (0,1)$ *such that the following holds. For any closed convex set* $K \subseteq \mathbb{R}^n$ *satisfying* $-K = K$ *and*

$$\min_{k=1}^{n} \min_{S\subseteq[n]:|S|=k} \mathrm{vol}_k(K \cap W_S) \geq 1,$$

and for any $\mathbf{y} \in (-1,1)^n$*, there exists an* $\mathbf{x} \in [-1,1]^n$ *with* $|\mathrm{fixed}(\mathbf{x})| \geq \lceil \epsilon_0 n\rceil$ *and* $\mathbf{x}-\mathbf{y} \in cK$*, where* $\mathrm{fixed}(\mathbf{x}) \coloneqq \{i \in [n] : |x_i| = 1\}$*.*

We are now ready to complete the proof of Theorem 7.

Proof of Theorem 7. It suffices to show that $\mathrm{disc}(\mathbf{A}) \lesssim \sqrt{n}\,\mathrm{detlb}(\mathbf{A})$, since this implies that for any submatrix $\mathbf{B}$ of $\mathbf{A}$ with k columns we also have

$$\mathrm{disc}(\mathbf{B}) \lesssim \sqrt{k}\ \mathrm{detlb}(\mathbf{B}) \leq \sqrt{n}\ \mathrm{detlb}(\mathbf{A}).$$

Using Lemma 44, we construct a sequence of partial colorings $\mathbf{x}_0 = \mathbf{0}, \ldots, \mathbf{x}_T \in \{-1, +1\}^n$, where $T \lesssim 1 + \log_{1/(1-\epsilon_0)}(n)$, each $\mathbf{x}_t \in [0, 1]^n$, and

$$\|\mathbf{A}(\mathbf{x}_t - \mathbf{x}_{t-1})\|_\infty \lesssim \sqrt{n(1-\epsilon_0)^{t-1}}\ \mathrm{detlb}(\mathbf{A}). \tag{2.8}$$

To construct $\mathbf{x}_1$, we apply Lemma 44 to $\mathbf{y} \coloneqq \mathbf{0}$, and $K \coloneqq \mathrm{volLB}(A) \cdot K_{\mathbf{A}}$. By the definition of $\mathrm{volLB}(A)$, this K satisfies the assumption of the lemma, and we let $\mathbf{x}_1$ equal the $\mathbf{x}$ guaranteed by the lemma. Since $\mathbf{x}_1 \in cK = c \cdot \mathrm{volLB}(A) \cdot K_{\mathbf{A}}$, by the definition of $K_{\mathbf{A}}$ we have that

$$\|\mathbf{A}\mathbf{x}_1\|_\infty \leq c \cdot \mathrm{volLB}(A) \asymp \mathrm{volLB}^*(A) \lesssim \sqrt{n}\ \mathrm{detlb}(\mathbf{A}),$$

where the last two inequalities follow, respectively, by Claim 41 and by Lemma 42. In general, to get the bound (2.8) for $\mathbf{x}_t - \mathbf{x}_{t-1}$ for $t \geq 2$, we set $S \coloneqq [n] \setminus \mathrm{fixed}(\mathbf{x}_{t-1})$, and apply Lemma 44 with $\mathbf{y} \coloneqq \mathbf{\Pi}_S \mathbf{x}_1$, and $K \coloneqq \mathrm{volLB}(\mathbf{A}_S) \cdot K_{\mathbf{A}_S}$, where $\mathbf{A}_S$ is the submatrix of $\mathbf{A}$ consisting of the columns indexed by S. If $\mathbf{x} \in [-1, +1]^S$ is the partial coloring guaranteed by the lemma, we define $\mathbf{x}_t$ by setting its coordinates in S to equal the corresponding coordinates in $\mathbf{x}$, and the remaining coordinates to equal the corresponding coordinates in $\mathbf{x}_{t-1}$. It is straightforward to check that $\mathrm{fixed}(\mathbf{x}_t) \geq (1 - (1-\epsilon_0)^t)n$ and (2.8) hold for all t. Moreover, once $t \geq T \geq 1 + \log_{1/(1-\epsilon_0)}(n)$, we must have $\mathbf{x}_t \in \{-1, +1\}^n$.

Having constructed $\mathbf{x}_1, \ldots, \mathbf{x}_T$, we observe that, by (2.8) and the triangle inequality,

$$\|\mathbf{A}\mathbf{x}\|_\infty \lesssim \sqrt{n} \cdot \mathrm{detlb}(\mathbf{B}) \left(1 + \sqrt{(1-\epsilon_0)} + \sqrt{(1-\epsilon_0)^2} + \cdots\right) \asymp \sqrt{n} \cdot \mathrm{detlb}(\mathbf{A}). \qquad \square$$

2.2.4 Upper Bound in Terms of VC Dimension

In this section we present another proof of a weak upper bound of the ratio between the hereditary discrepancy and determinant lower bound of a matrix $\mathbf{A}$ using the VC dimension of $\mathbf{A}$, denoted $\dim(\mathbf{A})$. We introduce some terminology from stochastic processes that will appear in our proof. For a metric space (T, d), let $\mathcal{N}(T, d, \epsilon)$ be the *covering number of* T i.e. $\mathcal{N}(T, d, \epsilon)$ is the smallest number of closed balls with centers in T and radii ϵ whose union covers T. Further, let $\|Y\|_{\psi_2}$ be the sub-gaussian norm of a real-valued random variable Y where $\|Y\|_{\psi_2} \coloneqq \inf\{t \geq 0 : \mathbb{E}\exp\left(Y^2/t^2\right) \leq 2\}$. Finally, for a class of real-valued functions $\mathcal{F}$ defined on a probability space (Ω, μ), where Ω is a finite set, we define the $L^2(\mu)$ norm by $\|f\|_{L^2(\mu)} \coloneqq \left(\sum_{\omega \in \Omega} |f(\omega)|^2 \mu(\omega)\right)^{1/2}$. If μ is the uniform measure on Ω, then we simple write L^2 rather than $L^2(\mu)$.

The following result is a consequence of well-known lemmas. We recount it here for completeness.

Lemma 45. *For any non-constant matrix* $\mathbf{A} \in \{0, 1\}^{m \times n}$,

$$\mathrm{herdisc}(\mathbf{A}) \lesssim \sqrt{n \cdot \dim(\mathbf{A})}.$$

Proof. It is enough to prove that $\mathrm{disc}(\mathbf{A}) \lesssim \sqrt{n \cdot \dim(\mathbf{A})}$ since, applying this inequality to any submatrix $\mathbf{B}$ consisting of a subset of k columns from $\mathbf{A}$ shows that, $\mathrm{disc}(\mathbf{B}) \lesssim \sqrt{k \cdot \dim(\mathbf{B})} \leq \sqrt{n \cdot \dim(\mathbf{A})}$.

We prove that $\mathrm{disc}(\mathbf{A}, \mathbf{x}) \lesssim \sqrt{n \cdot \dim(\mathbf{A})}$ is satisfied in expectation by a uniformly random coloring $\mathbf{x} \in \{-1,+1\}^n$ with entries $X_1, X_2, ...X_n$ which are independent Rademacher random variables (i.e., uniform in $\{-1,+1\}$). Let $\mathcal{F}$ denote the class of indicator functions defined by the rows of $\mathbf{A}$, i.e., for every $i \in [m]$, we define a function $f_i : [n] \to \{0,1\}$ given by $f_i(j) = a_{i,j}$. We will show that

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i f(i) \right| \lesssim \sqrt{\dim(\mathcal{F})}.$$

For each indicator function f, let the random variable $Z_f := \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i f(i) \right|$. Note that this differs from the discrepancy of the row indicated by f by a multiple of $\sqrt{n}$, i.e., $\mathrm{disc}(\mathbf{A}, \mathbf{x}) = \sqrt{n} \cdot \sup_{f \in \mathcal{F}} Z_f$. Consider the random process $(Z_f)_{f \in \mathcal{F}}$. We will apply Dudley's inequality (Lemma 46) to show that

$$\mathbb{E} \sup_{f \in \mathcal{F}} Z_f \lesssim \int_0^1 \sqrt{\log \mathcal{N}\left(\mathcal{F}, L^2, \epsilon\right)} d\epsilon. \tag{2.9}$$

In order to apply Dudley's inequality we must show that $(Z_f)_{f \in \mathcal{F}}$ has sub-gaussian increments. Note that, since $\|X_i\|_{\psi_2} \lesssim 1$, we have

$$\begin{aligned} \|Z_f - Z_g\|_{\psi} &= \frac{1}{\sqrt{n}} \left\| \left| \sum_{i=1}^{n} X_i f(i) \right| - \left| \sum_{i=1}^{n} X_i g(i) \right| \right\|_{\psi_2} \\ &\leq \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^{n} X_i (f-g)(i) \right\|_{\psi_2} \lesssim \left(\frac{1}{n} \sum_{i=1}^{n} (f-g)(i)^2 \right)^{1/2}, \end{aligned}$$

where the second step follows from the reverse triangle inequality, and the final step by Hoeffding's lemma. The right hand side is $\|f-g\|_{L^2}$ so when we apply Dudley's inequality as shown in Lemma 46, we obtain Equation 2.9.

Using Theorem 47, we bound the covering number with respect to the normalized L_2 norm as

$$\log \mathcal{N}\left(\mathcal{F}, L^2, \epsilon\right) \lesssim \dim(\mathcal{F}) \log \left(\frac{2}{\epsilon} \right).$$

Plugging the right hand side into the integral in Equation 2.9 and integrating, $\mathbb{E} \sup_{f \in \mathcal{F}} Z_f \lesssim \sqrt{\dim(\mathcal{F})}$. Recall that the discrepancy of the row indicated by f is $\sqrt{n} \cdot Z_f$, thus the hereditary discrepancy is bounded above as $\mathrm{herdisc}(\mathbf{A}) \lesssim \sqrt{n \dim(\mathcal{F})}$, as was our goal. □

Lemma 46. (Dudley's Inequality, Remark 8.1.5 [Ver18]). *Let $(X_t)_{t \in T}$ be a random process on a metric space (T, d) with sub-gaussian increments i.e. there exists a $K \geq 0$ such that $\|X_t - X_s\|_{\psi_2} \leq K d(t, s)$ for all $t, s \in T$. Then*

$$\mathbb{E} \sup_{t,s \in T} |X_t - X_s| \lesssim K \int_0^{\infty} \sqrt{\log \mathcal{N}(T, d, \epsilon)} d\epsilon.$$

Theorem 47. (Covering Numbers via VC Dimension, 8.3.18 [Ver18]). *Let $\mathcal{F}$ be a class of Boolean functions on a probability space (Ω, Σ, μ). Then, for every $\epsilon \in (0,1)$, we have*

$$\mathcal{N}\left(\mathcal{F}, L^2(\mu), e\right) \leq \left(\frac{2}{\epsilon}\right)^{C\cdot \dim(\mathcal{F})}$$

for an absolute constant C.

We note that Lemma 45 can likely be improved further, for example by following the techniques of Matoušek [Mat95], and carefully tracking constants.

To finish the proof, it remains to show a connection between VC dimension and the determinant lower bound. To do so, we show that a matrix $\mathbf{A} \in \{0,1\}^{m\times n}$ with large VC dimension must contain a submatrix with large determinant. This submatrix is a binary version of the Hadamard matrix, described next.

Let the 0-1 Hadamard matrix be the $\{0,1\}$ matrix obtained by applying the linear map $a \mapsto (a+1)/2$ to all of the entries in the standard ± 1 Hadamard matrix. Denote the $n \times n$ 0-1 and standard Hadamard matrices by $\tilde{\mathbf{H}}_n$ and $\mathbf{H}_n$ respectively. We prove the following.

Claim 48. $\left|\det\left(\tilde{H}_n\right)\right| \geq 2^{-n} \cdot n^{n/2}$.

Proof. Consider $\mathbf{H}_n$ and suppose w.l.o.g. that its first rows is the all ones row. Add this row to all the other rows. Observe that all the other rows now have entries in $\{0,2\}$. Scale them down by a factor of two. Adding one row to another does not change the determinant. Scaling a row scales the determinant by the same amount. Since $|\det(\mathbf{H}_n)| = n^{n/2}$, $\left|\det\left(\tilde{\mathbf{H}}_n\right)\right| = 2^{-n}\cdot n^{n/2}$. □

We can now finish the proof of Theorem 7 for $\mathbf{A} \in \{0,1\}^{m\times n}$. If $\mathbf{A}$ is a constant matrix (i.e., all its entries are equal), the bound is trivial, so we assume otherwise. The upper bound arises from the pair of inequalities $\mathrm{herdisc}(\mathbf{A}) \lesssim \sqrt{n \dim(\mathbf{A})}$ and $\sqrt{\dim(\mathbf{A})} \lesssim \mathrm{detlb}(\mathbf{A})$. The former inequality is achieved by a random coloring, as shown in Lemma 45. The latter follows by considering the power matrix $\mathbf{P}_{\dim(\mathbf{A})}$, which is a submatrix of $\mathbf{A}$. Since every $\dim(\mathbf{A}) \times \dim(\mathbf{A})$ 0-1 matrix is a submatrix of $\mathbf{P}_{\dim(\mathbf{A})}$, we can find also find $\tilde{\mathbf{H}}_{\dim(\mathbf{A})}$ as a submatrix of $\mathbf{P}_{\dim(\mathbf{A})}$, and, therefore, of $\mathbf{A}$. By Claim 48 we know that $\left|\det\left(\tilde{\mathbf{H}}_{\dim(\mathbf{A})}\right)\right| \geq 2^{-\dim(\mathbf{A})}\cdot(\dim(\mathbf{A}))^{\dim(\mathbf{A})/2}$. It follows that

$$\mathrm{detlb}(\mathbf{A}) \geq \left|\det\left(\tilde{\mathbf{H}}_{\dim(\mathbf{A})}\right)\right|^{1/\dim(\mathbf{A})} \gtrsim \sqrt{\dim(\mathbf{A})}.$$

Thus, the two inequalities together give us $\mathrm{detlb}(\mathbf{A}) \gtrsim \mathrm{herdisc}(\mathbf{A})/\sqrt{n}$ as required.

Chapter 3

Linear Discrepancy

In this chapter we discuss algorithmic and the computational complexity aspects of linear discrepancy from a joint work with Aleksandar Nikolov [LN20].

Our investigation proceeds in two directions: proving hardness results and finding both exact and approximate algorithms to evaluate the linear discrepancy of certain matrices. For the former, we show that linear discrepancy is NP-hard so we do not expect to find an efficient exact algorithm in the general case. Thus for the latter, we restrict our attention to matrices with a constant number of rows. We present a poly-time exact algorithm for matrices consisting of a single row and matrices with a constant number of rows and entries of bounded magnitude. We also present an exponential-time approximation algorithm for general matrices, and an algorithm that approximates linear discrepancy to within an exponential factor.

Recall the definition of linear discrepancy in Equation (1.2) and repeated below for ease of use. For a fixed $\mathbf{w} \in [0,1]^n$ we define $\mathrm{lindisc}(\mathbf{A},\mathbf{w})$ to be

$$\mathrm{lindisc}(\mathbf{A},\mathbf{w}) = \min_{\mathbf{x}\in\{0,1\}^n} \|\mathbf{A}\,(\mathbf{w}-\mathbf{x})\|_\infty. \tag{3.1}$$

A vector $\mathbf{w}^*$ which maximizes $\mathrm{lindisc}(\mathbf{A},\mathbf{w})$ is known as a *deep-hole* of $\mathbf{A}$ and $\mathrm{lindisc}(\mathbf{A})$ is the value of the linear discrepancy of $\mathbf{A}$ with respect to this deep-hole, i.e.,

$$\mathrm{lindisc}(\mathbf{A}) = \max_{\mathbf{w}\in[0,1]^n} \mathrm{lindisc}(\mathbf{A},\mathbf{w}). \tag{3.2}$$

Before stating our results, it is worth mentioning that linear discrepancy can also be seen as an analogue of the covering radius in lattice theory. Let $\Lambda \subset \mathbb{R}^n$ be a lattice, i.e. discrete additive subgroup of $\mathbb{R}^n$, and let us choose $\mathbf{b}_1,\ldots,\mathbf{b}_n$ to be a basis of Λ. Let $\mathbf{B}$ be a matrix with the $\mathbf{b_i}$ as its columns. The covering radius of Λ in the ℓ_p-norm is defined as

$$\rho(\Lambda) = \max_{\mathbf{y}\in\mathbb{R}^n}\min_{\mathbf{z}\in\Lambda}\|\mathbf{y}-\mathbf{z}\|_p = \max_{\mathbf{w}\in\mathbb{R}^n}\min_{\mathbf{x}\in\mathbb{Z}^n}\|\mathbf{B}\cdot(\mathbf{w}-\mathbf{x})\|_p = \max_{\mathbf{w}\in[0,1]^n}\min_{\mathbf{x}\in\mathbb{Z}^n}\|\mathbf{B}\cdot(\mathbf{w}-\mathbf{x})\|_p, \tag{3.3}$$

and is independent of the basis. This definition is equivalent to the the definition of $\mathrm{lindisc}(\mathbf{A})$, except that the minimum is over $\mathbb{Z}^n$ rather than $\{0,1\}^n$. Haviv and Regev showed that the covering radius problem (CRP) in the ℓ_p-norm is Π_2-hard to approximate within some fixed constant for all large enough p [HR06], and Guruswami, Micciancio, and Regev showed it can be approximated within a factor of $2^{O(n \log n / \log \log n)}$ for the case of $p = 2$ [GMR05].

3.1 Hardness Result

In this section, we show that linear discrepancy (LDS) is NP-Hard by reducing from monotone not-all-equal 3-SAT (MNAE3SAT) [Gol78] to each. The decision problem version of linear discrepancy we consider is defined below.

> [MNAE3SAT] **Monotone Not-All-Equal 3-SAT**
> Let U be a collection of variables $\{u_1, ..., u_n\}$ and $\mathcal{C}$ be a 3-CNF with clauses $\{C_1, ..., C_m\}$ such that $C_i = t_{i,1} \vee t_{i,2} \vee t_{i,3}$ for positive literals $t_{i,j}$.
> **Question:** Does there exist a truth assignment $\tau : U \to \{\mathsf{T}, \mathsf{F}\}$ such that $\mathcal{C}$ is satisfied and each clause has at least one true and one false literal?

> [LDS] **Linear Discrepancy**
> Let $\mathbf{A} \in \mathbb{Q}^{m \times n}$ be a matrix and $t \geq 0$ a rational value.
> **Question:** Is $\text{lindisc}(\mathbf{A}) \leq t$?

To see that this reduction is sound, suppose instead that $\mathcal{C}$ is a YES-instance of MNAE3SAT. Let τ^* be a satisfying truth assignment of $\mathcal{C}$. Let $\mathbf{x}^*$ be the indicator vector of the true variables in τ^*. Then

$$\text{disc}(\mathbf{A}) \leq \left\| \mathbf{A} \left(\frac{1}{2} \cdot \mathbb{1} - \mathbf{x}^* \right) \right\|_\infty = \frac{1}{2}$$

since every clause has exactly two elements with the same truth value. Thus $\mathbf{A}$ is a YES-instance of DS.

Hereditary discrepancy can be shown to be shard using the same reduction with $t = 1$. A NO-instance of MNAE3SAT translates into a NO-instance of HDS since $\text{herdisc}(\mathbf{A}) \geq \text{disc}(\mathbf{A}) = \frac{3}{2}$. A YES-instance of MNAE3SAT translates into a YES-instance of HDS as it did for DS.

Let τ^* be a satisfying assignment of $\mathcal{C}$ and $\mathbf{x}^*$ be the indicator vector of the true variables. Consider any subset of the variables $U' \subseteq U$. We will show that $\mathbf{A}_{U'}$, the matrix $\mathbf{A}$ restricted to the variables in U', satisfies $\text{disc}(\mathbf{A}_{U'}) \leq 1$. Consider a row of $\mathbf{A}_{U'}$. Either the row has fewer than three or exactly three non-zero entries. In the former case any assignment to the variables achieves discrepancy ≤ 1 for that row. In the latter case all variables in the associated clause are intact. The rows of $A_{U'}$ which have three non-zero entries correspond to a subset of the clauses in $\mathcal{C}$. Since τ^* is a satisfying assignment of $\mathcal{C}$, it must also be a satisfying assignment of this subset of clauses. Let $\mathbf{x}^*_{U'}$ be the vector $\mathbf{x}^*$ restricted to the variables in U' and $\frac{1}{2} \cdot \mathbb{1}_{U'}$ be the vector $\frac{1}{2} \cdot \mathbb{1}$ restricted to the variables in U'. Then

$$\text{herdisc}(\mathbf{A}) \leq \max_{U' \subset U} \left\| \mathbf{A}_{U'} \left(\frac{1}{2} \cdot \mathbb{1}_{U'} - \mathbf{x}_{U'} \right) \right\|_\infty \leq 1$$

since U' was an arbitrary subset of U.

Before we show that linear discrepancy is hard, we will show that the value of $\text{lindisc}(\mathbf{A})$ can be expressed using a polynomial number of bits in the bit complexity of a matrix for rational matrices.

Lemma 49. *For any matrix $\mathbf{A} \in \mathbb{Q}^{m\times n}$, there exists a deep hole for $\mathbf{A}$ with bit complexity polynomial in n and the bit complexity of $\mathbf{A}$, and, therefore, $\text{lindisc}(\mathbf{A})$ can be written in number of bits polynomial in n and the bit complexity of $\mathbf{A}$.*

Proof. Let $\mathbf{r}_i$ for $i \in [m]$ be the rows of $\mathbf{A}$, $\text{lindisc}(\mathbf{A}) = \lambda_A$, and $\mathbf{w}^*$ be a deep-hole of $\mathbf{A}$. For every $\mathbf{x} \in \{0,1\}^n$ there exists an $i \in [m]$ and $\sigma \in \{-1, 1\}$ such that $\sigma \mathbf{r}_i(\mathbf{w}^* - \mathbf{x}) \geq \lambda_A$. Let $\mathbf{b}_x = \sigma \mathbf{r}_i$ and consider the following linear program over the variables $\mathbf{w} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$:

$$\begin{aligned} \text{Maximize: } & \lambda \\ \text{Subject to: } & \mathbf{b}_{\mathbf{x}}(\mathbf{w} - \mathbf{x}) \geq \lambda && \text{for all } \mathbf{x} \in \{0,1\}^n \\ & \mathbf{0} \leq \mathbf{w} \leq \mathbf{1} \end{aligned}$$

Let λ^* be the optimum value of this linear program. First note that $\lambda_A \leq \lambda^*$ since $(\mathbf{w}^*, \lambda)$ satisfies the constraints. Next we show that $\lambda_A \geq \lambda^*$. Suppose, towards contradiction, that $\lambda_A < \lambda^*$. Then there exists $\mathbf{w}' \in [0,1]^n$ such that

$$\|\mathbf{A}(\mathbf{w}' - \mathbf{x})\|_\infty \geq \mathbf{b}_x(\mathbf{w}' - \mathbf{x}) \geq \lambda^* > \lambda_A$$

for every $\mathbf{x} \in \{0,1\}^n$. Since $\lambda_A = \text{lindisc}(\mathbf{A})$, we cannot have $\text{lindisc}(\mathbf{A}, \mathbf{w}') > \lambda_A$. Thus $\lambda^* = \text{lindisc}(\mathbf{A})$. Since this LP has n variables, the number of bits required to express the linear discrepancy and some deep-hole $\mathbf{w}^*$ of $\mathbf{A}$ are polynomial in n and the bit complexity of the largest entry of $\mathbf{A}$ [Sch98]. □

Theorem 8 (Linear Discrepancy Hardness)**.** *Given an $m \times n$ matrix $\mathbf{A}$ with rational entries, and a rational number t, deciding whether $\text{lindisc}(\mathbf{A}) \leq t$, is NP-hard and is contained in the class Π_2.*

Proof. Note first that the fact that LDS is contained in Π_2 is a straightforward consequence of Lemma 49: the "for-all" quantifier is over potential deep holes $\mathbf{w} \in [0,1]^n$ of the appropriate polynomially bounded bit complexity, and the "exists" quantifier is over $\mathbf{x} \in \{0,1\}^n$.

Next we prove hardness. Let 3-CNF $\mathcal{C}$ be a MNAE3SAT instance as described above. The corresponding LDS instance will be the incidence matrix $\mathbf{A} \in \{0,1\}^{m\times n}$ of $\mathcal{C}$: column $\mathbf{a}_j$ of $\mathbf{A}$ corresponds to variable u_j and row $\mathbf{r}_i$ of $\mathbf{A}$ corresponds to clause C_i, and $A_{i,j} = 1$ if and only if variable u_j appears in clause C_i. Let the target t in the LDS problem be $\frac{3}{2} - \epsilon$ for $\epsilon > 0$ to be determined later.

Consider first that case that $\mathcal{C}$ is a NO-instance of MNAE3SAT i.e. for every truth assignments τ, there exists a clause C_i whose literals all get the same truth assignment. Each $\mathbf{x} \in \{0,1\}^n$ corresponds to a truth assignment. If $x_i = 1$ (resp. $x_i = 0$) then u_i is true (resp. u_i is false). Let C_j be the clause whose literals have the same truth value. Then

$$\text{lindisc}(\mathbf{A}) \geq \text{lindisc}(\mathbf{A}, (1/2)\cdot \mathbb{1}) \geq \left| \mathbf{r}_j \left(\frac{1}{2} \cdot \mathbb{1} - \mathbf{x} \right) \right| = \frac{3}{2} > \frac{3}{2} - \epsilon,$$

so $\mathbf{A}$ is a NO-instance of LDS.

Consider next the case that $\mathcal{C}$ is a YES-instance of $\mathsf{MNAE3SAT}$, and let τ be a satisfying assignment. Suppose $\mathbf{w}^* \in [0,1]^n$ is a deep-hole of $\mathbf{A}$. If $w_i^* = \frac{1}{2}$ for all $i \in [n]$ then

$$\operatorname{lindisc}(\mathbf{A}) = \operatorname{lindisc}(\mathbf{A}, (1/2)\cdot \mathbb{1}) = \operatorname{disc}(\mathbf{A}) \leq \left\| \mathbf{A}\left(\frac{1}{2}\cdot \mathbb{1} - \mathbf{x}^*\right)\right\|_\infty = \frac{1}{2}$$

since every clause has exactly two elements with the same truth value. Thus $\mathbf{A}$ is a YES-instance of LDS as long as we choose $\epsilon \leq 1$. Suppose then that $\mathbf{w}^* \neq \frac{1}{2}\mathbb{1}$, and let ϵ be a lower bound on the smallest non-zero gap between w_i^* and $1/2$ i.e. for all $w_i^* \neq \frac{1}{2}$,

$$\left| w_i^* - \frac{1}{2}\right| \geq \epsilon.$$

By Lemma 49, which implies that $\mathbf{w}^*$ has polynomial bit complexity, we know that we can choose such an ϵ of polynomial bit complexity. We will show that $\operatorname{lindisc}(\mathbf{A}, \mathbf{w}^*) \leq \frac{3}{2} - \epsilon$ by constructing a colouring $\mathbf{x}^*$. Let

$$x_i^* = \begin{cases} \operatorname{rd}(w_i^*) & \text{if } w_i^* \neq \frac{1}{2} \\ \tau(u_i) & \text{otherwise} \end{cases}$$

where $\operatorname{rd}(w_i^*)$ is w_i^* rounded to the closest integer and u_i is the variable corresponding to column i. Let $\mathbf{r}$ be a row of matrix $\mathbf{A}$ with non-zero entries in columns i, j, and k. We bound the discrepancy of row $\mathbf{r}$ based on the number of rounded variables R_v among $\{x_i, x_j, x_k\}$.

$R_v = 0$: Since none of the variables are rounded, $w_i^* = w_j^* = w_k^* = \frac{1}{2}$ and

$$|\mathbf{r}\,(\mathbf{x}^* - \mathbf{w}^*)| = \left| \left(x_i^* - \frac{1}{2}\right) + \left(x_j^* - \frac{1}{2}\right) + \left(x_k^* - \frac{1}{2}\right)\right| = \frac{1}{2}$$

since τ is a satisfying assignment.

$R_v = 1$: W.l.o.g assume that that x_i^* is the rounded value and $w_j^* = w_k^* = \frac{1}{2}$. Then

$$|\mathbf{r}\,(\mathbf{x}^* - \mathbf{w}^*)| = \left| (x_i^* - w_i^*) + \left(x_j^* - \frac{1}{2}\right) + \left(x_k^* - \frac{1}{2}\right)\right| \leq \left(\frac{1}{2} - \epsilon\right) + 1 = \frac{3}{2} - \epsilon.$$

$R_v = 2$: W.l.o.g assume that x_i^* and x_j^* are the rounded values and $w_k^* = \frac{1}{2}$. Then

$$|\mathbf{r}\,(\mathbf{x}^* - \mathbf{w}^*)| = \left| (x_i^* - w_i^*) + \left(x_j^* - w_j^*\right) + \left(x_k^* - \frac{1}{2}\right)\right| \leq 2\cdot\left(\frac{1}{2} - \epsilon\right) + \frac{1}{2} = \frac{3}{2} - 2\epsilon.$$

$R_v = 3$: All three values are rounded so

$$|\mathbf{r}\,(\mathbf{x}^* - \mathbf{w}^*)| = \left| (x_i^* - w_i^*) + \left(x_j^* - w_j^*\right) + (x_k^* - w_k^*)\right| \leq 3\cdot\left(\frac{1}{2} - \epsilon\right) = \frac{3}{2} - 3\epsilon.$$

Since $\mathbf{r}$ was an arbitrary row of $\mathbf{A}$, $\operatorname{lindisc}(\mathbf{A}) = \operatorname{lindisc}(\mathbf{A}, \mathbf{w}^*) \leq \frac{3}{2} - \epsilon$ as required. This completes the reduction. $\square$

3.2 Algorithms

In the following we consider restrictions and variants of linear discrepancy for which we are able to give poly-time algorithms. The first subsection considers matrices with a single row. The second subsection considers matrices $\mathbf{A} \in \mathbb{Z}^{d\times n}$ with constant d and entry of largest magnitude δ. In that case, we compute lindisc$(\mathbf{A})$ in time $O\left(d(2n\delta)^{d^2}\right)$. The third subsection presents a poly-time 2^n approximation to lindisc$(\mathbf{A})$ for $\mathbf{A} \in \mathbb{Q}^{m\times n}$.

3.2.1 Row Matrix

We begin by developing some intuition for the linear discrepancy of a one-row matrix, $\mathbf{A} = [a_1, ..., a_n]$. For now, let us make the simplifying assumption that the entries of $\mathbf{A}$ are non-negative and sorted in decreasing order. Define the *subset sums* of $\mathbf{A}$ to be the multi-set $\mathcal{S}(\mathbf{A}) = \{s_1, ..., s_{2^n}\}$ where each $s_i = \mathbf{A}\mathbf{x}$ for exactly one $\mathbf{x} \in \{0,1\}^n$. Enumerate the element of $\mathcal{S}(\mathbf{A})$ in non-decreasing order, i.e. $s_i \le s_{i+1}$. If $\ell_A = 2 \cdot \text{lindisc}(\mathbf{A})$, then ℓ_A is the width of the largest gap between consecutive entries in $\mathcal{S}(\mathbf{A})$.

Suppose $\mathbf{A}_i = [a_1, ..., a_i]$. Let us consider how $\mathcal{S}(\mathbf{A}_i)$ and lindisc(A_i) change for the first couple of values of i. Clearly, $\mathcal{S}(\mathbf{A}_1) = [0, a_1]$ and lindisc$(\mathbf{A}_1) = \frac{a_1}{2}$. $\mathcal{S}(\mathbf{A}_2)$ is the disjoint union of $\mathcal{S}(\mathbf{A}_1)$ and $\mathcal{S}(\mathbf{A}_1)$ shifted to the right by a_2. Since $a_1 \ge a_2$, $\mathcal{S}(\mathbf{A}_2) = [0, a_2, a_1, a_1 + a_2]$ where the largest gap is of size $\max(a_2, a_1 - a_2)$. See Figure 3.1. In general, the entries of $\mathcal{S}(\mathbf{A}_i)$ consists of two copies of $\mathcal{S}(\mathbf{A}_{i-1})$ with one shifted to the right by a_i. The gaps in $\mathcal{S}(\mathbf{A}_i)$ are gaps previously in $\mathcal{S}(\mathbf{A}_{i-1})$ or between an element of $\mathcal{S}(\mathbf{A}_{i-1})$ and one in $\{a_i + s : s \in \mathcal{S}(\mathbf{A}_{i-1})\}$.

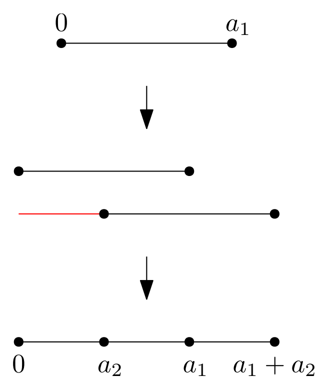


Figure 3.1: Obtaining $\mathcal{S}(\mathbf{A}_2)$ from $\mathcal{S}(\mathbf{A}_1)$ when $a_1 \ge a_2$.

A similar structure occurs for general matrices with real valued entries with two caveats: (1) the previous interval is shifted left or right depending on the sign of the current entry (negative and positive respectively) and (2) the smallest entry of $\mathcal{S}(\mathbf{A})$ is not zero but the sum of the negative entries in $\mathbf{A}$.

Lemma 50. *Suppose $\mathbf{A}_{k-1} = [a_1, ..., a_{k-1}]$ with entries in $\mathbb{R}$ and $|a_i| \ge |a_{i+1}|$. Let the largest gap in $\mathcal{S}(\mathbf{A}_{k-1})$ be of size ℓ_{k-1}. Then, for $\mathbf{A}_k = [a_1, ..., a_{k-1}, a_k]$ where $|a_k| \le |a_i|$ for all $i \in [k-1]$, the largest gap in $\mathcal{S}(\mathbf{A}_k)$ is of size* $\max(|a_k|, \ell_{k-1} - |a_k|)$.

Proof. Again, it is important to remember that the entries of $\mathcal{S}(\mathbf{A}_k)$ are exactly those in $\mathcal{S}(\mathbf{A}_{k-1})$

along with those in $\{a_k + s : s \in \mathcal{S}(\mathbf{A}_{k-1})\}$. Let $\ell = \max(|a_k|, \ell_{k-1} - |a_k|)$.

We first show that $2 \cdot \mathrm{lindisc}(\mathbf{A}_k) \leq \ell$ by showing that gaps between consecutive entries in $\mathcal{S}(\mathbf{A}_k)$ have size at most ℓ. If (s_j, s_{j+1}) is a consecutive pair in $\mathcal{S}(\mathbf{A}_{k-1})$ such that $s_{j+1} - s_j > \ell$, then s_j and s_{j+1} are no longer consecutive in $\mathcal{S}(\mathbf{A}_k)$, since $s_j \leq s_j + a_k \leq s_{j+1}$ if $a_k > 0$ and $s_j \leq s_{j+1} + a_k \leq s_{j+1}$ if $a_k < 0$. See Figure 3.2. Then, the gap given by any such pair gets split into gaps of size at most $\max\{|a_k|, s_{j+1} - s_j - |a_k|\} \leq \ell$, where the inequality holds because $s_{j+1} - s_j \leq \ell_{k-1}$. It follows that the size of each gap in $\mathcal{S}(\mathbf{A}_k)$ is at most ℓ.

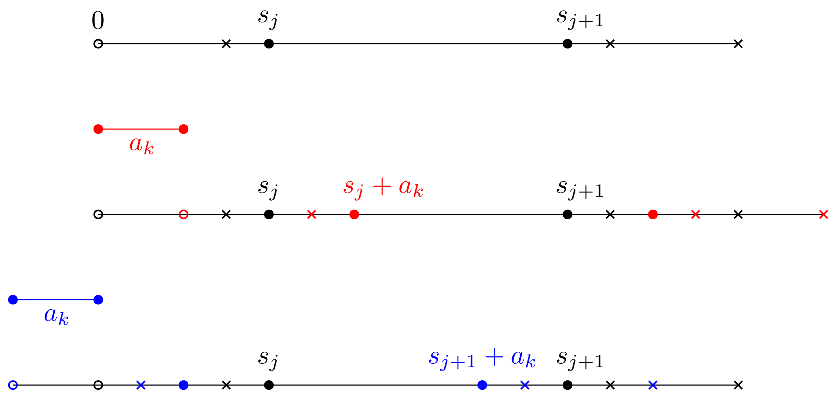


Figure 3.2: All consecutive pairs in $\mathcal{S}(\mathbf{A}_{k-1})$ of size greater than $|a_k|$ will be divided into two or more consecutive pairs in $\mathcal{S}(\mathbf{A}_k)$. The red interval indicates what happens when $a_k > 0$. The blue interval indicates what happens when $a_k < 0$.

Next we will show that $2 \cdot \mathrm{lindisc}(\mathbf{A}_k) \geq \ell$ by producing a pair of consecutive entries in $\mathcal{S}(\mathbf{A}_k)$ which achieves gap ℓ. Suppose $\ell = |a_k|$. Recall that s_0 is the smallest subset sum of all entries in $\mathbf{A}_k$, which equals the sum of all negative entries in $\mathbf{A_k}$. Then it is easy to check that s_1 equals $s_0 + |a_k|$, where we recall that a_k is the entry in $\mathbf{A}_k$ with minimum absolute value. Therefore, $(s_0, s_0 + |a_k|)$ is a consecutive pair in $\mathcal{S}(\mathbf{A}_k)$. This means that if $\ell = |a_k|$, then we are done, as we have produced a pair with gap ℓ.

When $\ell = \ell_{k-1} - |a_k| > |a_k|$, we split our analysis into two cases: (1) $a_k > 0$ and (2) $a_k < 0$.

In the former case, let (s_{j^*}, s_{j^*+1}) be a consecutive pair in $\mathcal{S}(\mathbf{A}_{k-1})$ that achieves gap ℓ_{k-1} and suppose, towards a contradiction, that $s_{j^*} + a_k$ and s_{j^*+1} do not appear consecutively in $\mathcal{S}(\mathbf{A}_k)$. Then there must be some $s \in \mathcal{S}(\mathbf{A}_k)$ such that $s_{j^*} + a_k < s < s_{j^*+1}$. Note that s cannot be an element of $\mathcal{S}(\mathbf{A}_{k-1})$ since s_{j^*} and s_{j^*+1} are consecutive in $\mathcal{S}(\mathbf{A}_{k-1})$, so $s - a_k$ must be an element of $\mathcal{S}(\mathbf{A}_{k-1})$. However, since $s > s_{j^*} + a_k$, we have $s - a_k > s_{j^*}$. This is a contradiction since s_{j^*} and s_{j^*+1} are consecutive entries in $\mathcal{S}(\mathbf{A}_{k-1})$. See Figure 3.3. Thus $(s_{j^*} + a_k, s_{j^*+1})$ must be a consecutive pair in $\mathcal{S}(\mathbf{A}_k)$.

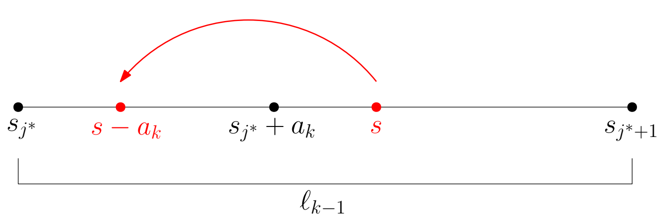


Figure 3.3: Suppose $a_k < \ell_{k-1} - a_k$ and there exists $s \in \mathcal{S}(\mathbf{A}_k)$ such that $s_{j^*} + a_k < s < s_{j^*+1}$.

The latter case, when $a_k < 0$, is similar. Again there exists a pair of consecutive entries (s_{j^*}, s_{j^*+1}) in $\mathcal{S}(\mathbf{A}_{k-1})$ which achieves gap ℓ_{k-1}. Suppose, towards contradiction, that s_{j^*} and $s_{j^*+1} - |a_k|$ do not appear consecutively in $\mathcal{S}(\mathbf{A}_k)$. Then there must be some $s \in \mathcal{S}(\mathbf{A}_k)$ such that $s_{j^*} < s < s_{j^*+1} - |a_k|$. Again, s cannot be an element of $\mathcal{S}(\mathbf{A}_{k-1})$ since s_{j^*} and s_{j^*+1} are consecutive in $\mathcal{S}(\mathbf{A}_{k-1})$, so $s+|a_k|$ must be an element of $\mathcal{S}(\mathbf{A}_{k-1})$. However since $s < s_{j^*+1} - |a_k|$, we have $s + |a_k| < s_{j^*+1}$. This is a contradiction since s_{j^*} and s_{j^*+1} are consecutive entries in $\mathcal{S}(\mathbf{A}_{k-1})$. □

Lemma 50 has the following curious corollary.

Corollary 51. *Let* $\mathbf{A} = [a_1, ..., a_n]$ *and* $\mathbf{A}' = [|a_1|, ..., |a_n|]$. *Then* $\operatorname{lindisc}(\mathbf{A}) = \operatorname{lindisc}(\mathbf{A}')$.

Lemma 50 and Corollary 51 suggest an algorithm: replace the entries of $\mathbf{A}$ by their magnitudes. Sort $\mathbf{A}$. Consider each entry in turn and update the largest gap accordingly. See Algorithm 1.

Theorem 9 (Linear Discrepancy for One Row Matrix)**.** *For any matrix* $\mathbf{A} \in \mathbb{R}^{1\times n}$, $\operatorname{lindisc}(\mathbf{A})$ *can be computed in time* $O(n \log n)$.

Proof. By Corollary 51 it is sufficient to consider row matrices with non-negative entries. Suppose that $\mathbf{A} = [a_1, ..., a_n]$ is such a matrix with entries sorted in decreasing order. Algorithm 1 correctly outputs the linear discrepancy for matrices with a single entry. Let $\mathbf{A}_i = [a_1, ..., a_i]$. Lemma 50 gives us a recursive method for computing the largest gap in $\mathcal{S}(\mathbf{A}_{i+1})$ from the largest gap in $\mathcal{S}(\mathbf{A}_i)$. Since $\operatorname{lindisc}(\mathbf{A})$ is half the size of the largest gap in $\mathcal{S}(\mathbf{A})$, Algorithm 1 computes $\operatorname{lindisc}(\mathbf{A})$ as required. □

Algorithm 1: Linear discrepancy of row matrix.

Input: Matrix $\mathbf{A} \in \mathbb{Q}^{1\times n}$.
Output: $\operatorname{lindisc}(\mathbf{A})$.

for i *from* 1 *to* n **do**
  $\mathbf{A}[i] \leftarrow |a_i|$
sort $\mathbf{A}$ in decreasing order
$\ell \leftarrow a_1$
for i *from* 2 *to* n **do**
  $\ell \leftarrow \max(a_i, \ell - a_i)$
return $\frac{\ell}{2}$

Thus, for any row matrix $\mathbf{A}$ with n elements, we can find $\operatorname{lindisc}(\mathbf{A})$ in time $O(n \log n)$. From the geometric intuition discussed in Section 1.2, we have that finding the largest gap between any two lattice points in $\mathbf{A}$, can be computed in polynomial time. Note, however, that the task of determining if the *smallest* gap is equal to zero is NP-hard [WY92]. Further, there is an intimate relationship between oracles which approximate the smallest gap (via the Number Balancing Problem) and oracles which approximate Minkowski's Theorem the Shortest Vector Problem [HRRY17].

3.2.2 One Row Linear Discrepancy Rounding

Let $\operatorname{lindisc}(\mathbf{A}) = \ell$. By the definition of linear discrepancy, for every $\mathbf{w} \in [0,1]^n$ there exists an $\mathbf{x} \in \{0,1\}^n$ such that $\|\mathbf{A}(\mathbf{w} - \mathbf{x})\|_\infty \leq \ell$. In-fact, if $\mathbf{w}$ is not a deep-hole, there exists an $\mathbf{x}$ which satisfies $\|\mathbf{A}(\mathbf{w}-\mathbf{x})\|_\infty < \ell$. However it is not obvious that finding such an $\mathbf{x}$ can be done efficiently i.e. in polynomial time with respect to the bit complexity of $\mathbf{A}$ and n. By reducing from the subset-sum problem, we observe that it is difficult to compute $\mathrm{lindisc}(\mathbf{A},\mathbf{w})$ let alone find an $\mathbf{x}$ which minimizes $\|\mathbf{A}(\mathbf{w}-\mathbf{x})\|_\infty \leq \ell$.

Theorem 10 (Linear Discrepancy for One Row Matrix Approximation)**.** *For any matrix* $\mathbf{A} \in \mathbb{Q}^{1\times n}$ *and any* $\mathbf{w} \in ([0,1]\cap\mathbb{Q})^n$*, we can find an* $\mathbf{x} \in \{0,1\}^n$ *such that* $\|\mathbf{A}(\mathbf{w}-\mathbf{x})\|_\infty \leq \mathrm{lindisc}(\mathbf{A})$ *in time* $O(n \log n)$*.*

Proof. To begin, let $\mathbf{A} = [a_1, ..., a_n]$ for positive a_i in non-increasing order. We will consider $\mathbf{A}$ with arbitrary entries at the end. Let $w = \mathbf{A}\mathbf{w}$. As before, let $\mathcal{S}(\mathbf{A}) = [s_0, ..., s_{2^n-1}]$ be the subset-sums of $\mathbf{A}$ where each $s_i = \mathbf{A}\mathbf{x}$ for an $\mathbf{x} \in \{0,1\}^n$ and $s_i \leq s_{i+1}$ for all i. Recall that $2 \cdot \mathrm{lindisc}(\mathbf{A})$ is the largest gap between any two consecutive entries in $\mathcal{S}(\mathbf{A})$. Our algorithm will find a pair of subset sums containing w. If we can show that the size of the interval between these two subset sums is no more than the gap between some two consecutive entries in $\mathcal{S}(\mathbf{A})$, then the closest subset sum to w among these two will be within $\mathrm{lindisc}(\mathbf{A})$ of w.

Just as in Algorithm 1, we refine the interval between two subset sums containing w by incrementally adding the entries of $\mathbf{A}$ in decreasing order. Initially our interval is $g_0 = [0, \sum_{i=1}^n a_i]$. We maintain the invariants: (1) $w \in g_i$ for all i, and (2) the end-points of g_i are subset sums.

Suppose $w \in g_i = [u, v]$ and we are considering a_i. If $u + a_i > w$ then set $v \leftarrow \min(v, u + a_i)$. Otherwise let $u \leftarrow u + a_i$. Algorithm 2 computes this interval and the associated vectors $\mathbf{u}$ and $\mathbf{v}$ representing its endpoints.

Consider the values of u and v at the end of the algorithm. We claim that the final interval $[u, v]$ is at most the width of some gap between two consecutive terms in $\mathcal{S}(\mathbf{A})$, the array of all subset sums of $\mathbf{A}$. Notice $u = a_1u_1 + \cdots + a_nu_n$ where $\mathbf{u} = [u_1, ..., u_n]$ is an endpoint of the interval once Algorithm 2 completes.

We partition $\mathbf{u}$ into maximal blocks where all entries in the same block have the same value i.e. $[u_1, u_2, ..., u_{\ell_1}], ..., [u_{\ell_r+1}, u_{\ell_r+2}, ..., u_n]$ such that $u_{\ell_i+1} = u_{\ell_i+2} = \cdots = u_{\ell_{i+1}}$ for $i = 0, 1, ..., r-1$ where $\ell_0 = 0$.

We claim that Algorithm 2 outputs an interval containing w whose width is at most the distance between some two consecutive entries in $\mathcal{S}(\mathbf{A})$. The proof is by induction on r, the number of blocks. In the base case, $r = 1$ and there is only one block. Thus $u = 0$ or $u = \sum a_i$. In the case where $u = 0$, we must have $a_i > w$ for all $i \in [n]$, and $v = a_n$. Thus $w \in [0, a_n]$ with consecutive elements 0 and a_n of $\mathcal{S}(\mathbf{A})$. In the latter case when $u = \sum a_i$, we can output $\mathbf{w}$ since it is already a subset sum.

Suppose next that the claim holds for all matrices where the algorithm outputs a vector $\mathbf{u}$ with k blocks, and we will show that it still holds for a matrix $\mathbf{A}$ whose output $\mathbf{u}$ has $k+1$ blocks. Let $\mathbf{u}' = [u_1, ..., u_{\ell_{k+1}}]$ and $\mathbf{v}' = [v_1, ..., v_{\ell_{k+1}}]$ be the final vectors after running the algorithm on $\mathbf{A}' = [a_1, ..., a_{\ell_{k+1}}]$. Further let $u' = \sum_{i=1}^{\ell_{k+1}} a_iu_i$ and $v' = \sum_{i=1}^{\ell_{k+1}} a_iv_i$. By the induction hypothesis, the width of $[u', v']$ is at most the distance between some two consecutive elements in the list of subset sums of $\mathbf{A}'$. The last block of $\mathbf{u}$ is $[u_{\ell_{k+1}+1}, ..., u_n]$. The entries of this block are either all zeros or all ones. Consider each case in-turn.

First suppose $u_{\ell_{k+1}+1} = \cdots = u_n = 0$. Since none of the a_i for $i = \ell_{k+1}+1, ..., n$ were added to u, it must be the case that $u' + a_i > w$ for all such i. Thus the interval $[u, v] = [u', \min(v', u' + a_n)]$ has width at most a_n. Since 0 and a_n are consecutive in $\mathcal{S}(\mathbf{A})$, as $|a_n|$ is the entry with the smallest magnitude in $\mathbf{A}$, the output interval satisfies our requirements.

Next suppose $u_{\ell_{k+1}+1} = \cdots = u_n = 1$. It must be the case that $u = u' + a_{\ell_{k+1}+1} + \cdots + a_n \leq w$. Observe that $a_{\ell_{k+1}}$ is in the k^{th} block and so $u_{\ell_{k+1}} = 0$. Let $[u'', v'']$ be our interval after processing the $k-1^{\text{st}}$ block i.e. $u'' = \sum_{i=1}^{\ell_k} a_i u_i$ and $v'' = \sum_{i=1}^{\ell_k} a_i v_i$. Notice that since none of the entries in the k^{th} block were added to u'', we must have $u'' + a_i > w$ for all $i = \ell_k + 1, ..., \ell_{k+1}$. In such cases, we always update $v'' \leftarrow \min(v'', u'' + a_i)$ after each such i, thus the interval $[u', v']$ has width at most $a_{\ell_{k+1}}$. Thus it suffices to show that $a_{\ell_{k+1}+1} + \cdots + a_n$ and $a_{\ell_{k+1}}$ are consecutive in $\mathcal{S}(\mathbf{A})$. First note that $a_{\ell_{k+1}+1} + \cdots + a_n \leq a_{\ell_{k+1}}$ since $u' + a_{\ell_{k+1}+1} + \cdots + a_n \leq w \leq v' \leq u' + a_{\ell_{k+1}}$. The two subset sums then are also consecutive, since $a_i > a_{\ell_{k+1}}$ for all $i < \ell_{k+1}$.

Now consider the case where $\mathbf{A}$ can have both positive and negative entries. Without loss of generality we can assume that none of the entries are zero. Let $A_- = \{a_i \in \mathbf{A} : a_i < 0\}$ and $A_+ = \{a_i \in \mathbf{A} : a_i > 0\}$. It suffices to set $u_0 = \sum_{a \in A_-} a$ and $v_0 = \sum_{a \in A_+} a$ and let $\mathbf{u}$ and $\mathbf{v}$ be the indicator vectors of $\mathbf{A}_-$ and $\mathbf{A}_+$ respectively. The remainder of the algorithm is identical except that the matrix should be sorted in decreasing order of *magnitude* and every time an element $a_i \in \mathbf{A}_-$ is added to u, its entry in $\mathbf{u}$ should be set to zero. □

Alternative proof of Theorem 10. Let $\mathbf{A} = [a_1, ..., a_n]$ with $a_1 \geq a_2 \geq \cdots a_n$. It turns out that the linear discrepancy solution structure is the *largest difference between a_i and the sum of all smaller elements $a_{i+1} + \cdots + a_n$*. The proof is by induction.

Let $\mathbf{A}^{(i)} = [a_1, ..., a_i]$ and $s_{i,j} = \sum_{k=i}^{j} a_k$, where $s_{i+1,i} = 0$. Recall that by our original algorithm $\ell_k = 2\text{lindisc}(\mathbf{A}^{(k)}) = \max(\ell_{k-1} - a_k, a_k)$. Now we will show that $\max(\ell_{k-1} - a_k, a_k) = \max_{i \in [k]} a_i - s_{i+1,k}$.

In the base case, $k = 1$ and $a_1 = a_1 - s_{2,1}$. Suppose the claim is true for integers up to k. By in the induction hypothesis we have $\ell_k = \max_{i \in [k]} a_i - s_{i+1,k}$. We want to show that $\ell_{k+1} = \max_{i \in [k+1]} a_i - s_{i+1,k+1}$. By our original algorithm we know that $\ell_{k+1} = \max(\ell_k - a_{k+1}, a_{k+1})$. This is equivalent to

$$\max\left(\left(\max_{i \in [k]} a_i - s_{i+1,k}\right) - a_{k+1}, a_{k+1}\right).$$

However note that $s_{i+1,k} + a_{k+1} = s_{i+1,k+1}$, i.e. adding up all the numbers from $i+1$ to $k+1$. Further $a_{k+1} = a_{k+1} + 0 = a_{k+1} - s_{k+2,k+1}$. Thus

$$\ell_{k+1} = \max_{i \in [k+1]} a_i - s_{i+1,k+1}$$

as required. Plugging in $k = n$, we have that $\ell_n = 2 \cdot \text{lindisc}(\mathbf{A}) = \max_{i \in [n]}(a_i - s_{i+1,n})$. □

3.2.3 Constant Rows with Bounded Matrix Entries

Let $\mathbf{A} \in \mathbb{Z}^{d \times n}$ with $\max_{i,j} |A_{i,j}| \leq \delta$. Let $Z = \mathbf{A}[0,1]^d$ be the zonotope of $\mathbf{A}$ and let $T = [-n\delta, n\delta]^d \cap \mathbb{Z}^d$ be the set of all integer lattice points of Z. The following algorithm computes lindisc($\mathbf{A}$) in

Algorithm 2: Finding a close subset sum to $\mathbf{Aw}$.

Input: A vector $\mathbf{w} \in [0,1]^n$ and a row matrix $\mathbf{A} = [a_1, ..., a_n]$ of positive integers sorted in increasing order.

Output: A vector $\mathbf{x} \in \{0,1\}^n$ such that $\|\mathbf{A}(\mathbf{w}-\mathbf{x})\|_\infty \leq \text{lindisc}(\mathbf{A})$.

$\mathbf{A} \leftarrow$ SORT-DECREASING($\mathbf{A}$)
$\mathbf{u} \leftarrow$ ZEROS(n)
$\mathbf{v} \leftarrow$ ONES(n)
$w \leftarrow \mathbf{Aw}$, $u \leftarrow \mathbf{Au}$, $v \leftarrow \mathbf{Av}$
return v if $w = v$
for $k = 1..n$ **do**
  if $u + a_k > w$ **then**
    $v \leftarrow \min(v, u + a_k)$
    if $v = u + a_k$ **then**
      $\mathbf{v} \leftarrow$ COPY($\mathbf{u}$)
      $\mathbf{v}[k] \leftarrow 1$
  else
    $u \leftarrow u + a_k$
    $\mathbf{u}[k] \leftarrow 1$
return u *if* u *is closer to* w *else* **v**

polynomial time with respect to n for fixed d and δ. The algorithm makes use of Lemma 52 which is stated below and proved in Section 3.4.

Lemma 52 (LEC in Higher Dimensions)**.** *Let V be a set of n points in $\mathbb{R}^d$ for some fixed constant d. The LEB of V, in both ℓ_2- and ℓ_∞-norms, can be computed in time $O(n^d)$.*

Theorem 11 (Linear Discrepancy for Matrices with Constantly Many Rows)**.** *For any matrix $\mathbf{A} \in \mathbb{Z}^{d\times n}$ where d is some fixed constant and $\max_{i,j} |A_{i,j}| \leq \delta$, $\text{lindisc}(\mathbf{A})$ can be computed in time $O\left(d(n\delta)^{d^2+d}\right)$.*

Proof. For every one of the $(2n\delta+1)^d$ integral points $\mathbf{b} \in T$, compute whether $\mathbf{Ax} = \mathbf{b}$ for some $\mathbf{x} \in \{0,1\}^n$ using dynamic programming. This procedure generalizes dynamic programming algorithms for knapsack and subset sum and will be outlined in the following. Let $\mathbf{a}_1, ..., \mathbf{a}_n$ be the columns of $\mathbf{A}$. Construct a matrix $\mathbf{M}$ with dimensions $[-n\delta, n\delta]^d \times n$. Cell $(\mathbf{v}, i)$ of $\mathbf{M}$ contains the indicator $[\mathbf{M}(\mathbf{v}-\mathbf{a}_i, i-1) \vee \mathbf{M}(\mathbf{v}, i-1)]$; this corresponds to a linear combination of the first $i-1$ columns of $\mathbf{A}$ which adds up to $\mathbf{v}-\mathbf{a}_i$ or a linear combination of the first $i-1$ columns which adds up to $\mathbf{v}$. The first column of $\mathbf{M}$ is the indicator vector for $\{\mathbf{a}_1\}$. Computing the entries of $\mathbf{M}$ takes time $O(2n\delta)^{d+1}$. $\mathbf{M}(\mathbf{b}, n)$ indicates the feasibility of $\mathbf{Ax} = \mathbf{b}$. Computing this for all $\mathbf{b}$ takes time $O(2n\delta)^{d+1}$. Let $S \subseteq T$ be the set of points $\mathbf{b}$ in Z such that $\mathbf{Ax} = \mathbf{b}$ for some $\mathbf{x} \in \{0,1\}^n$, and set $|S| = N$.

Apply Lemma 52 to the points of S in ℓ_∞-norm. The output is some radius r and point $\mathbf{x}^*$ such that the ℓ_∞-ball centered at $\mathbf{x}^*$ with radius r is the largest such ball with center inside the convex hull of S not containing any points of S. Note that r is in-fact the linear discrepancy of $\mathbf{A}$. Since r and $\mathbf{x}^*$ can be computed in time $O(N^d)$, $\text{lindisc}(\mathbf{A})$ can be computed in time $O(2n\delta)^{d^2+d}$. $\square$

3.3 Poly-time Approximation Algorithm

Next, we prove Theorem 12 restated below.

Theorem 12 (Approximate Linear Discrepancy for Matrices)**.** *For any matrix $\mathbf{A} \in \mathbb{Q}^{m\times n}$, the linear discrepancy of $\mathbf{A}$ can be approximated in polynomial time within a factor of 2^{n+1}.*

Recall that $\mathrm{rd}(\mathbf{w})$ is the function which rounds each coordinate of $\mathbf{w}$ to its nearest integer (with ties broken arbitrarily). Let the operator norms of a matrix $\mathbf{A}$ be:

$$\|\mathbf{A}\|_{p\to q} = \max_{\mathbf{x}\in\mathbb{R}^n\setminus\{0\}} \frac{\|\mathbf{A}\mathbf{x}\|_q}{\|\mathbf{x}\|_p}.$$

Note that

$$\mathrm{lindisc}(\mathbf{A}) \le \max_{\mathbf{w}\in[0,1]^n} \|\mathbf{A}(\mathbf{w}-\mathrm{rd}(\mathbf{w}))\|_\infty \le \frac{1}{2}\max_{\mathbf{z}\in[-1,1]^n}\|\mathbf{A}\mathbf{z}\|_\infty = \frac{1}{2}\|\mathbf{A}\|_{\infty\to\infty}.$$

To bound $\mathrm{lindisc}(\mathbf{A})$ from below, we show that $\|\mathbf{A}\|_{\infty\to\infty} \le 2^{n+1}\cdot\mathrm{lindisc}(\mathbf{A})$. This completes the proof of the theorem, since $\|\mathbf{A}\|_{\infty\to\infty}$ equals the largest ℓ_1 norm of any row of $\mathbf{A}$, and can be computed in polynomial time.

Let us try to interpret the statement $\|\mathbf{A}\|_{\infty\to\infty} \le 2^{n+1}\cdot\mathrm{lindisc}(\mathbf{A})$. Note that $\|\mathbf{A}\mathbf{z}\|_\infty$ is equal to the Minkowski $\mathcal{P}$-norm $\|\mathbf{z}\|_{\mathcal{P}}$ for $\mathcal{P} = \{\mathbf{x} : \|\mathbf{A}\mathbf{x}\|_\infty \le 1\}$ i.e. $\|\mathbf{z}\|_{\mathcal{P}} = \inf\{t \ge 0 : \mathbf{z} \in t\mathcal{P}\}$ so

$$\|\mathbf{A}\|_{\infty\to\infty} = \max_{\mathbf{z}\in[-1,1]^n}\|\mathbf{A}\mathbf{z}\|_\infty = \max_{\mathbf{z}\in[-1,1]^n}\|\mathbf{z}\|_{\mathcal{P}}.$$

By interpreting $\mathbf{z}$ as the difference of two vectors $\mathbf{x},\mathbf{x}' \in [0,1]^n$ we have that

$$\|\mathbf{A}\|_{\infty\to\infty} = \max_{\mathbf{z}\in[-1,1]^n}\|\mathbf{z}\|_{\mathcal{P}} = \max_{\mathbf{x},\mathbf{x}'\in[0,1]^n}\|\mathbf{x}-\mathbf{x}'\|_{\mathcal{P}}.$$

It is an easy, and well-known fact that $\mathrm{lindisc}(\mathbf{A})$ is the smallest t such that $[0,1]^n \subseteq \bigcup_{\mathbf{x}\in\{0,1\}^n}(\mathbf{x}+\mathcal{P})$; see [Mat09]. We then just need to show that the diameter of the unit hyper-cube with respect to the Minkowski $\mathcal{P}$-norm is no more than this scale-factor t times $O(2^n)$. We prove the following more general statement.

Lemma 53. *Let $\mathcal{K}$ be a convex symmetric polytope and $S \subset \mathbb{R}^n$ be convex. Suppose there exist N elements $x_1, ..., x_N \in S$ such that*

$$S \subseteq \bigcup_{x_i} x_i + t\mathcal{K}.$$

Then $\max_{x,x'\in S}\|x-x'\|_{\mathcal{K}} \le 2tN$.

Proof. Fix any two points x and x' in S. Let $\mathcal{P}_i$ be the polytope $x_i + t\mathcal{K}$. Since S is convex, the line segment $\lambda x + (1-\lambda)x'$ for $\lambda \in [0,1]$ is in S. Therefore $\lambda x + (1-\lambda)x'$ intersects a sequence of polytopes $\mathcal{P}_{k_1}, ..., \mathcal{P}_{k_r}$ with centres $x_{k_1}, ..., x_{k_r}$, such that any two consequtive polytopes in the sequence intersect. Since the polytopes are convex, we can assume that they appear in the sequence at most once, so $r \le N$. By the triangle inequality we have

$$\begin{aligned}\|x - x'\|_K &= \|(x - x_{k_1}) + (x_{k_1} - x_{k_2}) + \cdots + (x_{k_r} - x')\|_K \\ &\le \|x - x_{k_1}\|_K + \|x_{k_1} - x_{k_2}\|_K + \cdots + \|x_{k_r} - x'\|_K \\ &\le t + 2t(N-1) + t = 2tN\end{aligned}$$

where the last inequality follows as $x \in \mathcal{P}_{k_1}$, $x' \in \mathcal{P}_{k_r}$, and $\|x_{k_i} - x_{k_{i+1}}\|_K \le 2t$. □

Proof of Theorem 12. In Lemma 53, set $\mathcal{K}$ to be the parallelepiped defined by $\mathbf{A}$, $S = [0,1]^n$, $t = \text{lindisc}(\mathbf{A})$, and $\{x_1, ..., x_N\} = \{0,1\}^n$. □

3.4 Largest Empty Ball Problem

Let V be a set of n points in the plane and let $\text{ch}(V)$ denote the convex hull of V. The largest empty circle problem (LEC), takes V and outputs both a radius r and point $\mathbf{x}^* \in \text{ch}(V)$ such that the circle centered at $\mathbf{x}^*$ with radius r is the largest empty circle not containing any point of V. We generalize this problem to other norms and to higher dimensions as follows: V is a set of n points in $\mathbb{R}^d$, and the goal is to compute a point $\mathbf{x}^*$ in $\text{ch}(V)$ such that $\mathbf{x}^* + rB$ does not contain any point of V, where B is the unit ball of either the ℓ_2^d or the ℓ_∞^d norm. In the following we present an algorithm which solves this largest empty ball (LEB) problem.

Lemma 52 (LEC in Higher Dimensions)**.** *Let V be a set of n points in $\mathbb{R}^d$ for some fixed constant d. The LEB of V, in both ℓ_2- and ℓ_∞-norms, can be computed in time $O(n^d)$.*

Proof. We use the following terminology. Define a face F of the Voronoi diagram $\text{vd}(V)$ of V to be a subset of $\mathbb{R}^d$ such that, for some $S \subseteq V$, and every $\mathbf{x} \in F$, S are the points in V closest to $\mathbf{x}$. In particular, this means that any $\mathbf{x} \in F$ is equidistant from all points in S.

The algorithm of Toussaint [Tou83] computes the LEB of n points V in the plane with respect to the ℓ_2-norm as follows,

1. Compute $\text{vd}(V)$. Note that $\text{vd}(V)$ is the union of Voronoi faces of dimension k, the set of which we denote $\text{vd}_k(V)$, over all $k = 0, ..., d-1$.

2. Compute the convex hull of V, denoted $\text{ch}(V)$. Let h be the number of facets of $\text{ch}(V)$.

3. Preprocess the points of $\text{ch}(V)$ so that queries of the form "Is a point x in $\text{ch}(V)$?" can be answered in time $O(\log h)$. For every $v \in \text{vd}_0(V)$, determine if $v \in \text{ch}(V)$. Let $C_1 = \{v \in \text{vd}_0(V) : v \in \text{ch}(V)\}$.

4. Determine the intersection points of faces in $\text{vd}_k(V)$ with faces of $\text{ch}(V)$ of co-dimension k, for pairs of such faces that intersect at a unique point. Let C_2 be the set of all such intersection points.

5. For all points $v \in C_1 \cup C_2$, find the largest empty circle centered at v. Output a v which maximizes this radius.

We find the analogue of each step for points in $\mathbb{R}^d$ with respect to the ℓ_2-norm, and then adapt the algorithm to the ℓ_∞-norm.

In the following let $N = n^{\lceil d/2 \rceil}$. The complexity, i.e. total number of faces of every dimension, of the ℓ_2-Voronoi diagram in $\mathbb{R}^d$ for fixed d is $O(N)$ and can be computed in time $O(N + n \log n)$ by a classic result of Chazelle [Cha93]. The complexity of $\mathrm{ch}(V)$ is $O(N)$ and can also be computed in time $O(N + n \log n)$.

To determine the set C_1 of Voronoi intersection points inside the convex hull, we let $\mathcal{H}$ be the set of bounding hyperplanes of $\mathrm{ch}(V)$. Assume, without loss of generality, that $\mathrm{ch}(V)$ contains the origin, and, for each $H \in \mathcal{H}$, let H^- be the half-space with H as its boundary containing the origin. Then $\mathrm{ch}(V) = \bigcap_{H \in \mathcal{H}} H^-$. We simply test, for each Voronoi intersection point $\mathbf{v}$, whether $\mathbf{v} \in H^-$ for each $H \in \mathcal{H}$, in total time $O(N)$. Since there are at most $O(N)$ Voronoi intersection points, we can find C_1 in time $O(N^2)$.

To determine the set C_2 of all unique intersection points of k-faces of $\mathrm{vd}_k(V)$ and faces of $\mathrm{ch}(V)$ of co-dimension k will require solving several linear systems. Note that the points in each face F in $\mathrm{vd}_k(V)$ satisfy $d-k$ equality constraints $\langle \mathbf{a}_1, \mathbf{x} \rangle = b_1, \langle \mathbf{a}_k, \mathbf{x} \rangle = b_k$ for linearly independent vectors $\mathbf{a}_1, ... \mathbf{a}_k \in \mathbb{R}^d$. Similarly, the points in each face of co-dimension k of $\mathrm{ch}(V)$ satisfy k linearly independent equality constraints. Since there are at most $O(2^d N) = O(N)$ faces of $\mathrm{ch}(V)$, there are at most that many faces of $\mathrm{ch}(V)$ of co-dimension k. We can then go over all Voronoi faces F of dimension k, and all faces G of $\mathrm{ch}(V)$ of co-dimension k, and solve the corresponding system of $(d-k)+k = d$ linear equations. If the system has a unique solution, we check if that solution is in $F \cap G$, and, if so, we add it to C_2. Thus, for constant d, the size of C_2 and the time to compute it are bounded bounded above by $O(N \cdot 2^d N) = O(N^2)$.

In total there are at most $O(N + N^2)$ points in $C_1 \cup C_2$ which can be computed in time $O(N^2)$. Thus solving the largest empty ball problem in dimension d for constant d takes time $O(n^d)$.

Next we consider the largest empty ball problem in ℓ_∞-norm. The convex hull remains the same, so we just have to consider the Voronoi diagram with respect to the ℓ_∞-norm. Again, constructing the Voronoi diagram can be done in expected time $O(n^{\lceil d/2 \rceil} \log^{d-1} n)$ using the randomized algorithm of Boissonnat et al. [BSTY98]. Next we consider the number of intersections between the Voronoi diagram and the convex hull. First note that Voronoi diagrams with respect to the ℓ_∞-norm need not consist of only hyperplanes and their intersections. Indeed, in $\mathbb{R}^d$, for two points with the same y-coordinate, there exists regions with affine dimension two which are equidistant to both points. To remedy this, we assume that no two points in V have the same i-th coordinate, for any $i \in [d]$. This is without loss of generality, by perturbing the points in V slightly. It remains to consider the complexity of each bisector in ℓ_∞-norm. By Claim 54, in constant dimension d, each such bisector can have at most $O(d^2)$ facets. Therefore, the complexity of any face of the Voronoi diagram, being the intersection of at most d bisectors, is bounded by a function of d. Thus the bounds of the ℓ_2-norm algorithm still hold, up to constant factors that depend on d. □

Claim 54. (Bound on Number of Facets of ℓ_∞ Bisectors.) *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ be such that assume that $u_i \neq v_i$ for all $i \in [d]$. Then the bisector $\{\mathbf{x} : \|\mathbf{x} - \mathbf{u}\|_\infty = \|\mathbf{x} - \mathbf{v}\|_\infty\}$ has at most $O(d^2)$ facets.*

Proof. Let $\mathbf{x}$ be a point in the bisector at ℓ_∞ distance r from $\mathbf{u}$ and $\mathbf{v}$. Pick coordinates i and j and signs σ and τ in $\{-1, +1\}$ such that

$$\sigma_i(x_i - u_i) = \tau_j(x_j - v_j) = r. \tag{3.4}$$

Moreover, let us make this choice so that either $i \neq j$ or $\sigma_i \neq \tau_i$. This is always possible, since, otherwise, the assumption on $\mathbf{u}$ and $\mathbf{v}$ is violated. Then, (3.4) defines a hyperplane in $\mathbb{R}^d$, namely $H_{i,j,\sigma,\tau} = \{\mathbf{x} : \sigma_i x_i - \tau_j x_j = \sigma_i u_i - \tau_j v_j\}$. Note that there are at most $\binom{2d}{2} \in O(d^2)$ such hyperplanes, and each $\mathbf{x}$ in the bisector lies in at least one of them. Moreover, a point $\mathbf{x}$ in $H_{i,j,\sigma,\tau}$ lies in the bisector if and only if it satisfies the inequalities

$$\begin{aligned}
|x_k - u_k| &\leq \sigma_i(x_i - u_i) \quad \forall k \in [d], \\
|x_k - v_k| &\leq \tau_j(x_j - v_j) \quad \forall k \in [d].
\end{aligned}$$

Thus, the bisector is the union of $(d-1)$-dimensional convex polyhedra, one per each of the $O(d^2)$ hyperplanes $H_{i,j,\sigma,\tau}$. □

3.5 Open Problems

Because of the similarity between the closest vector problem and linear discrepancy, we suspect that linear discrepancy is also Π_2-complete, and the hardness result of Theorem 8 is, in this sense, not tight. Recall from the introduction of this chapter, we noted that Haviv and Regev showed CRP is Π_2-hard to approximate to with-in a factor of $\frac{3}{2}$. We conjecture that a similar hardness of approximation result should hold for linear discrepancy. Since the publication of our paper, Mansurangsi [Man21] showed that linear discrepancy is *indeed* hard to approximate to within a factor of 9/8.

We suspect that the algorithm used to prove Theorem 9 can be generalized to matrices $\mathbf{A} \in \mathbb{Q}^{d \times n}$ with running time $\tilde{O}(n^d)$. This would be a substantial improvement on the $O\left(d(n\delta)^{d^2}\right)$ running time algorithm used to prove Theorem 11, and would be independent of the magnitude of the largest entry of $\mathbf{A}$.

It is also interesting to extend the largest empty ball algorithm from Lemma 52 to other ℓ_p norms, or even arbitrary norms, given appropriate access to the norm ball. Currently, this seems rather difficult as Voronoi diagrams with respect to the ℓ_p-norm for $p \in (2, \infty)$ are poorly behaved. For the standard ℓ_2-norm Voronoi diagram in $\mathbb{R}^d$, it is the case that $d+1$ affinely independent vertices are equidistant to exactly one point. This is no longer the case even in $\mathbb{R}^3$ for ℓ_4-norm [IKLM95]. In particular, there exists a set of four vertices such that the intersection of their pair-wise bisectors has size three. The situation is even worse for general strictly convex norms. There exists such norms where the pair-wise bisectors of a set of four points in $\mathbb{R}^3$ can have arbitrarily many intersections.

We currently also have no evidence that the approximation factor in Theorem 12 is tight. One possibility is that there exists an approximation preserving reduction from the closest vector problem in lattices to linear discrepancy. This would show that one cannot expect a significant improvement to Theorem 12 without also improving the best polynomial time approximation to the covering radius, which is currently also exponential in the dimension n. On the other hand, we also conjecture that the approximation factor in Theorem 12 can be taken to be a function of $\min\{m, n\}$, or even of the rank of the matrix $\mathbf{A}$.

Chapter 4

Balanced Graph Partitions

This chapter is based on a part of a joint work with Evi Micha, Aleksandar Nikolov, and Nisarg Shah [LMNS23].

Recall that hedonic games are partitions of agents into coalitions where the preference of each agent only depends on the other agents in the coalition to which they belong. Such games are symmetric and boolean if agents have boolean utilities for one another and their utilities are symmetric. Along with restrictions on the number of coalitions, these games have been studied before under other objectives, such as swap stability [BMM22] and Pareto optimality [CFH19]. Envy-freeness has been studied recently in the hedonic games literature [Pet16; BY19], again with possibly negative utilities. Another concept similar to envy-freeness is *Nash-stability* [BJ02; OBT12], which requires that no agent be happier by *joining* another part (rather than by swapping places with an agent in another part).[1] This problem is also studied in the case where the parts are required to be of almost the same size [BTV10]. However, since such partitions do not always exist, the literature primarily focuses on the computational complexity of distinguishing between graphs which have partitions which are envy-free — we recall the definition below — and those which are not.

Instead, our focus is on providing worst-case guarantees on the necessary violation of envy-freeness, as is commonly done in the literature on fair resource allocation [LMMS04; CKMPSW19; ACIW19]. We make a connection to discrepancy theory [CST14] to establish a constructive $O(\sqrt{n})$ bound though the discrepancy of the graph may be overkill. In addition to achieving the goal of distributing each agent's friends as evenly as possible between the parts (i.e. she will not have many more friends in another part than her own part), the discrepancy bound also ensures that she does not have many more friends in her own part than in any other part. The latter restriction, a flipped version of the satisfactory partition problem, has also been studied separately as the co-satisfactory or unfriendly partition problem [AMP90]. Manurangsi and Suksompong [MS22] use discrepancy theory in a similar problem with n agents partitioned into k groups, but with agents having utilities over goods being allocated to the groups, not over the other agents.

We recount the standard graph setting here: Consider a set $V = [n]$ of agents who are members of a social network. The network is represented by an undirected graph $G(V, E)$, where the agents

[1] The two differ only when the other part consists entirely of the agent's friends.

are the nodes and an edge $(i, i') \in E$ indicates friendship between agents i and i'. This induces the utility function of agent i, denoted $u_i : V \to \{0, 1\}$, where $u_i(i') = 1$ if $(i, i') \in E$ and 0 otherwise. Let $N_G(i)$ denote the set of neighbors of agent i in G, i.e., $N_G(i) = \{i' \in V : (i, i') \in E\}$. We refer to $d_G(i) = |N_G(i)|$ as the degree of agent i. We omit G when it is clear from the context.

A *k-partition* of V is given by $X = (X_0, \ldots, X_{k-1})$, where $X_j \cap X_{j'} = \emptyset$ for all distinct $j, j' \in [k]$; $X_j \neq \emptyset$ for all $j \in [k]$; and $\cup_{j \in [k]} X_j = V$. We may refer to an individual group X_j as a *part*. With slight abuse of notation, we denote by $X(i)$ the part X_j to which agent i belongs (i.e., $i \in X_j$). We assume that $n \geq k$, so a k-partition exists. The utility of agent i for $S \subseteq V$ is denoted by, with slight abuse of notation, $u_i(S)$. We assume that utilities are additive, i.e., $u_i(S) = \sum_{i' \in S} u_i(i') = |S \cap N(i)|$.

For $r \geq 0$, a k-partition X is EF-r if for every pair of agents $i, i' \in V$, $u_i(X(i)) \geq u_i(X(i') \cup \{i\} \setminus \{i'\}) - r$. Further, we take the $u_i(S)$ for a subset $S \subset V$ to be $u_i(S) = d_S(i)$. The *satisfactory partition problem* asks if there exists a balanced k-partition which is EF-0.

4.1 Envy-Freeness

Before finding k-partitions which are approximately envy-free, we show that EF-1 cannot always be guaranteed even for $k = 2$.

Theorem 55. *Even when $k = 2$, a 2-partition that is EF-1 does not always exist.*

Proof. Consider the $K_{3,3,3}$ graph which consists of three set of three nodes each, denoted by $C_1 = \{c_{11}, c_{12}, c_{13}\}$, $C_2 = \{c_{21}, c_{22}, c_{23}\}$ and $C_3 = \{c_{31}, c_{32}, c_{33}\}$, respectively, and every node of each set is adjacent to every node in the other two sets.

For the sake of contradiction, assume that $X = (X_0, X_1)$ is a partition of the graph that is EF-1. Since the graph is 6-regular, we can see that $|X_0| \geq 4$ and $|X_1| \geq 4$, as if an agent i is assigned to a part with only at most two of its neighbours, then the other four of its neighbours are assigned to the other part along with an agent i' which is not neighbour of i, and then i envies i' for more than one agent. Without loss of generality, we assume that $|X_0| = 4$. If X_0 contains three nodes of the same set, then we can easily see that this partition is not EF-1, as each of them is assigned to the same group with at most one of its neighbours. As there are three sets and X_0 contains four agents, we see that two agents of the same set, say c_{11} and c_{12}, are assigned to X_0. Then these two agents are in the same part along with at most two of its neighbours, while all the remaining nodes are assigned to X_1. Then, c_{11} and c_{12} envy c_{13} for more than one agents, which is a contradiction. $\square$

To obtain non-trivial bounds on envy-freeness for higher values of k, we turn to the discrepancy theory literature. Intuitively, we want to color the elements of a set using k colors such that each pre-determined subset has an approximately equal number of elements of each color. Formally, we are given a universe $\Omega = [n]$ and a set system $\mathcal{S} = \{S_0, \cdots, S_{m-1}\}$, where $S_i \subseteq [n]$ for each $i \in [m]$. The k-color discrepancy of a coloring $\chi : \Omega \to [k]$ on the set system $\mathcal{S}$ is defined as

$$\mathrm{disc}_k(\mathcal{S}, \chi) = \max_{j \in [k], i \in [m]} \left| \left| \chi^{-1}(j) \cap S_i \right| - |S_i|/k \right| .$$

The k-discrepancy of $\mathcal{S}$ is then the minimum k-color discrepancy over all χ:

$$\mathrm{disc}_k(\mathcal{S}) = \min_{\chi:\Omega\to[k]} \mathrm{disc}_k(\mathcal{S},\chi).$$

Here we have that $\mathrm{disc}_k(\mathcal{S}) = O(\sqrt{\frac{n}{k}\ln(km/n)})$ for any set system $\mathcal{S}$ and a k-coloring achieving this bound can be computed in polynomial time [HS14, Corollary 44].

In this setting, where $\Omega = V = [n]$, a k-coloring $\chi : \Omega \to [k]$ induces a k-partition X given by $X_j = \chi^{-1}(j)$ for all $j \in [k]$.[2] Further, if we consider the set system $\mathcal{S}$ where $S_i = N_G(i)$ for each $i \in [n]$ (i.e., with $m = n$), then we are guaranteed that agent i can have at most $2\mathrm{disc}_k(\mathcal{S},\chi)$ more neighbors in any other part than in her own part, implying EF-$(2\mathrm{disc}_k(\mathcal{S},\chi))$. The above discrepancy bound then immediately yields the existence of a k-partition that is EF-$O(\sqrt{\frac{n}{k}\ln k})$. However, this may not be balanced.

To fix this, we add another set $S_n = V$ to our set system; we now have $m = n+1$, which does not asymptotically change the discrepancy bound. Now, we obtain a k-partition X that is also approximately balanced: $||X_j| - |X_{j'}|| = O(\sqrt{\frac{n}{k}\ln k})$ for all $j, j' \in [k]$. By arbitrarily moving $O(\sqrt{\frac{n}{k}\ln k})$ agents between parts, we can make it perfectly balanced, while only increasing the EF approximation by $O(\sqrt{\frac{n}{k}\ln k})$. Thus, we have Theorem 17 which states,

Theorem 17 (Constructive Envy-Free Partitions). *For any $k \geq 2$, a k-partition that is EF-$O(\sqrt{\frac{n}{k}\ln k})$ is guaranteed to exist and can be computed in polynomial time.*

For discrepancy, the aforementioned upper bound is known to be almost tight: there is a lower bound of $\Omega(\sqrt{n/k})$ [HS14, Theorem 61]. However, for our "one-sided" envy-freeness guarantee, achieving a constant approximation remains an open question.

Conjecture 56. *Every graph admits an EF-2 bisection for all $k \geq 2$.*

4.2 Beyond Balancedness

An interesting variation of our problem is to drop the requirement of balancedness and simply seek k non-empty groups, i.e., imbalanced k-partitions. This variation was first introduced by [GK00] and, since then, it has been given much attention [BTV10] due to its importance to real-life applications such as clustering [FTT04; Sha04].

In this section, we briefly consider this case and study envy-freeness for unbalanced k-partitions. In particular, we provide a complete picture for $k = 2$ by making a connection to the literature on satisfactory partitions, and point out interesting open questions for $k \geq 3$.

First, we use the following result from the literature on satisfactory partitions, restated in our framework, to establish the existence of an EF-2 imbalanced partition when $k = 2$.

Theorem 57. [Sti96; BTV07] *Given a graph $G = (V,E)$ and functions $a, b : V \to \mathbb{N}$ such that $d(i) \geq a(i) + b(i) + 1$ for every $i \in V$, there exists an imbalanced 2-partition $X = (X_0, X_1)$ of V such that $u_i(X_0) \geq a(i)$ for each $i \in X_0$ and $u_i(X_1) \geq b(i)$ for all $i \in X_1$, and it can be computed in polynomial time.*

[2] Technically, we also need to ensure $X_j \neq \emptyset$, but this is guaranteed due to the discrepancy bound.

In our case, we use functions $a(i) = b(i) = \lfloor (d(i) - 1)/2 \rfloor$ for all $i \in V$. Note that these satisfy the condition $d(i) \geq a(i) + b(i) + 1$. Hence, the above result allows us to efficiently compute a 2-partition X satisfying $u_i(X(i)) \geq \lfloor (d(i) - 1)/2 \rfloor$ for all $i \in V$. Since there are only two parts, this also implies that for all $i, i' \in V$,

$$\begin{aligned} u_i(X(i')) - u_i(X(i)) &\leq d(i) - 2 \cdot \lfloor (d(i) - 1)/2 \rfloor \\ &\leq d(i) - 2 \cdot (d(i) - 2)/2 = 2, \end{aligned}$$

which implies that X is EF-2.

Corollary 58. *An imbalanced* 2*-partition that is EF-*2 *always exists and can be computed in polynomial time.*

Theorem 57 admits an extension to $k > 2$ parts, but in our case, this only guarantees that $u_i(X(i)) \geq \lfloor (d(i) - k + 1)/k \rfloor$ for all $i \in V$ [BTV07]. This does not meaningfully limit the number of neighbors that agent i has in another part and, therefore, fails to provide a non-trivial approximation to envy-freeness. That said, if one is interested in the slightly weaker guarantee of proportionality [Ste48], which, in our setting, would require $u_i(X(i)) \geq d(i)/k$, then this would provide an additive 1-approximation.

For the satisfactory partition problem, where the goal is to indeed minimize $u_i(X(i')) - u_i(X(i))$, as in the equation above, it is easy to see that an additive error of 2 is the best possible. Consider dividing any clique with an odd number of nodes into two parts. An agent i in the smaller part will have at least two more neighbors in the larger part than in her own part. However, this does not hold for envy-freeness: if i envisions *swapping places* with an agent i' from the other part, then $X(i') \cup \{i\} \setminus \{i'\}$ will only contain one more neighbor of i than $X(i)$ does. Nonetheless, notice that the example that is used in the proof of Theorem 55 can also be used to show that EF-1 cannot always be guaranteed even in the imbalanced case when $k = 2$.

However, if we restrict our attention to trees, we can achieve EF-1 even with balanced k-partitions for all $k \geq 2$ (see Theorem 18 restated below).

Algorithm 3: EF-1 Trees

$\forall j \in [k], X_j \leftarrow \emptyset$;
Phase 1: for $i \in N$ **do**
  $X_{i \bmod k} = X_{i \bmod k} \cup i$;
Phase 2: for $\ell = 2$ *to* d **do**
  for $i \in N$ *with level*$(i) = \ell$ *that is envious for more than one agents* **do**
    $i' \leftarrow$ an arbitrary child of i such that $X(i') = X(p(i))$;
    $X(i') = X(i') \cup \{i\} \setminus \{i'\}$;
    $X(i) = X(i) \cup \{i'\} \setminus \{i\}$;
return $X = (X_0, ..., X_{k-1})$

Theorem 18 (Compute EF-1 Balanced k-partition in Trees)**.** *For all $k \geq 2$ and every tree, we can find a balanced EF-*1 *k-partition in polynomial time.*

Proof. We show that Algorithm 3 returns a balanced EF-1 k-partition for every tree, in polynomial time. The algorithm works as follows. Let d denote the depth of the tree. Without loss of generality, suppose the tree is labelled as following. Agent 0 is at level 1, agent 1 is the left most node of level 2, agent 2 is the second leftmost node of level 1, and so on, while agent $n-1$ is the rightmost node of level d. Algorithm 3 first colors the nodes of the tree in a simple round-robin fashion to obtain EF-2 (in fact, it achieves a discrepancy bound of 2, whereby there are at most 2 more nodes of any color than of any other color), and then makes small edits to improve its guarantee to EF-1.

Suppose that at Line 5 of the algorithm, when $\ell = level(i)$, i is not envious for more than one agents. Then, when $\ell = level(i) + 1$, a child of i may be moved to the same part with i, but no child of i that is assigned to the same part with i is removed from it, while afterwards no neighbour of i is never moved to a different part. Hence, clearly, the partition remains EF-1 with respect to i.

Now, suppose i is envious for more than one agents. This means that before Line 4, $|X(i) \cap c(i)| = \lfloor |c(i)|/k \rfloor < |c(i,T)|/k$, and for some $i' \notin N(i)$, $|X(i') \cap c(i,T)| = \lceil |c(i,T)|/k \rceil$ and $X(i') = X(p(i))$. Then, i and one of her children that is assigned to $X(i')$ are swapped. Hence, i is currently assigned to the same group with at least $\lfloor |c(i,T)|/k \rfloor + 1$ of her neighbours while any other part still contains at most $\lceil |c(i,T)|/k \rceil$ neighbours of i. Thus, at Line 5 of the algorithm, when $\ell = level(i) + 1$, i is not envious for more than one agents, and by the same reasoning as above, we have that partition remains EF-1 with respect to i until the end of the algorithm. □

Generally it is hard to distinguish between EF-0 and EF-1 graphs [BTV06], however, we show that this is possible for trees.

Theorem 19 (Distinguishing EF-0 and EF-1 in Trees)**.** *For $k = 2$ and every tree, we can distinguish whether it has an EF-1 partition or not in polynomial time.*

Proof. We describe a dynamic program (DP) which distinguishes between EF-0 and EF-1 in a balanced k-partition for any constant k in a tree T with nodes V of size n. First consider the case for $k = 2$ (partitioning the nodes into two parts). See Algorithm 4. At the end, we point out the modifications necessary in order to obtain a DP for general k.

For each non-leaf node v_i in the tree, define a table M_i. Index M_i by $(c_0, c_1, c_2, d_0, d_1, d_2)$. Here c_0, c_1, and c_2 indicate the number of *children* of v_i in the zeroth, first, and unassigned parts while d_0, d_1, and d_2 indicate the number of *descendants* of v_i in the zeroth, first, and unassigned parts excluding the children respectively. The entry $M_i(c_0, c_1, c_2, d_0, d_1, d_2)$ stores true or false depending on whether there exists an EF-0 partition of the descendants of v_i matching the corresponding indices i.e. there are c_0 children in part zero, c_1 children in part one, etc.

From M_i, we construct a set $\mathcal{S}_i$ which contains the summary of all possible partitions of v_i. Each entry of $\mathcal{S}_i$ takes the form (s, p_0, p_1, p_2). It indicates the partition of the subtree rooted at v_i and the number of descendants(including children) of v_i in the zeroth, first, and unassigned parts respectively. In particular, define $\mathcal{S}_i$ as follows. Let $c(i,T)$ be all children of v_i. For all true entries $M_i(c_0, c_1, c_2, d_0, d_1, d_2)$ where $c_0 + c_1 + c_2 = |c(i,T)|$,

- if $c_0 + c_2 > c_1 + 1$, add $(+1, c_0 + c_2 + d_0 + d_2, c_1 + d_1, 0)$ to $\mathcal{S}_i$. In this case at least one more than half of the children of i can be in the first part, so i can be in the first part as well. All previously unassigned children, and thus descendants, of i will be put into the first part.

Algorithm 4: Distinguishing EF-0 and EF-1 in a Balanced 2-partition for Trees

$v_1 \leftarrow$ root of T;
SUMMARIZE-NODE(v_1);
for $(s, p_0, p_1, p_2) \in v_1.\mathcal{S}$ **do**
 if $s = +1$ *and* $|p_0 + 1 - p_1| \leq 1$ **then**
  return *True*;
 else if $s = -1$ *and* $|p_1 + 1 - p_0| \leq 1$ **then**
  return True;
 else if $s = 0$ *and* $(|p_0 + 1 + p_2 - p_1| \leq 1$ *or* $|p_1 + 1 + p_2 - p_0| \leq 1)$ **then**
  return *True*;
 return *False*;
`Function(SummarizeNode(`v*: node*`))` **if** v *has only leaf nodes as children* **then**
 return $v.\mathcal{S} = \{(+1, |C_v(T)|, 0, 0), (-1, 0, |C_v(T)|, 0)\}$;
$n_c \leftarrow$ number of children of v;
$n_d \leftarrow$ number nodes in the subtree rooted at v;
$v.M \leftarrow$ ZEROES$(n_c, n_c, n_c, n_d, n_d)$;
$v.M(0, 0, 0, 0, 0, 0) \leftarrow$ True;
for $u \in C_v(T)$ **do**
 SUMMARIZE-NODE(u);
 for $v.M(c_0, c_1, c_2, d_0, d_1, d_2) =$ *True and* $c_0 + c_1 + c_2 + d_0 + d_1 + d_2$ *maximal and* $(s, p_0, p_1, p_2) \in u.\mathcal{S}$ **do**
  if $s = +1$ **then**
   $v.M(c_0 + 1, c_1, c_2, d_0 + p_0, d_1 + p_1, d_2) \leftarrow$ True;
  else if $s = -1$ **then**
   $v.M(c_0, c_1 + 1, c_2, d_0 + p_0, d_1 + p_1, d_2) \leftarrow$ True;
  else if $s = 0$ **then**
   $v.M(c_0, c_1, c_2 + 1, d_0 + p_0, d_1 + p_1, d_2 + p_2) \leftarrow$ True;
$v.\mathcal{S} = \emptyset$;
for $v.M(c_0, c_1, c_2, d_0, d_1, d_2) =$ *True and* $c_0 + c_1 + c_2 = |C_v(T)|$ **do**
 if $c_0 + c_2 > c_1 + 1$ **then**
  add $(+1, c_0 + c_2 + d_0 + d_2, c_1 + d_1)$ to $v.\mathcal{S}$;
 if $c_1 + c_2 > c_1 + 1$ **then**
  add $(-1, c_0 + d_0, c_1 + c_2 + d_1 + d_2)$ to $v.\mathcal{S}$;
 if $c_0 + c_2 = c_1$ *or* $c_0 + c_2 + 1 = c_1$ *or* $c_1 + c_2 = c_0$ *or* $c_1 + c_2 + 1 = c_0$ **then**
  add $(0, c_0 + d_0, c_1 + d_1, c_2 + d_2)$ to $v.\mathcal{S}$;

- if $c_1 + c_2 > c_0 + 1$, add $(-1, c_0 + d_0, c_1 + d_1 + c_2 + d_2, 0)$ to $\mathcal{S}_i$ for reasons similar to the above.
- if $c_0 + c_2 = c_1$, $c_0 + c_2 + 1 = c_1$, $c_1 + c_2 = c_0$, or $c_1 + c_2 + 1 = c_0$ add $(0, c_0 + d_0, c_1 + d_1, c_2 + d_2)$ to $\mathcal{S}_i$. Here, the part containing i determines the part of its unassigned ancestors must belong to. At the same time, the parent of i might determine the part of i since i could have an equal number children in the zeroth and first part. Thus we temporarily leave i (and its unassigned descendants) unassigned.

Note that for every true if-statement above we add the corresponding entry to $\mathcal{S}_i$. Thus we could add at most two entries to $\mathcal{S}_i$ for every entry of M_i.

In the base case, let i be a node with $|c(i,T)|$ children all leaves. Then it suffices to construct the summary $\mathcal{S}_i$ directly,

$$\mathcal{S}_i = \{(+1, |c(i,T)|, 0, 0), (-1, 0, |c(i,T)|, 0)\}.$$

Generally, for a vertex i with $n_c = |c(i,T)|$ children, where each child u has and associated summary $\mathcal{S}_u$, we construct the table M_i as follows. Initially, set $M_i(0,0,0,0,0)$ to true. For $u \in |c(i,T)|$, consider each entry in $\mathcal{S}_u$ with every true $M_i(c_0, c_1, c_2, d_0, d_1, d_2)$. For a particular $(s, p_0, p_1, p_2) \in \mathcal{S}_u$,

- if $s = +1$, then $M_i(c_0 + 1, c_1, c_2, d_0 + p_0, d_1 + p_1, d_2)$ is true. Since $s \neq 0$, $p_2 = 0$.
- similarly, if $s = -1$, then set $M_i(c_0, c_1 + 1, c_2, d_0 + p_0, d_1 + p_1, d_2)$ to true.
- if $s = 0$, then set $M_i(c_0, c_1, c_2 + 1, d_0 + p_0, d_1 + p_1, d_2 + p_2)$ to true.

Let v_1 be the root of T. To determine if T has a balanced EF-0 2-partition, consider all entries (s, p_0, p_1, p_2) of $\mathcal{S}_1$. If $s = +1$ and $|c_0 + 1 - c_1| \leq 1$ or if $s = -1$ and $|c_1 + 1 - c_0| \leq 1$, then a balanced EF-0 2-partition exists (recall that in both these cases $c_2 = 0$). If $s = 0$, then if $|c_0 + 1 + c_2 - c_1| \leq 1$ or $|c_1 + 1 + c_2 - c_0| \leq 1$ then a balance EF-0 2-partition exists.

We use induction to show the correctness of this algorithm. The base case is true by inspection. To show that the recurrence (from the summaries S_u of the children u of v_i to the summary of v_i) is valid, let $|c(i,T)| = n_c$ as above.

Suppose each $\mathcal{S}_{v_j}$ contains all the possible EF-0 partitions of the sub-tree $SB(v_j)$ rooted at v_j for each $v_j \in c(i,T)$. We show that every valid EF-0 partition of the sub-tree $SB(v_i)$ rooted at v_i can be formed from the $\mathcal{S}_{v_j}$s. Consider an EF-0 partition of $SB(v_i)$, denoted $\chi_i : V(SB(v_i)) \to \{+1, -1\}$. Wlog. assume that $\chi(v_i) = 1$. For all children $v_j \in c(i,T)$, such that $\chi(v_j) = -1$, χ restricted to $SB(v_j)$, denoted as $\chi|_{v_j}$, is a valid EF-0 partition so appears in $\mathcal{S}_{v_j}$. Next consider those $v_j \in c(i,T)$, such that $\chi(v_j) = 1$. If half or more of the children of v_j are assigned 1 by χ, then again χ is a valid EF-0 partition restricted to $SB(v_j)$ and $\chi|_{v_j}$ must appear in $\mathcal{S}_{v_j}$. Thus it remains to consider the case where fewer than half of the children of v_j are assigned 1 by χ. Since χ is a valid EF-0 partition for $SB(v_i)$, it must be the case that number of children of v_j assigned 1 by χ is less than or equal to $\lfloor |c(j,T)|/2 \rfloor + 1$. Construct a EF-0 partition of $SB(v_j)$ by taking $\chi|_{v_i}$ and unassigning v_j. Further, recursively unassign all descendants of v_j which no longer belong to the same part as at least half of its neighbors. Note that this EF-0 partition appears in $\mathcal{S}_{v_j}$. Since the restriction of χ to $SB(v_j)$ appears in each $\mathcal{S}_{v_j}$ for each child $v_j \in c(i,T)$ and all unassigned nodes are eventually put in the same part as their parent, we see that χ will appear as a valid EF-0 partition in $\mathcal{S}_{v_i}$.

Note that $|\mathcal{S}_i| \leq 2n^4$ for every vertex v_i. Filling out each entry of M_i requires looking at all n^6 entries of M_i and comparing them to the values in $\mathcal{S}_i$, the summary of child $u \in c(i,T)$ of v_i. The running time for this is at most $2n^{10}$. Since we do this for every node in the tree, the total running time is $O(n^{11})$.

In order to transform the above into a DP for $k > 2$, we need to make the following changes to M_i and S_i. Let M_i have entries indexed by $(c_0, ..., c_k, d_0, ..., d_k)$ where c_j indicates the number of children of v_i in part j (independent of the part containing v_i) and d_j is the number of descendants in part j not counting the children. As before, c_k counts the number of unassigned children and d_k counts the number of unassigned descendants not counting the children of v_i. Similarly, for $\mathcal{S}_i$, entries are of the form $(s, p_0, p_1, ..., p_k)$ where $s \in \{0, ..., k-1\}$ indicates the part containing v_i (0 means that v_i is unassigned) and p_k indicates the number of descendants in part j (d_k is the number of unassigned descendants of v_i).

The process for constructing the summaries $\mathcal{S}_i$ and filling out the tables M_i is also similar to the $k = 2$ case. Suppose that M_i is filled out. For every true entry $M_i(c_0, ..., c_k, d_0, ..., d_k)$ where $c_0 + \cdots + c_k = |c(i,T)|$,

- if $c_{\ell^*} + u > c_\ell + 1$ for every $\ell \neq \ell^*$, add $(\ell^*, c_0 + d_0, ..., c_{\ell^*} + d_{\ell^*} + d_k, ..., c_{k-1} + d_{k-1}, 0)$ to $\mathcal{S}_i$.
- if $c_{\ell^*} + c_k = c_\ell$ or $c_{\ell^*} + c_k = c_\ell + 1$ for every $\ell \neq \ell^*$, then add $(0, c_0 + d_0, ..., c_{k-1} + d_{k-1}, c_k + d_k)$ to $\mathcal{S}_i$.

Suppose $c(i,T)$ are the children of i and $\mathcal{S}_u$ is the summary corresponding to $u \in c(i,T)$. Consider every entry $(s, p_0, ..., p_k)$ of $\mathcal{S}_u$. For each such entry, consider every $M_i(c_0, ..., c_k, d_0, ..., d_k)$ which is true. If $s = j$ for $j \geq 1$, set $M_i(c_0, ..., c_j + 1, ..., c_k, d_0 + p_0, ..., d_j + p_j + p_k, ..., d_{k-1} + p_{k-1}, d_k)$ to true. If $s = 0$, then set $M_i(c_0, ..., c_j, c_k + 1, d_0 + p_0, ..., d_{k-1} + p_{k-1}, d_k + p_k)$ to true. To determine if an EF-0 balanced k-partition of the tree exists, consider every entry $(s, p_0, p_1, ..., p_k)$ of $\mathcal{S}_1$ where v_1 is the root. For every entries whose $s \geq 1$, if $|d_s + 1 - d_j| \leq 1$ for every $j \neq s$, then such a partition exist. If $s = 0$, then if there exists an j such that $|d_j + 1 + d_k - d_\ell| \leq 1$ for every $\ell \neq j$, then such a partition exists. Note that the running time of this algorithm is on the order of $O(n^{2k+7})$. $\square$

4.3 Open Problems

In this chapter, we considered the problem of partitioning n agents into k almost equal-sized groups, when the agents have binary preferences, induced by a social network. We designed algorithms which approximately satisfy the envy-free fairness guarantees. We leave open the questions: Does an EF-2 partition always exist?

There are natural ways to extend our model. One can consider more general preferences than symmetric and binary. Symmetric weighted preferences are particularly interesting as while one can verify that our positive result of min 2-cut carries over this case, our guarantees of min k-cut for $k \geq 3$ are not easily expandable beyond the binary case. Moreover, if the agents are described by a number of attributes, the construction of fair and diverse groups is another interesting direction.

Chapter 5

Balanced Friendly Partitions in Random Graphs

In this chapter we cover another joint work with Aleksandar Nikolov currently in submission. In particular, we prove Theorem 21, Theorem 25, Theorem 26, and Theorem 27 where γ-friendly k-partitions are defined in Definition 16 for undirected graphs. Suppose instead that G is a directed graph. Then, these definitions can be extended to apply to G by replacing the number of neighbours of $v \in V(G)$ in the set $P \subseteq V(G)$, denoted $d_P(v)$, by the number of *out*-neighbours of the vertex v in P. With a slight abuse of notations, we will also use $d_P(v)$ to represent this value when dealing with directed graphs. Finally, note that a partition is *balanced* if no two of its parts differ by more than one in size.

Definition 16 (γ-Friendly k-Partitions)**.** *Given a graph $G = (V, E)$ and a partition $\pi = (P_1, \dots, P_k)$ of V into non-empty parts, we say that π is an*

- average γ-friendly k-partition *if, for every $v \in P_i$, $d_{P_i}(v) \geq \frac{1}{k-1} \sum_{j \neq i} d_{P_j}(v) + \gamma$;*
- max γ-friendly k-partition *if, for every $v \in P_i$, $d_{P_i}(v) \geq \max_{j \neq i} d_{P_j}(v) + \gamma$.*
- sum γ-friendly k-partition *if, for every $v \in P_i$, $d_{P_i}(v) \geq \sum_{j \neq i} d_{P_j}(v) + \gamma$.*

If $k = 2$, these three definitions are equivalent, so we say that $v \in P_i$ is a γ-friendly vertex if $d_{P_i}(v) \geq d_{P_j}(v) + \gamma$ for $j \neq i$.

We restate the theorems below for ease of use. Input graphs to these theorems are drawn from the Erdös-Rényi digraph random graph model defined in Definition 15.

Definition 15 (Erdös-Rényi Digraph Model)**.** *$\mathcal{G}_B(n)$ is a distribution on random digraphs with n vertices. To construct $G \sim \mathcal{G}_B(n)$, take the complete graph on n vertices and for each of the $n(n-1)$ directed edges, add the edge to G independently with probability* $\mathrm{Bern}(1/2)$*. Note that G does not have self loops.*

Further, for random graphs, we want to know if they contain average balanced γ-friendly k-partitions.

Theorem 21 (γ-Friendly Bisections in Erdös-Rényi Random Digraphs)**.** *Let $G \sim \mathcal{G}_B(2n)$ as in Definition 15. Then, for all constant integers $\gamma \leq -1$, G has a γ-friendly bisection with uniform positive probability. Further, with high probability, G does not have a balanced γ-friendly for integer $\gamma \geq 0$.*

The proof of Theorem 21 makes use of a corresponding result for graphs drawn from a Gaussian random graph model, $\mathcal{G}_N(2n)$. See Definition 59.

Definition 59. (Gaussian Graph Model). *$\mathcal{G}_N(n)$ is a distribution on random digraphs with n vertices. To construct $G \sim \mathcal{G}_N(n)$, take the complete graph on n vertices and for each of the $n(n-1)$ directed edges, add the edge with edge-weight drawn independently and identically from the Gaussian distribution $\mathcal{N}(1/2, 1/4)$.*

In particular we show Lemma 60 and use local limit results to approximate $G \sim \mathcal{G}_B(2n)$ using $G \sim \mathcal{G}_N(2n)$ with high accuracy in order to prove Theorem 21.

Lemma 60. *Let $G \sim \mathcal{G}_N(2n)$ as in Definition 59. For any $\delta > 0$, if we define $\gamma := -1/2 - \delta$, then G has a γ-friendly bisection w.u.p.p. Further, w.h.p, G is not γ-friendly when $\gamma \geq -1/2$.*

In order to extend Theorem 21 to the random graph model in Definition 15 with $p \neq 1/2$ satisfying $np(1-p) \to \infty$, it suffices to define a random Gaussian graph model similar to Definition 59 with edge weights of the complete digraph drawn from $\mathcal{N}(p, p(1-p))$. Then we can prove a version of Lemma 60 where $G \sim \mathcal{G}_N(2n)$ for this modified graph model. The current proof of Lemma 60 can be reused with only slight modifications to serve this purpose. Going forward, we focus on the case where $p = 1/2$.

The structure of proving a result for $G \sim \mathcal{G}_B(n)$ by first proving a result for $G \sim \mathcal{G}_N(n)$ will be a reoccurring motif in our other theorems.

For Theorem 25, we will additionally need Assumption 24 as discussed in the introduction.

Assumption 24. *For any integer $k > 2$ and constant $\delta > 0$, define the function $f : [0, 1] \to [0, 1]$ defined over the variable a as*

$$f(a) := \mathbb{P}\left[\sigma_1 Z_1 + \sigma_2 Z_2 \geq (c_k - \delta) \wedge \sigma_1 Z_1 - \sigma_2 Z_2 \geq (c_k - \delta)\right], \tag{1.6}$$

where Z_1 and Z_2 are standard Gaussians, $c_k := \Phi^{-1}\left(1 - \frac{1}{k}\right)$, $\sigma_1 := \sqrt{\frac{(1+a)k-2}{2(k-1)}}$, and $\sigma_2 := \sqrt{\frac{(1-a)k}{2(k-1)}}$. Further, define the function g

$$g(\mathbf{A}) = \prod_{i,j \in [k]} \left(\frac{f(a_{i,j})}{a_{i,j}}\right)^{a_{i,j}} \tag{1.7}$$

defined on the $k \times k$ doubly stochastic matrix $\mathbf{A}$ with entry in row i and column j denoted $a_{i,j}$.

The function $\ln g(\mathbf{A})$ has a unique maximum over the set of doubly stochastic matrices at $\mathbf{A}^* := \frac{1}{k}\mathbb{J}$.

Theorem 25 (Average γ-Friendly k-Partition)**.** *For any constant integer $k > 2$, let $G \sim \mathcal{G}_B(kn)$, $c_k := \Phi^{-1}(1 - 1/k)$ where Φ is the CDF of the standard normal distribution, and $\sigma = \sqrt{\frac{nk}{4(k-1)}}$. If Assumption 24 holds, then for any $\delta > 0$, with high probability G has an average $\sigma(c_k - \delta)$-friendly balanced k-partition.*

Conversely, with high probability, G will not have an average $\sigma \cdot c_k$-friendly balanced k-partition.

Theorem 26 (Maximum γ-Friendly k-Partition)**.** *For constant $k > 2$, let $G \sim \mathcal{G}_B(kn)$. Then, if $\gamma \geq 0$, with high probability G does not have any max γ-friendly balanced k-partitions.*

Theorem 27 (Sum γ-Friendly k-Partition)**.** *For constant $k > 2$, let $G \sim \mathcal{G}_B(kn)$. Then,* even with *$\gamma \lesssim_k -n$, with high probability G does not have any sum γ-friendly balanced k-partitions.*

For each of the above theorems, we define a corresponding lemma for graphs drawn from $\mathcal{G}_N(kn)$, namely, Lemma 61, Lemma 62, Lemma 63 stated below.

Lemma 61. *For constant k where $k > 2$, let $G \sim \mathcal{G}_N(kn)$. Suppose that Assumption 24 holds. Then, with $\gamma \coloneqq \sigma\left(c_k - \delta\right)$ where $c_k \coloneqq \Phi^{-1}\left(1 - 1/k\right)$, $\sigma \coloneqq \sqrt{\frac{nk}{4(k-1)}}$, and $\delta > 0$, G has an average γ-friendly balanced k-partition w.u.p.p.*

Conversely, if $\gamma \coloneqq \sigma c_k$, then w.h.p. G will not have an average, γ-friendly balanced k-partition.

Lemma 62. *For constant k where $k > 2$, let $G \sim \mathcal{G}_N$ with $|V| = kn$. Then, if $\gamma \geq 0$, G does not have any max γ-friendly balanced k-partitions w.h.p.*

Lemma 63. *For constant k where $k > 2$, let $G \sim \mathcal{G}_N$ with $|V| = kn$. Then, if $\gamma \lesssim_k -n$, G does not have any sum γ-friendly balanced k-partitions w.h.p.*

We prove Lemma 61 in Section 5.2.2 using the second moment method. To derive Theorem 25 from it, we not only use the local limit theorems mentioned above, but also a method of amplifying the uniformly positive probability result to a high probability result that we adapt from [MSS23]. The converse components of Lemma 61, Lemma 62, and Lemma 63 are instead shown using the first moment method and we do so in Section 5.2.1.

Our work suggests a number of interesting open problems. See Table 5.1 for a summary of what is known and what is open about the existence of friendly balanced partitions in random directed and undirected graphs. In addition, a natural open problem is to either complement our existence results with algorithms to compute friendly balanced partitions, or to show evidence that this is computationally hard.

In the first two sections, Section 5.1 and Section 5.2, we prove the lemmas pertaining to graphs $G \sim \mathcal{G}_N(2n)$, namely Lemma 60 which we will use as a model to prove Lemma 61, Lemma 62, and Lemma 63. Then, in Section 5.3, we will use a modified version of a result of Minzer, Sah, and Sawhney [MSS23, Theorem 2.2] to transfer these $G \sim \mathcal{G}_N(2n)$ results to those corresponding to theorems for $G \sim \mathcal{G}_B(2n)$, namely Theorem 21, Theorem 25, Theorem 26, and Theorem 27. We conclude the main body of work in Section 5.4 with a summary of what is known so far and what remains to be explored in the domain of γ-friendly balanced k-partitions in random digraphs. Section 5.5 and Section 5.6 contain tricky function bounds and proofs of local limit lemmas which are necessary but difficult to parse on first-reading.

Before we begin, there are a few additional pieces of terminology we must introduce. Let $\mathcal{N}(\mu, \sigma)$ the normal distribution with mean μ and variance σ. Further, let H be the discrete entropy function of a random variable Y. If Y takes values $y \in \mathcal{D}$, then

$$H(Y) = \sum_{y \in \mathcal{D}} \Pr[Y = y] \log \frac{1}{\Pr[Y = y]}.$$

Further, we recall a well know approximation of multinomial coefficients which we include here for completeness.

Lemma 64. *For integer $k \geq 2$ the multinomial $\binom{kn}{n,\ldots,n} = \frac{k^{nk}}{(2\pi n)^{(k-1)/2}}\left(1 + O_k\left(\frac{1}{n}\right)\right)$.*

Proof. We have by Stirling's approximation,

$$\binom{kn}{n,\ldots,n} = \frac{(kn)!}{(n!)^k} = \frac{\sqrt{k}\cdot k^{nk}}{(2\pi n)^{(k-1)/2}}\left(1 - O_k\left(\frac{1}{n}\right)\right) \qquad \square$$

5.1 Bisections in the Gaussian Case

To prove Lemma 60, i.e., the existence of a γ-friendly bisection with uniform positive probability, we use the second moment method. In particular, we show that the number of γ-friendly bisections, X_γ, is non-zero with uniform positive probability by showing that $\mathbb{E}X_\gamma \gg 1$ and that $(\mathbb{E}X_\gamma)^2 \geq c \cdot \mathbb{E}X_\gamma^2$ for some constant c. Then, by the Paley-Zygmund inequality, we have that

$$\mathbb{P}\left[X_\gamma > 0\right] \geq \frac{(\mathbb{E}X_\gamma)^2}{\mathbb{E}X_\gamma^2} \geq c. \tag{5.1}$$

We show $\mathbb{E}X_\gamma \gg 1$ in Lemma 65 and $(\mathbb{E}X_\gamma)^2 \geq c \cdot \mathbb{E}X_\gamma^2$ in Lemma 67.

To show that with high probability there does not exist a γ-friendly bisection when $\gamma = -1/2$, we use the first moment to show that $\mathbb{E}X_\gamma = o(1)$ in the second part of Lemma 65.

Lemma 65. (First Moment γ-Friendly Bisection in $\mathcal{G}_N(2n)$). *Let $G \sim \mathcal{G}_N(2n)$. Let X_γ be the number of γ-friendly bisections of G for any $\gamma = -(1/2+\delta)$ where $\delta > 0$. Then*

$$\mathbb{E}X_\gamma \asymp_\delta \frac{e^{4\delta\sqrt{n/\pi}}}{\sqrt{\pi n}}. \tag{5.2}$$

Instead, when $\gamma = -1/2$, $\mathbb{E}X_\gamma = o(1)$.

Proof. Assume that $V(G) = \{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$, and let $\rho = (P_1, P_2)$ be a bisection where $P_i = \{x_{i,j} : j \in [n]\}$ for $i \in \{1,2\}$. Fix some vertex $x_{1,1}$ and let $E_\gamma(\rho; x_{1,1})$ be the event that $x_{1,1}$ is γ-friendly with respect to ρ. Further, let $E_\gamma(\rho)$ be the event that *all* vertices are γ-friendly with respect to ρ. Since all the edges are directed and independent, $\mathbb{P}E_\gamma(\rho; x_{1,1}) = \mathbb{P}\left[E_\gamma(\rho; x_{1,1})\right]^{2n}$ and it suffices to compute $\mathbb{P}E_\gamma(\rho; x_{1,1})$.

Let $X_{i,j}$ be the random variable for the weight of the directed edge from vertex $x_{1,1}$ to $x_{i,j}$ where $X_{1,1} = 0$ and $X_{i,j} \sim \mathcal{N}(1/2, 1/4)$. Let $S_i = \sum_{j=1}^n X_{i,j}$ be the sum of the edges from $x_{1,1}$ to the nodes in part P_i. Note that $S_1 \sim \mathcal{N}(n/2 - 1/2, n/4 - 1/4)$ and $S_2 \sim \mathcal{N}(n/2, n/4)$ as $x_{1,1}$ has a weighted edge incident to all nodes except itself in P_1 and all nodes in P_2. Then $\mathbb{P}E_\gamma(\rho; x_{1,1}) = \mathbb{P}(S_1 - S_2 \geq \gamma)$ where $S_1 - S_2 \sim \mathcal{N}(-1/2, n/2 - 1/4)$. By standardizing $S_1 - S_2$, we have

$$\mathbb{P}E_\gamma(\rho; x_{1,1}) = \mathbb{P}\left[S_1 - S_2 \geq \gamma\right] = \mathbb{P}\left[\frac{S_1 - S_2 + 1/2}{\sqrt{n/2 - 1/4}} \geq \frac{-\delta}{\sqrt{n/2 - 1/4}}\right] = \mathbb{P}\left[Z \geq \frac{-\delta}{\sqrt{n/2 - 1/4}}\right]$$

where Z is a standard Gaussian random variable. We can bound $\mathbb{P}E_\gamma(\rho; x_{1,1})$ below by

$$\mathbb{P}E_\gamma(\rho; x_{1,1}) > \frac{1}{2} + \frac{\delta}{\sqrt{n/2}} \cdot \phi\left(\frac{\delta}{\sqrt{n/2}}\right) \geq \frac{1}{2} + \frac{\delta \cdot e^{-\delta^2/n}}{\sqrt{\pi n}} \geq \frac{1}{2} + \frac{\delta}{\sqrt{\pi n}}\left(1 - \frac{\delta^2}{n}\right).$$

and similarly we can bound $\mathbb{P}E_\gamma(\rho; x_{1,1})$ above by

$$\mathbb{P}E_\gamma(\rho; x_{1,1}) \leq \frac{1}{2} + \frac{\delta}{\sqrt{n/2}} \cdot \phi(0) < \frac{1}{2} + \frac{\delta}{\sqrt{\pi n}}.$$

Together, we have the following approximation of $\mathbb{P}E_\gamma(\rho; x_{1,1})$ up to $O(1/n)$ error,

$$\mathbb{P}E_\gamma(\rho; x_{1,1}) = \frac{1}{2} + \frac{\delta}{\sqrt{\pi n}} + \delta^2 \cdot O\left(\frac{1}{n}\right) \tag{5.3}$$

Since $\mathbb{E}X_\gamma = \sum_\rho \mathbb{P}E_\gamma(\rho) = \binom{2n}{n} \cdot \mathbb{P}\left(E_\gamma(\rho; x_{1,1})\right)^{2n}$, $\mathbb{E}X_\gamma$ is bounded below by

$$\mathbb{E}X_\gamma \gtrsim \frac{2^{2n}}{\sqrt{\pi n}}\left(\frac{1}{2} + \frac{\delta \cdot e^{-\delta^2/n}}{\sqrt{\pi n}}\right)^{2n} \geq \frac{\exp\left(\delta\left(1 - \frac{\delta^2}{n}\right)\sqrt{\frac{8n}{\pi}}\right)}{\sqrt{\pi n}}$$

and above by

$$\mathbb{E}X_\gamma \lesssim \frac{2^{2n}}{\sqrt{\pi n}}\left(\frac{1}{2} + \frac{\delta}{\sqrt{\pi n}}\right)^{2n} \leq \frac{\exp\left(\delta\sqrt{\frac{8n}{\pi}}\right)}{\sqrt{\pi n}}$$

Thus, when $n \to \infty$, we have

$$\mathbb{E}X_\gamma \asymp \frac{\exp\left(\delta\sqrt{\frac{8n}{\pi}}\right)}{\sqrt{\pi n}} \gg 1$$

Note that when $\delta = 0$, we have that $\mathbb{P}E_\gamma(\rho; x_{1,1}) = 1/2$. Following the same calculations as above, $\mathbb{E}X_\gamma \asymp 1/\sqrt{\pi n} = o(1)$ as required. $\square$

Claim 66. *Let $G \sim \mathcal{G}_N(2n)$ as in Definition 59. Let $\rho_1 = (P_{1,1}, P_{1,2})$ and $\rho_2 = (P_{2,1}, P_{2,2})$ be two bisections of $V(G)$ and let $|P_{1,1} \cap P_{2,1}| = |P_{1,2} \cap P_{2,2}| = \alpha n$. Then for*

$$\psi_n(\alpha) := f_n(\alpha)^\alpha f_n(1-\alpha)^{1-\alpha} \tag{5.4}$$

with

$$f_n(\alpha) := \frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{\delta^2}{\pi n}, \tag{5.5}$$

$\mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right] \leq \psi_n(\alpha)^{2n}$ where $E_\gamma(\rho)$ is the event that G is γ-friendly with respect to ρ and $\gamma := -(1/2 + \delta)$ as in Lemma 60.

Proof. We compute $\mathbb{P}\left[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})\right]$ where $E_\gamma(\rho; x_{1,1})$ is the event that some vertex $x_{1,1}$ is γ-friendly with respect to bisection ρ. Assume that $x_{1,1} \in P_{1,1}$ of ρ_1 and $x_{1,1} \in P_{2,1}$ of ρ_2.

As in the proof of the first moment, let $X_{i,j}$ be the random variable for the weight of the directed edge $x_{1,1}$ to $x_{i,j}$. Note that there are αn vertices in common between $P_{1,1}$ and $P_{2,1}$ as well as between $P_{1,2}$ and $P_{2,2}$. Further there are $(1-\alpha)n$ vertices in common between $P_{1,1}$ and $P_{2,2}$ as well as between $P_{1,2}$ and $P_{2,1}$. Let

$$R_+ = \sum_{x_{i,j} \in P_{1,1} \cap P_{2,1}} X_{i,j} - \sum_{x_{i,j} \in P_{1,2} \cap P_{2,2}} X_{i,j} \tag{5.6}$$

$$R_- = \sum_{x_{i,j} \in P_{1,1} \cap P_{2,2}} X_{i,j} - \sum_{x_{i,j} \in P_{1,2} \cap P_{2,1}} X_{i,j} \tag{5.7}$$

where $R_+ \sim \mathcal{N}(-1/2, n\alpha/2 - 1/4)$ and $R_- \sim \mathcal{N}(0, n(1-\alpha)/2)$. The γ-friendly events $E_\gamma(\rho_1; x_{1,1})$ and $E_\gamma(\rho_2; x_{1,1})$ are equivalent to the inequalities $R_+ + R_- \geq \gamma$ and $R_+ - R_- \geq \gamma$ being true respectively. Note that these two inequalities define two half-spaces in R_+ and R_-, so the probability $\mathbb{P}[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})]$ is equal to the Gaussian measure of the wedge formed by the intersection of these two half-spaces which satisfy the two inequalities. Standardize R_+ and R_- as $\overline{R}_+ := \left(R_+ + \frac{1}{2}\right)\sqrt{\frac{4}{2\alpha n - 1}}$ and $\overline{R}_- := R_- \sqrt{\frac{2}{(1-\alpha)n}}$. Then, in terms of the standardized random variables, we have

$$E_\gamma(\rho_1; x_{1,1}) = \left[\sqrt{\frac{2\alpha n - 1}{4}} \cdot \overline{R}_+ + \sqrt{\frac{(1-\alpha)n}{2}} \cdot \overline{R}_- \geq -\delta\right] \tag{5.8}$$

$$E_\gamma(\rho_2; x_{1,1}) = \left[\sqrt{\frac{2\alpha n - 1}{4}} \cdot \overline{R}_+ - \sqrt{\frac{(1-\alpha)n}{2}} \cdot \overline{R}_- \geq -\delta\right] \tag{5.9}$$

It follows that $\mathbb{P}[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})]$ is equal to the *standard* Gaussian measure of the wedge formed by the intersection of the half-spaces defined by the inequalities in Equation (5.8) and (5.9). See Figure 5.1.

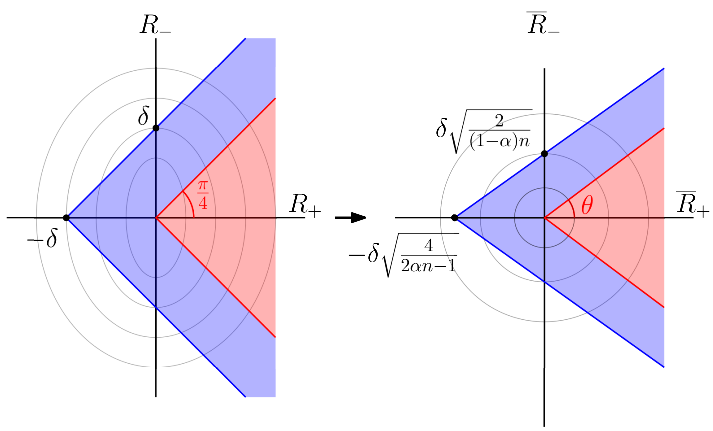


Figure 5.1: R_+ and R_- are as defined in Equations (5.6) and (5.7). $\overline{R}_+$ and $\overline{R}_-$ are the same random variables after standardizing. The left image depicts the wedge whose Gaussian measure is equal to $\mathbb{P}[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})]$. The right image depicts the same wedge after standardizing.

We can decompose this wedge into three regions defined by the following linear expressions:

$$L_1 := \sqrt{\frac{2\alpha n - 1}{4}} \cdot \overline{R}_+ + \sqrt{\frac{(1-\alpha)n}{2}} \cdot \overline{R}_-, \; L_2 := \sqrt{\frac{2\alpha n - 1}{4}} \cdot \overline{R}_+ - \sqrt{\frac{(1-\alpha)n}{2}} \cdot \overline{R}_-,$$
$$L_1^\perp := \sqrt{\frac{(1-\alpha)n}{2}} \cdot \overline{R}_+ - \sqrt{\frac{2\alpha n - 1}{4}} \cdot \overline{R}_-, \text{ and } L_2^\perp := \sqrt{\frac{(1-\alpha)n}{2}} \cdot \overline{R}_+ + \sqrt{\frac{2\alpha n - 1}{4}} \cdot \overline{R}_-.$$

The first region $\mathcal{R}_1$ is the intersection of the half-spaces $L_1 \geq 0$ and $L_2 \geq 0$. The second region $\mathcal{R}_2$

consist of two "partial strips" defined by $L_1 \in [-\delta, 0]$ in the half-space $L_1^\perp \geq 0$ and $L_2 \in [-\delta, 0]$ in the half-space $L_2^\perp \geq 0$. The final region $\mathcal{R}_3$ is the subset of the intersection of $L_1 \geq -\delta$ and $L_2 \geq -\delta$ not contained in the union of $\mathcal{R}_1$ and $\mathcal{R}_2$ i.e. the region bounded by $WVOX$ in Figure 5.2.

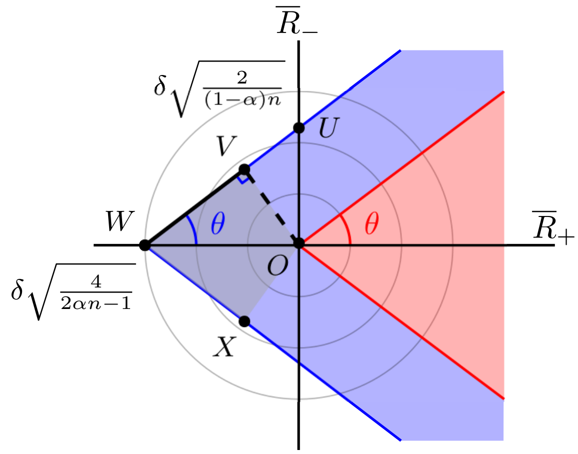


Figure 5.2: The three regions under consideration are $\mathcal{R}_1$ (red), $\mathcal{R}_2$ (blue), and $\mathcal{R}_3$ (gray). Note that the width of the blue strip is $|VO| = \delta\sqrt{\frac{2}{n}}$ and the length of one triangle composing $\mathcal{R}_3$ is $|WV| = \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\sqrt{n}}$.

We bound the standard Gaussian measure of each region in turn. For $\mathcal{R}_1$, let θ be half of the angle of the wedge formed by $L_1 \geq 0$ and $L_2 \geq 0$. Note that up to multiplicative error on the order of $1 + O(1/n)$, $\sin\theta = \sqrt{\alpha}$, $\cos\theta = \sqrt{1-\alpha}$, and $\tan\theta = \sqrt{\frac{\alpha}{1-\alpha}}$. Since the standard Gaussian measure is rotationally symmetric, $\mu(\mathcal{R}_1) = \frac{\theta}{\pi} = \arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)\frac{1}{\pi}$.

We bound $\mu(\mathcal{R}_2)$ from above by reducing the two dimensional standard Gaussian in $\left(\overline{R}_+, \overline{R}_-\right)$ to a one dimensional standard Gaussian. Notice that the two partial strips which make up $\mathcal{R}_2$ have Gaussian measure equal to the strip $L_2 \in [-\delta, 0]$. Thus, we can express the Gaussian measure of $\mathcal{R}_2$ as

$$\mu\left(\mathcal{R}_2\right) = \int_0^{|VO|} \phi\left(\ell_2\right) d\ell_2$$

where $|VO| = \delta\sin(\theta)\sqrt{\frac{2}{\alpha n}} = \delta\sqrt{\frac{2}{n}}$ is the width of the strip $L_2 \in [-\delta, 0]$. It follows that $\mu\left(\mathcal{R}_2\right) \leq \frac{\delta}{\sqrt{\pi n}}$.

We bound $\mu(\mathcal{R}_3)$ from above by computing the area of $\mathcal{R}_3$ (region bounded by $WVOX$), and multiplying it by the maximum of the density for all $(x, y) \in \mathcal{R}_3$. Since $|WV| = \delta\cos(\theta)\sqrt{\frac{2}{\alpha n}} = \delta\sqrt{\frac{2}{n}} \cdot \sqrt{\frac{1-\alpha}{\alpha}}$, $|WV| = \delta\sqrt{\frac{2}{n}}$, and $|WO|$ form a right-angle triangle. Thus $\mathcal{R}_3$ has area $|VO| \cdot |WV| = \delta^2\left(\frac{2}{n}\right) \cdot \sqrt{\frac{1-\alpha}{\alpha}}$. Note that the Gaussian density ϕ evaluated on $\mathcal{R}_3$ is maximized at $(x, y) = (0, 0)$, so $\mu\left(\mathcal{R}_3\right) \leq \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{\delta^2}{\pi n}$.

Together we have the following upper bound for $\mu(\mathcal{R}_1) + \mu(\mathcal{R}_2) + \mu(\mathcal{R}_3)$:

$$f_n(\alpha) \coloneqq \frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{\delta^2}{\pi n} \geq \mu(\mathcal{R}_1) + \mu(\mathcal{R}_2) + \mu(\mathcal{R}_3), \tag{5.10}$$

and thus $\mathbb{P}\left[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})\right] = \mu(\mathcal{R}_1) + \mu(\mathcal{R}_2) + \mu(\mathcal{R}_3) \leq f_n(\alpha)$.

The above upper-bound for the probability that $x_{1,1}$ is γ-friendly with respect to bisections ρ_1 and ρ_2 assumed that $x_{1,1} \in P_{1,1}$ of ρ_1 and $P_{2,1}$ of ρ_2. Over all n vertices in $P_{1,1}$, αn will be in $P_{2,1}$, while the remaining $(1-\alpha)n$ will be in $P_{1,1}$ for ρ_1 but end up in $P_{2,2}$ for ρ_2. Repeating the above computations for such vertices $x_{1,1} \in P_{1,1} \cap P_{2,2}$, we find that

$$\mathbb{P}\left[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})\right] \leq f_n(1-\alpha). \tag{5.11}$$

To see this, observe that $\mathbb{P}\left[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})\right]$ is equal to the Gaussian measure of the intersections of the half-spaces $R_+ + R_- \geq \gamma$ and $-R_+ + R_- \geq \gamma$ where R_+ and R_- are defined in Equations (5.6) and (5.7) with $X_{i,j}$ as the random variable for the directed edge from $x_{1,1}$ to $x_{i,j}$. Again we can decompose this wedge into three regions and bound the Gaussian measure of each as we have done above. Thus, define

$$\psi_n(\alpha) \coloneqq f_n(\alpha)^\alpha f_n(1-\alpha)^{1-\alpha}. \tag{5.12}$$

It follows that $\mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right] \leq \psi_n(\alpha)^{2n}$. $\square$

Using Claim 66, we can bound the probability that X_γ is far from its expectation in Lemma 67. We will first state Lemma 67 before we prove it at the end of the section as it will require a modified Laplace method, defined in Lemma 68.

Lemma 67. (Second Moment γ-Friendly Bisection). *With X_γ as defined in Lemma 65, $(\mathbb{E}X_\gamma)^2 \geq c \cdot \mathbb{E}X_\gamma^2$ for a universal constant c.*

We use a modified version of the Laplace method to bound the sum in Equation (5.26) (Lemma 3, [AM06]). Typically, this will require taking the limit as $n \to \infty$, but we take extra precaution as the joint probability ψ_n is a function of n.

Lemma 68. (Modified Laplace Lemma). *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a family of positive, twice-differentiable functions on $[0,1]$ which are symmetric about $1/2$. Let $q \geq 1$ and define $0^0 \equiv 1$ for*

$$Q_n = \sum_{z=0}^{n} \binom{n}{\alpha n}^q \psi_n(\alpha)^{qn}. \tag{5.13}$$

Define a family $\{g_n\}_{n\in\mathbb{N}}$ of functions on $[0,1]$ where

$$g_n(\alpha) \coloneqq \frac{\psi_n(\alpha)}{\alpha^\alpha(1-\alpha)^{1-\alpha}}. \tag{5.14}$$

Let $\alpha_{\max} = \frac{1}{2}$ and for $\xi \in O(1/\sqrt{n})$, define the interval $I_\xi = [\alpha_{\max} - \xi, \alpha_{\max} + \xi] \subset [0,1]$. If

1. *$g_n(\alpha_{\max}) > g_n(\alpha)\left(1 + \frac{c_1}{\sqrt{n}}\right)$ for all $\alpha \in [0,1] \backslash I_\xi$ and some constant $c_1 > 0$,*
2. *$\ln g_n$ is uni-modal on I_ξ and maximized at $\alpha_{\max}$,*
3. *$(\ln g_n)''(\alpha_{\max}) < 0$, and*

4. *in the Taylor expansion of* $\ln g_n(\alpha)$ *about* $\alpha = \alpha_{\max}$*:*

$$\ln g_n(\alpha_{\max}) + (\ln g_n)'(\alpha_{\max})(\alpha - \alpha_{\max}) + (\ln g_n)''(\alpha_{\max})\frac{(\alpha - \alpha_{\max})^2}{2} + R(\alpha - \alpha_{\max})^3, \tag{5.15}$$

$R(\alpha - \alpha_{\max})^3 \leq c'(\alpha - \alpha_{\max})^3$ *for some constant* c'.

Then there exists a constant c *which is a function of* $\alpha_{\max}$*,* $g_n(\alpha_{\max})$*, and* $g_n''(\alpha_{\max})$ *such that*

$$Q_n < cn^{-(q-1)/2} g_n(\alpha_{\max})^n.$$

In Lemmas 69 and 73 below, we will show that g_n defined in (5.14) as

$$g_n(\alpha) := \frac{\psi_n(\alpha)}{\alpha^\alpha(1-\alpha)^{1-\alpha}},$$

satisfies the prerequisites of Lemma 68. Instead of dealing with $g_n(\alpha)$ directly, it will be useful to consider instead the function

$$\ln g_n(\alpha) = \alpha \ln\left(\frac{f_n(\alpha)}{\alpha}\right) + (1-\alpha)\ln\left(\frac{f_n(1-\alpha)}{1-\alpha}\right),$$

where f_n is defined in Equation (5.5) and repeated below for convenience:

$$f_n(\alpha) = \frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{\delta^2}{\pi n}.$$

The derivatives of $\ln g_n(\alpha)$ are then

$$\begin{aligned}(\ln g_n)' &= \frac{d}{d\alpha}\left(\alpha \ln f_n(\alpha) + (1-\alpha)\ln f_n(1-\alpha) - \alpha\ln\alpha - (1-\alpha)\ln(1-\alpha)\right) \\ &= \left(\ln f_n(\alpha) - \ln f_n(1-\alpha) + \frac{\alpha f_n'(\alpha)}{f_n(\alpha)} - \frac{(1-\alpha)f_n'(1-\alpha)}{f_n(1-\alpha)} + \ln\frac{(1-\alpha)}{\alpha}\right)\end{aligned} \tag{5.16}$$

$$\begin{aligned}(\ln g_n)'' &= 2\frac{f_n'(\alpha)}{f_n(\alpha)} + 2\frac{f_n'(1-\alpha)}{f_n(1-\alpha)} + \frac{\alpha f_n''(\alpha)}{f_n(\alpha)} + \frac{(1-\alpha)f_n''(1-\alpha)}{f_n(1-\alpha)} \\ &\quad - \frac{\alpha\left(f_n'(\alpha)\right)^2}{f_n(\alpha)^2} - \frac{(1-\alpha)(f_n'(1-\alpha))^2}{f_n(1-\alpha)^2} - \frac{1}{\alpha(1-\alpha)}.\end{aligned} \tag{5.17}$$

When $\alpha \in [0.01, 0.99]$, we can simplify $g_n(\alpha)$ slightly by simplifying $f_n(\alpha)$. Observe that the first term of $f_n(\alpha)$ is independent of n, the second is independent of α, while the last is of order $O(1/n)$ if α is constant. If we define $f(\alpha) = \frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi}$, then $g_n(\alpha)$ is

$$\begin{aligned}g_n(\alpha) &\leq \frac{\left(f(\alpha) + \frac{\delta}{\sqrt{\pi n}} + \frac{200\delta^2}{\pi n}\right)^\alpha \cdot \left(f(1-\alpha) + \frac{\delta}{\sqrt{\pi n}} + \frac{200\delta^2}{\pi n}\right)^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} \\ &= \frac{\left(f(\alpha) + \frac{\delta}{\sqrt{\pi n}}\right)^\alpha \cdot \left(f(1-\alpha) + \frac{\delta}{\sqrt{\pi n}}\right)^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}}\left(1 + O_k\left(\frac{1}{n}\right)\right)\end{aligned}$$

We then define

$$\tilde{g}_n(\alpha) := \frac{\left(f(\alpha) + \frac{\delta}{\sqrt{\pi n}}\right)^{\alpha} \cdot \left(f(1-\alpha) + \frac{\delta}{\sqrt{\pi n}}\right)^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}, \tag{5.18}$$

as an approximation of $g_n(\alpha)$ for our second moment applications. In this case, let

$$f_n(\alpha) = \frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}}. \tag{5.19}$$

The first and second derivatives of f_n are then

$$f_n'(\alpha) = \frac{1}{2\pi\sqrt{\alpha(1-\alpha)}}, \text{ and} \tag{5.20}$$

$$f_n''(\alpha) = \frac{2\alpha - 1}{4\pi\left(\alpha(1-\alpha)\right)^{3/2}}. \tag{5.21}$$

Lemma 69. *Let $\alpha_{\max} = 1/2$ and g_n as defined in Equation (5.14). For constant $\delta \geq 0$, $\xi \leq 2\delta/\sqrt{\pi n}$, and $\alpha \in [0,1] \backslash I_\xi$, $g_n(\alpha_{\max}) > g_n(\alpha)\left(1 + O\left(\frac{1}{n}\right)\right)$.*

Proof. By symmetry, it suffices to only consider $[0, \alpha_{\max} - \xi)$. We decompose it into three intervals $[0, 0.01]$, $[0.01, 0.1]$, and $[0.1, \alpha_{\max} - \xi)$. The outline of the body of the proof is as follows.

On $[0.0.1)$, Claim 70 shows that g_n is monotonically decreasing by showing that $g_n' < 0$. Thus it will suffice to show that $g_n(\alpha_{\max}) > \left(1 + O\left(\frac{1}{n}\right)\right) \lim_{\alpha \to 0^+} g_n(\alpha)$. On $[0.01, 0.1)$, Claim 71 shows that $\tilde{g}_n$ given in Equation (5.18) is convex by showing that $\tilde{g}_n'' < 0$. Thus $g_n(\alpha)$ is bounded above on this interval will be subsumed by showing that g_n is bounded above on $[0, 0.01]$ and $[0.1, \alpha_{\max - \xi}]$. Finally, on $[0.1, \alpha_{\max} - \xi)$, Claim 72 shows that g_n is monotonically increasing by showing that $(g_n)' > 0$. Thus, $g_n(\alpha_{\max}) > g_n\left(\alpha_{\max} - \xi\right)\left(1 + O\left(\frac{1}{n}\right)\right)$.

Note that we can bound $g_n\left(\alpha_{\max}\right)$ from above by

$$g_n\left(\alpha_{\max}\right) = 2\psi_n\left(\frac{1}{2}\right) = 2f_n\left(\frac{1}{2}\right) = 2\left(\frac{1}{4} + \frac{\delta}{\sqrt{\pi n}} + \frac{\delta^2}{\pi n}\right) = \frac{1}{2} + \frac{2\delta}{\sqrt{\pi n}} + \frac{2\delta^2}{\pi n}.$$

In the following we will show that $\frac{1}{2} + \frac{2\delta}{\sqrt{\pi n}} + \frac{2\delta^2}{\pi n} \geq g_n(\alpha)\left(1 + O\left(\frac{1}{n}\right)\right)$ for α on each of the different intervals. The claims below are proved in the appendix.

Claim 70. *$g_n(\alpha)$, as defined in Equation (5.14), is monotonically decreasing on $[0, 0.01)$.*

Since g_n' is monotonically decreasing by Claim 70 on $[0, 0.01)$, $g_n(\alpha)$ achieves its maximum value for $\lim_{\alpha \to 0^+} g_n(\alpha) = \frac{1}{2} + \frac{\delta}{\sqrt{\pi n}}$. Thus, for every $\alpha \in [0, 0.1)$, $g_n(\alpha_{\max}) \geq \left(1 + O\left(\frac{1}{n}\right)\right) \lim_{\alpha \to 0^+} g_n\left(\alpha\right)$.

Claim 71. *$g_n(\alpha)$, as defined in Equation (5.14), is convex for $\alpha \in [0.01, 0.1]$.*

Since g_n'' is convex on $[0.01, 0.1)$ by Claim 71, its maximum value will be obtained at one of its endpoints. These are considered in the previous and next intervals.

Claim 72. *g_n, as defined in Equation (5.14), is monotonically increasing on $[0.1, \alpha_{\max} - \xi)$.*

Finally, we consider the interval $[0.1, \alpha_{\max} - \xi)$. By Claim 72, we have that $g_n(\alpha)$ is monotonically increasing on this interval. Thus $g_n(\alpha)$ is maximized at the right end-point $\alpha = \alpha_{\max} - \xi$. At this end-point we have

$$f_n(\alpha_{\max} - \xi) = \frac{\arctan\left(1 + \frac{2\xi}{1-2\xi} + O\left(\xi^2\right)\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} = \frac{1}{4} + \frac{\xi}{\pi(1-2\xi)} + O\left(\xi^2\right) + \frac{\delta}{\sqrt{\pi n}},$$

$$f_n(\alpha_{\max} + \xi) = \frac{\arctan\left(1 - \frac{2\xi}{1+2\xi} + O\left(\xi^2\right)\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} = \frac{1}{4} - \frac{\xi}{\pi(1+2\xi)} + O\left(\xi^2\right) + \frac{\delta}{\sqrt{\pi n}},$$

by considering the Taylor expansions of $\sqrt{1-x} = 1 - \frac{x}{2} - O\left(x^2\right)$ about $x = 0$ and $\arctan(1-x) = \frac{4}{\pi} - \frac{x}{2} - O\left(x^2\right)$ about $x = 0$. Plugging these into $g_n\left(\alpha_{\max}\right)$ we get,

$$\begin{aligned}
g_n(\alpha_{\max} - \xi) &= \left(\frac{f_n(\alpha_{\max} - \xi)}{\alpha_{\max} - \xi}\right)^{\alpha_{\max}-\xi} \left(\frac{f_n(\alpha_{\max} + \xi)}{\alpha_{\max} + \xi}\right)^{\alpha_{\max}+\xi} \\
&= \sqrt{\frac{f_n(\alpha_{\max} - \xi) f_n(\alpha_{\max} + \xi)}{1/4 - \xi^2}} \left(\frac{f_n(\alpha_{\max} + \xi)\left(\frac{1}{2} - \xi\right)}{f_n(\alpha_{\max} - \xi)\left(\frac{1}{2} + \xi\right)}\right)^{\xi} \\
&= \sqrt{\frac{f_n(\alpha_{\max} - \xi) f_n(\alpha_{\max} + \xi)}{1/4 - \xi^2}} \left(\frac{\frac{1}{4} + \frac{\xi}{\pi(1-2\xi)} + O\left(\xi^2\right) + \frac{\delta}{\sqrt{\pi n}}}{\frac{1}{4} - \frac{\xi}{\pi(1+2\xi)} + O\left(\xi^2\right) + \frac{\delta}{\sqrt{\pi n}}} \left(1 - \frac{6\xi}{1+2\xi}\right)\right)^{\xi} \\
&= \sqrt{\frac{f_n(\alpha_{\max} - \xi) f_n(\alpha_{\max} + \xi)}{1/4 - \xi^2}} \left(\left(1 - \frac{8\xi}{\pi(1+2\xi)}\right) \cdot \left(1 - \frac{6\xi}{1+2\xi}\right)\right)^{\xi} \\
&= \sqrt{\frac{f_n(\alpha_{\max} - \xi) f_n(\alpha_{\max} + \xi)}{1/4 - \xi^2}} \exp\left(\xi\left(1 - \frac{8\xi}{\pi(1+2\xi)} - \frac{6\xi}{1+2\xi}\right)\right) \\
&= \sqrt{\left(\frac{1}{2} - \frac{2\xi}{\pi(1+2\xi)} + \frac{\sqrt{2}\delta}{\sqrt{\pi n}}\right)\left(\frac{1}{2} + \frac{2\xi}{\pi(1-2\xi)} + \frac{\sqrt{2}\delta}{\sqrt{\pi n}}\right)} \left(1 + \xi + O\left(\xi^2\right)\right) \\
&= \sqrt{\frac{1}{4} - O(\xi^2)} \left(1 + \xi + O\left(\xi^2\right)\right) = \frac{1}{2} + O(\xi).
\end{aligned}$$

Comparing $g_n(\alpha_{\max} - \xi)$ with $g_n(\alpha_{\max})$, $g_n(\alpha_{\max}) \geq g_n(\alpha)\left(1 + O(1/\sqrt{n})\right)$ on this interval as well. □

Lemma 73. *Let $\alpha_{\max} = 1/2$, $\xi = o(1)$, and $I_\xi = [\alpha_{\max} - \xi, \alpha_{\max} + \xi]$. For g_n as defined in Equation (5.14) on I_ξ,*

1. *$\ln g_n$ is unimodal with mode achieved at $\alpha_{\max}$,*
2. *$(\ln g_n)''(\alpha_{\max}) + c < 0$ for some positive universal constant c, and*
3. *in the Taylor expansion of $\ln g_n(\alpha)$ about $\alpha_{\max}$:*

$$\ln g_n\left(\alpha_{\max}\right) + \left(\ln g_n\right)'\left(\alpha_{\max}\right)\left(\alpha - \alpha_{\max}\right) + \left(\ln g_n\right)''\left(\alpha_{\max}\right)\frac{\left(\alpha - \alpha_{\max}\right)^2}{2} + R\left(\alpha - \alpha_{\max}\right)^3, \tag{5.22}$$

$R\left(\alpha - \alpha_{\max}\right)^3 \leq c'\left(\alpha - \alpha_{\max}\right)^3$ for some constant c'.

Proof. To show that $\ln g_n$ is uni-modal on I_ξ with mode achieved at $\alpha_{\max}$, it suffices to refer back to the second part of Lemma 69 where we showed that $\ln g_n'(\alpha) \geq 0$ for $\alpha \in [0.1, \alpha_{\max} - \xi)$. Observe that the monotonicity of $g_n'(\alpha)$ can be extended to the interval $[0.1, \alpha_{\max})$. By symmetry, we also have $(\ln g_n)'(\alpha) \le 0$ for $(\alpha_{\max}, 0.9]$. At $\alpha_{\max}$, $(\ln g_n)'(\alpha_{\max}) = 0$ so $\alpha_{\max}$ is indeed the mode of $\ln g_n$.

For the second property we compute $(\ln g_n)''(\alpha_{\max})$. By Equation (5.17), we have that

$$(\ln g_n)''(\alpha_{\max}) = \frac{4f_n'(1/2)}{f_n(1/2)} + \frac{f_n''(1/2)}{f_n(1/2)} - \frac{(f_n'(1/2))^2}{f_n(1/2)^2} - 4$$

Using Equations (5.19), (5.20), (5.21), we have that $f_n(1/2) = \frac{1}{4} + \frac{\delta}{\sqrt{\pi n}}$, $f_n'(1/2) = \frac{1}{\pi}$, and $f_n''(1/2) = 0$. Thus

$$\begin{aligned}
(\ln g_n)''(1/2) &= \left(\frac{4}{\pi\left(\frac{1}{4} + \frac{\delta}{\sqrt{\pi n}}\right)}\right) - \left(\frac{1}{\pi\left(\frac{1}{4} + \frac{\delta}{\sqrt{\pi n}}\right)}\right)^2 - 4 \\
&= \left(\frac{4}{\pi}\right)\left(4 - \frac{1}{\pi\left(\frac{1}{4} + \frac{\delta}{\sqrt{\pi n}}\right)}\right)\left(1 + O\left(\frac{1}{n}\right)\right) - 4 \\
&= \left(\frac{4}{\pi}\right)\left(\frac{4(\pi-1)}{\pi}\right)\left(1 + O\left(\frac{1}{n}\right)\right) - 4 \\
&\le 2.16\left(\frac{4}{3}\right)^2 - 4 = 3.84 - 4 = -0.16
\end{aligned}$$

It follows that the second property is true if $c > 0.16$.

For the third property we will bound the remainder term of the Taylor expansion, $R(\alpha - \alpha_{\max})^3$, by bounding $|(\ln g_n)'''(c)|$ from above for every $c \in I_\xi$. Take the second derivative of $\ln g_n$ from Equation (5.17) and compute the third derivative,

$$\begin{aligned}
(\ln g_n)'''(\alpha) = {}& \frac{3\cdot f_n''(\alpha)}{f_n(\alpha)} - \frac{3(f_n'(\alpha))^2}{(f_n(\alpha))^2} + \frac{\alpha f_n'''(\alpha)}{f_n(\alpha)} - \frac{3\alpha f_n'(\alpha) f_n''(\alpha)}{(f_n(\alpha))^2} \\
& - \frac{3\cdot f_n''(1-\alpha)}{f_n(1-\alpha)} + \frac{3(f_n'(1-\alpha))^2}{(f_n(1-\alpha))^2} - \frac{(1-\alpha)f_n'''(1-\alpha)}{f_n(1-\alpha)} + \frac{3(1-\alpha)f_n'(1-\alpha)f_n''(1-\alpha)}{(f_n(1-\alpha))^2} \\
& + \frac{1-2\alpha}{\alpha^2(1-\alpha)^2}
\end{aligned} \tag{5.23}$$

Recall, from Equation (5.19), (5.20), and (5.21) the definitions of f_n, f_n', and f_n''. For α in the interval $I_\xi = [\alpha_{\max} - \xi, \alpha_{\max} + \xi]$, we have the following

$$\begin{aligned}
f_n(\alpha) &= \frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} \implies f_n(\alpha) \asymp \frac{1}{4} \\
f_n'(\alpha) &= \frac{1}{2\pi\sqrt{\alpha(1-\alpha)}} \implies f_n'(\alpha) \asymp \frac{1}{\pi} \\
f_n''(\alpha) &= \frac{(2\alpha - 1)}{4\pi\left(\alpha(1-\alpha)\right)^{3/2}} \implies |f_n''(\alpha)| \le \frac{2\xi}{\pi} \\
f_n'''(\alpha) &= \frac{(8\alpha^2 - 8\alpha + 3)}{8\pi\left(\alpha(1-\alpha)\right)^{5/2}} \implies f_n'''(\alpha) \asymp 4
\end{aligned}$$

since δ is a constant. Recall that $\xi = O\left(\frac{1}{\sqrt{n}}\right)$. By plugging the above into the definition of $(\ln g_n)'''$, we have that $|(\ln g_n)'''(\alpha)| = O(1)$ for $\alpha \in I_\xi$ as required. $\square$

Proof of Lemma 68. We group together the terms of Q_n and show that the sum of each group is bounded above by $O\left(n^{-(q-1)/2} g_n(\alpha_{\max})^n\right)$. First, recall the following inequalities from Lemma 3 ([AM06]) which bound the terms of Q_n by a function of $g_n(\alpha)$: Equation (5.24) when α is bounded away from zero and one by a constant and Equation (5.25) when α is close to zero and one,

$$\binom{n}{z}^q \psi_n(z/n)^{qn} < \frac{g_n(\alpha)^{qn}}{(2\pi\alpha(1-\alpha)n)^{q/2}} \left(1+\frac{1}{n}\right)^q, \tag{5.24}$$

$$\binom{n}{z}^q \psi_n\left(z/n\right)^{qn} < (8\pi n)^{-q/2} g_n(\alpha)^{qn}. \tag{5.25}$$

We subdivide the interval $[0,1]$ into $[0,0.1) \cup [0.1, \alpha_{\max}-\xi) \cup [\alpha_{\max}-\xi, \alpha_{\max}+\xi] \cup (\alpha_{\max}+\xi, 0.9] \cup (0.9, 1]$. Since the functions $g_n(\alpha)$ are symmetric about $\alpha_{\max}$, it suffices to show that the sum of the terms on the first three intervals are bounded above by $O\left(n^{-(q-1)/2} g_n(\alpha_{\max})^n\right)$. On intervals $[0, 0.1)$ and $[0.1, \alpha_{\max}-\xi)$ we have that $g_n(\alpha_{\max}) \geq g_n(\alpha)\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)$. By Equation (5.25), the sum on $[0, 0.1)$ can be bounded by

$$\begin{aligned}
\sum_{\alpha\in[0,0.1)} \binom{n}{\alpha n}^q \psi_n(\alpha)^{qn} &< \sum_{\alpha\in[0,0.1)} (8\pi n)^{-q/2} g_n(\alpha)^{qn} \\
&< \frac{n \cdot g_n\left(\alpha_{\max}\right)^{qn}}{(8\pi n)^{q/2}\left(1+\frac{c_1}{\sqrt{n}}\right)^{qn}} \\
&= n(8\pi n)^{-q/2} e^{-O(q\sqrt{n})} \cdot g_n(\alpha_{\max})^{qn} \\
&\ll c n^{-(q-1)/2} g_n(\alpha_{\max})^{qn}.
\end{aligned}$$

Similarly, by Equation (5.24), the sum on $[0.1, \alpha_{\max}-\xi)$ can be bounded by

$$\begin{aligned}
\sum_{\alpha\in[0.1,\alpha_{\max}-\xi)} \binom{n}{\alpha n}^q \psi_n(\alpha)^{qn} &< \sum_{\alpha\in[0.1,\alpha_{\max}-\xi)} \frac{g_n(\alpha)^{qn}}{(2\pi\alpha(1-\alpha)n)^{q/2}} \\
&< \frac{n g_n\left(\alpha_{\max}\right)^{qn} e^{q/n}}{(2\pi\alpha(1-\alpha)n)^{q/2}} \\
&\ll c n^{-(q-1)/2} g_n\left(\alpha_{\max}\right)^{qn}.
\end{aligned}$$

Using Equation (5.24) on the interval $[\alpha_{\max}-\xi, \alpha_{\max}+\xi]$, we have that

$$\begin{aligned}
\sum_{z\in n\cdot I_\xi} \binom{n}{z}^q \psi_n(\alpha)^{qn} &< (2\pi\alpha_{\max}(1-\alpha_{\max})n)^{-q/2} \sum_{\alpha\in I_\xi} g_n(\alpha)^{qn} \\
&\leq (2\pi\alpha_{\max}(1-\alpha_{\max})n)^{-q/2}\left(qn\int_{I_\xi} g_n(\alpha)^{qn} d\alpha + g_n(\alpha_{\max})^{qn}\right)
\end{aligned}$$

Substitute $g_n(\alpha)^{qn} = e^{qn\ln g_n(\alpha)}$ and expand $\ln g_n$ by its Taylor approximation about $\alpha_{\max}$ as shown in Equation (5.15). Since g_n is unimodal and maximize at $\alpha_{\max}$ on I_ξ, $e^{qn\ln g_n}$ is as well. It follows that $\ln g_n'(\alpha_{\max}) = 0$. Observe that

$$\begin{aligned}\frac{e^{qn \ln g_n(\alpha)}}{e^{qn \ln g_n(\alpha_{\max})}} &= \exp qn \cdot \left(\frac{\left(\ln g_n(\alpha_{\max})(\alpha-\alpha_{\max})^2\right)''}{2} + R(\alpha-\alpha_{\max})^3 \right) \\ &= \left(\exp qn \cdot \left(\frac{\left(\ln g_n(\alpha_{\max})(\alpha-\alpha_{\max})^2\right)''}{2} \right)\right) \cdot \left(1 + qn \cdot R(\alpha-\alpha_{\max})^3\right) \\ &= \left(\exp qn \cdot \left(\frac{\left(\ln g_n(\alpha_{\max})(\alpha-\alpha_{\max})^2\right)''}{2} \right)\right) \cdot \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right),\end{aligned}$$

since $\alpha \in I_\xi$ and $\xi = O\left(\frac{1}{\sqrt{n}}\right)$. By the Laplace method (Section 3.3, [BC89]),

$$\begin{aligned}\int_{I_\xi} e^{qn \ln g_n(\alpha)} d\alpha &= e^{qn \ln g_n(\alpha_{\max})} \int_{-\infty}^{\infty} e^{qn (\ln g_n)''(\alpha_{\max}) \frac{x^2}{2}} dx \left(1 + O\left(\frac{1}{n}\right)\right) \\ &= g_n(\alpha_{\max})^{qn} \sqrt{\frac{2\pi}{qn \left|(\ln g_n)''(\alpha_{\max})\right|}} \left(1 + O\left(\frac{1}{n}\right)\right)\end{aligned}$$

Together, we have the bound

$$Q_n \leq \frac{qn^{(1-q)/2} g_n(\alpha_{\max})^{qn}}{(2\pi\alpha_{\max}(1-\alpha_{\max})))^{q/2}} \left(\sqrt{\frac{2\pi}{n\left|(\ln g_n)''(\alpha_{\max})\right|}} + o(1) \right). \qquad \square$$

Proof of Lemma 67. We consider two bisections ρ_1 and ρ_2 on the set of vertices $V = \{x_{i,j} : i \in \{1,2\}$ and $j \in [n]\}$. Let $\rho_1 = (P_{1,1}, P_{1,2})$ and $\rho_2 = (P_{2,1}, P_{2,2})$. Further, let $E_\gamma(\rho_i)$ be the event that ρ_i is a γ-friendly bisection. We compute $\mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right]$.

Group together pairs of (ρ_1, ρ_2) by the number of entries upon which they agree in each part. In particular, let $|P_{1,1} \cap P_{2,1}| = |P_{1,2} \cap P_{2,2}| = \alpha n$ for some $\alpha \in (0,1)$. Note that the probability $\mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right]$ *only depends on* α. There are $\binom{2n}{n}$ choices for ρ_1 and, if we let $z := \alpha n$, $\binom{n}{z}^2$ choices for ρ_2. Thus,

$$\mathbb{E} X_\gamma^2 = \binom{2n}{n} \sum_{z=0}^{n} \binom{n}{z}^2 \mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right]. \tag{5.26}$$

The ratio $\frac{\mathbb{E} X_\gamma^2}{(\mathbb{E} X_\gamma)^2}$ is then

$$\frac{\mathbb{E} X_\gamma^2}{(\mathbb{E} X_\gamma)^2} = \frac{\sum_{z=0}^{n} \binom{n}{z}^2 \mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right]}{\left(\binom{2n}{n} \mathbb{P} E_\gamma(\rho_1)\right)^2}. \tag{5.27}$$

By Claim 66, $\psi_n(\alpha)^{2n} \geq \mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right]$ where ψ_n is defined in Equation (5.4). Further, with $Q_n = \sum_{z=0}^{n} \binom{n}{z}^2 \psi_n(z/n)^{2n}$, we see that sums of the form Q_n are bounded above by $c \cdot \binom{n}{n/2}^2 \psi_n(1/2)^{2n}$ by Lemma 68.

Compare the denominator to $c \cdot \binom{n}{n/2}^2 \psi_n(1/2)^{2n}$. For a fixed vertex $x_{1,1}$, $\mathbb{P} E_\gamma(\rho_1; x_{1,1}) = \frac{1}{2} + \frac{\delta}{\sqrt{\pi n}} + O\left(\frac{1}{n}\right)$ by Lemma 65 Equation (5.3) and $\psi_n(1/2) = \frac{1}{4} + \frac{\delta}{\sqrt{\pi n}} + \frac{\delta^2}{\pi n}$, so we have $(\mathbb{P} E_\gamma(\rho_1))^2 \approx \psi_n(1/2)^{2n}$.

It follows that

$$\frac{\mathbb{E}X_\gamma^2}{(\mathbb{E}X_\gamma)^2} \lesssim \binom{2n}{n}^{-1} \cdot \psi_n(1/2)^{-2n} \cdot \left(\sum_{z=0}^{n} \binom{n}{z}^2 \cdot \psi_n\left(\frac{z}{n}\right)^{2n} \right) = c. \qquad \square$$

5.2 Multiple Equal Parts

Next, we generalize the problem to k balanced parts. Throughout this section, draw G from distribution $\mathcal{G}_N(kn)$ as in Definition 59. Further, let the vertices of G be $V(G) = \{x_{i,j}\colon\, i \in [k],\, j \in [n]\}$. As we saw in Definition 16, there are three different ways to generalize the γ-friendly bisection problem to k parts: average, maximum, or sum. We consider each in turn.

Let X_γ be the random variable which counts the number of γ-friendly balanced k-partitions. We compute the asymptotics of the first moment of X_γ for each type of generalization and — for those types where $\mathbb{E}X_\gamma$ is bounded away from zero — second moments of X_γ. In particular, for average, balanced k-partitions (Lemma 61), we show that γ-friendly exist w.u.p.p. when $\gamma \lesssim_k \sqrt{n}$ by showing that $\mathbb{E}X_\gamma$ is bounded away from zero (Lemma 74) and $\mathbb{E}X_\gamma^2 / \left(\mathbb{E}X_\gamma\right)^2$ is bounded above by a constant (Lemma 80). In the next section, we will improve this to a w.h.p. result.

For maximum, balanced k-partitions (Lemma 62), we show that there does not exist *any* γ-friendly partitions when $\gamma \geq 0$ by showing that $\mathbb{E}X_\gamma$ vanishes (Lemma 75). It is unknown if γ-friendly balanced k-partitions will appear when γ is taken to be a small negative constant possibly dependent on k. We discuss this further in Section 5.4.

Similarly, for sum, balanced k-partitions (Lemma 63), we show that there does not exist *any* γ-friendly partitions *even when* $\gamma \lesssim_k -n$ by again showing that $\mathbb{E}X_\gamma$ vanishes (Lemma 76).

5.2.1 First Moment

Without loss of generality, let $\rho = (P_1, ..., P_k)$ be a random balanced k-partition where $P_i = \{x_{i,j} : j \in [n]\}$. As in the bisection case, fix some vertex, say $x_{1,1}$, and let $E_\gamma(\rho; x_{1,1})$ be the event that $x_{1,1}$ is γ-friendly with respect to ρ. Let $E_\gamma(\rho)$ be the event that ρ is γ-friendly for all vertices. We compute the probability that $E_\gamma(\rho)$ occurs for each type of generalization. Let $X_{i,j}$ be the random variable for the weight of the directed edge from vertex $x_{1,1}$ to $x_{i,j}$ where $X_{1,1} = 0$ and $X_{i,j} \sim \mathcal{N}(1/2, 1/4)$. Let $S_i = \sum_{j=1}^{n} X_{i,j}$ be the sum of weights of the edges from $x_{1,1}$ to the nodes in part P_i. Note that $S_1 \sim \mathcal{N}(n/2 - 1/2, n/4 - 1/4)$ and $S_i \sim \mathcal{N}(n/2, n/4)$ for $i > 1$. For simplicity we define $S_1' \coloneqq S_1 + X$ where $X \sim \mathcal{N}(1/2, 1/4)$ instead of S_1. Note that $\mathbb{P}\left[|S_1 - S_1'| \geq \log n\right] \leq O(1/n)$. This difference is negligible since $\gamma \lesssim_k \sqrt{n}$ and we can reduce γ by $O(\log n)$ without changing its order of magnitude. In the following, *when we write S_1 we mean S_1'*.

Lemma 74. (First Moment Average γ-Friendly Balanced k-Partition). *For integer $k \geq 3$, let $c_k = \Phi^{-1}(1 - 1/k)$. Then, for any constant $\delta > 0$, when $\gamma \coloneqq (c_k - \delta)\sigma$ for $\sigma \coloneqq \sqrt{\frac{n}{4}\left(1 + \frac{1}{k-1}\right)}$, the number of γ-friendly balanced k-partitions, denoted X_γ, satisfies $\mathbb{E}X_\gamma = e^{\Omega_k(n)}$. Further, if $\gamma \geq c_k\sigma$, then $\mathbb{E}X_\gamma = o(1)$.*

Proof. From the definition of γ-friendly average balanced k-partitions, we have

$$\mathbb{P}E_\gamma(\rho; x_{1,1}) = \mathbb{P}\left[S_1 - \frac{1}{k-1}\sum_{i=2}^{k} S_i \geq \gamma\right],$$

where $\frac{1}{k-1}\sum_{i=2}^{k} S_i \sim \mathcal{N}\left(\frac{n}{2}, \frac{n}{4(k-1)}\right)$ and $S_1 - \frac{1}{k-1}\sum_{i=2}^{k} S_i$ is distributed like $\mathcal{N}\left(0, \sigma^2\right)$. Standardizing $S_1 - \frac{1}{k-1}\sum_{i=2}^{k} S_i$, we have that $\mathbb{P}E_\gamma(\rho; x_{1,1}) = \mathbb{P}[\sigma Z \geq \gamma] = \mathbb{P}[Z \geq (c_k - \delta)] = 1 - \Phi(c_k - \delta)$. Recall that c_k was chosen so that $1 - \Phi(c_k) = 1/k$. Using first order approximations and the concavity of the Φ function on the interval $(0, \infty)$, we have that

$$\frac{1 + c_1}{k} = 1 - (\Phi(c_k) - \phi(c_k - 1)\delta) > 1 - \Phi(c_k - \delta) > 1 - (\Phi(c_k) - \phi(c_k)\delta) = \frac{1 + c_2}{k} \tag{5.28}$$

where $c_1 = k\phi(c_k - 1)\delta$ and $c_2 = k\phi(c_k)\delta$.

To obtain an upper and lower bound on $\mathbb{E}X_\gamma$, sum over the $\binom{kn}{n,\ldots,n}\frac{1}{k!}$ balanced k-partitions.

$$\begin{aligned}
\mathbb{E}X_\gamma &= \sum_{\rho} \mathbb{P}E_\gamma\left(\rho; x_{1,1}\right)^{kn} \\
&= \binom{kn}{n, \ldots, n} \frac{\mathbb{P}\left[E_\gamma\left(\rho; x_{1,1}\right)\right]^{kn}}{k!} \\
&= \frac{k^{nk}\sqrt{k} \cdot \mathbb{P}\left[E_\gamma\left(\rho; x_{1,1}\right)\right]^{kn}}{(2\pi n)^{(k-1)/2} k!} \left(1 + O_k\left(\frac{1}{n}\right)\right) && \text{(Lemma 64)} \\
&= \frac{k^{nk}\sqrt{k} \cdot \mathbb{P}\left[Z \geq (c_k - \delta)\right]^{kn}}{(2\pi n)^{(k-1)/2} k!} \left(1 + O_k\left(\frac{1}{n}\right)\right) \\
&= \exp\left(k(1 - \ln k) - \frac{k-1}{2}\ln(2\pi n) + \frac{1}{2}\ln k\right) \cdot \\
&\qquad\qquad k^{nk}\mathbb{P}[Z \geq (c_k - \delta)]^{kn}\left(1 + O_k\left(\frac{1}{n}\right)\right)
\end{aligned}$$

Apply the bound on $\mathbb{P}[Z \geq c_k - \delta]$ from Equation (5.28) to obtain

$$e^{knc_1} \geq \exp\left(-k(1 - \ln k) + \frac{k-1}{2}\ln(2\pi n) - \frac{1}{2}\ln k\right)\mathbb{E}X_\gamma \geq e^{knc_2} \tag{5.29}$$

From the above, we can deduce that $\mathbb{E}X_\gamma \gg 0$ when $\gamma = (c_k - \delta)\sigma$.

Suppose instead that $\gamma = \sigma c_k$. Then, since $1 - \Phi(c_k) = \frac{1}{k}$, and we have

$$\mathbb{E}X_\gamma = \exp\left(k(1 - \ln k) - \frac{k-1}{2}\ln(2\pi n) + \frac{1}{2}\ln k\right)\left(1 + O_k\left(\frac{1}{n}\right)\right) = o(1). \qquad \square$$

Lemma 75. (First Moment Max γ-Friendly Balanced k-Partition). *For $\gamma = 0$, the expected number of maximum γ-friendly balanced k-partitions is $\mathbb{E}X_\gamma = o(1)$.*

Proof. Again, we compute $\mathbb{P}E_\gamma(\rho; x_{1,1})$ for some vertex $x_{1,1}$. Let $S_1' \sim \mathcal{N}(n/2, n/4)$, and let $S_1 \sim \mathcal{N}(n/2 - 1/2, n/4 - 1/4)$ as it was originally defined. Then S_1' and $S_1 + \frac{1}{2}$ are two normal random variables with the same mean, and by Theorem 1.3 in [DMR23] (see also [AAL23]) we have that the total variation distance between them is bounded by $\mathrm{d}_{tv}(S_1', S_1 + \frac{1}{2}) \leq \frac{3}{2n}$. This means that there is a coupling between S_1' and $S_1 + \frac{1}{2}$ so that $\mathbb{P}[S_1 + \frac{1}{2} \neq S_1'] \leq \frac{3}{2n}$, and, therefore, $\mathbb{P}[S_1 \geq S_1'] \leq \frac{3}{2n}$. We then have that

$$\mathbb{P}\left[S_1 \geq \max_{i=2}^{k} S_i\right] \leq \mathbb{P}\left[S_1' \geq \max_{i=2}^{k} S_i\right] + \mathbb{P}[S_1 \geq S_1'] \leq \mathbb{P}\left[S_1' \geq \max_{i=2}^{k} S_i\right] + \frac{3}{2n},$$

since the events $S_1' < \max_{i=2}^k S_i$ and $S_1 < S_1'$ imply $S_1 < \max_{i=2}^k S_i$. Notice now that $S_1', S_2, ..., S_k$ are independent and identically distributed, and the event $S_1' \geq \max_{i=2}^k S_i$ is equivalent to the event $S_1' = \max\{S_1', S_2, ..., S_k\}$ which occurs with probability $\frac{1}{k}$ by symmetry. Therefore, we have $\mathbb{P}\left[S_1 \geq \max_{i=2}^k S_i\right] \leq \frac{1}{k} + \frac{3}{2n}$, which is equivalent to $\mathbb{P} E_\gamma(\rho; x_{1,1}) \leq \frac{1}{k} + \frac{3}{2n}$. The rest of the proof of the lemma follows as in the proof of Lemma 74. □

Lemma 76. (First Moment Sum γ-Friendly Balanced k-Partition). *For any $\delta > 0$, $\gamma = -\frac{n(k-2+\delta)}{2}$, the expected number of sum γ-friendly balanced k-partitions is $\mathbb{E}X_\gamma = o(1)$.*

Proof. The computations here are similar to those of Lemma 74, though we compare S_1 with $\sum_{i=2}^k S_i$ instead of $\frac{1}{k-1}\sum_{i=2}^k S_i$. Note that $S_1 - \sum_{i=2}^k S_i \sim \mathcal{N}\left(\frac{-1}{2} - \frac{n(k-2)}{2}, \frac{kn}{4} - \frac{1}{4}\right)$. By standard concentration inequalities, the probability that $|S_1 - \sum_{i=2}^k S_i|$ differs from the mean by more than $c\sqrt{kn \log kn}$ is inverse exponential with respect to c. Thus $\mathbb{P}\left(S_1 - \sum_{i=2}^k S_i \geq \gamma\right) \ll \frac{1}{k}$. □

5.2.2 Second Moment

Similar to the bisection case (Lemma 67), consider two balanced k-partitions ρ_1 and ρ_2 on the set of kn vertices with $\rho_1 = (P_{1,1}, ..., P_{1,k})$ and $\rho_2 = (P_{2,1}, ..., P_{2,k})$. We compute $\mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right]$ in Lemma 79 by summing over all possible overlaps between $P_{1,i}$ and $P_{2,i}$. In particular, let $\mathbf{Z}$ be the $k \times k$ matrix whose entries $z_{i,j}$ denote represents the *number* of vertices in part $P_{1,i}$ of ρ_1 which end up in part $P_{2,j}$ of ρ_2 i.e. $z_{i,j} := |P_{1,i} \cap P_{2,j}|$. Further, let $\mathbf{A}$ be a $k \times k$ matrix, whose entry $a_{i,j}$ is the *fraction* of vertices in part $P_{1,i}$ of ρ_1 which end up in part $P_{2,j}$ of ρ_2 i.e. $a_{i,j} := \frac{z_{i,j}}{n}$. Note that $\mathbf{A}$ is an element of the Birkhoff Polytope $\mathbf{P}_n \subseteq \mathbb{R}^{n \times n}$ which is the polytope of all doubly stochastic matrix.

We define several important functions which will appear throughout this section: $f : (0,1) \to \mathbb{R}$, $\psi : \mathbf{P}_k \to \mathbb{R}$, and $g : \mathbf{P}_k \to \mathbb{R}$. Define $f(a)$ as,

$$f(a) := \mathbb{P}[\sigma_1 Z_1 + \sigma_2 Z_2 \geq (c_k - \delta) \wedge \sigma_1 Z_1 - \sigma_2 Z_2 \geq (c_k - \delta)], \tag{5.30}$$

for standard Gaussians Z_1, Z_2, $c_k := \Phi^{-1}(1 - 1/k)$, $\sigma_1 := \sqrt{\frac{(1+a)k-2}{2(k-1)}}$, and $\sigma_2 := \sqrt{\frac{k(1-a)}{2(k-1)}}$ (these terms are exactly the same as those which appear in Assumption 24).

Define $\psi(\mathbf{A})$ as,

$$\psi(\mathbf{A}) := \prod_{i,j \in [k]} f(a_{i,j})^{a_{i,j}}. \tag{5.31}$$

Finally, define $g(\mathbf{A})$ as,

$$g(\mathbf{A}) := \frac{\psi(\mathbf{A})}{\prod_{i,j \in [k]} a_{i,j}^{a_{i,j}}} = \prod_{i,j \in [k]} \left(\frac{f(a_{i,j})}{a_{i,j}}\right)^{a_{i,j}}. \tag{5.32}$$

Before using the second moment method, we recall the definition of a high dimensional Laplace method in Theorem 77 and show that it applies to a special function in Lemma 78.

Theorem 77. (High-Dimensional Laplace Method, [Won01]). *Consider the Laplace-type integral*

$$J(n) = \int_D h(\mathbf{x}) e^{-n\zeta(\mathbf{x})} d\mathbf{x}. \tag{5.33}$$

If $\mathbf{x}_0$ is an interior point of $D \in \mathbb{R}^m$ and if the following hold:

1. *$J(n)$ converges absolutely for all $n \geq n_0$.*
2. *In L_2-norm, for every $\epsilon > 0$ we have $d(\epsilon) > 0$ where*

$$d(\epsilon) = \inf\{\zeta(\mathbf{x}) - \zeta(\mathbf{x}_0) : x \in D \text{ and } |\mathbf{x} - \mathbf{x}_0| \geq \epsilon\}.$$

3. *The Hessian matrix*

$$\left(\nabla^2 \zeta\right)(\mathbf{x}_0) = \left(\frac{\partial^2 \zeta}{\partial x_i \partial x_j}\right)\Big|_{\mathbf{x}=\mathbf{x}_0},$$

is positive definite.

Then for $n \to \infty$ the Laplace-type integral of Equation 5.33 has the Laplace approximation

$$J(n) \sim \left(\frac{2\pi}{n}\right)^{m/2} h(\mathbf{x}_0) \left(\det\left(\nabla^2\zeta\right)(\mathbf{x}_0)\right)^{-1/2} \exp\left(-n\zeta\left(\mathbf{x}_0\right)\right). \tag{5.34}$$

Lemma 78. *Suppose Assumption 24 holds. Let $\zeta(\mathbf{A}) := -\ln g(\mathbf{A})$ with g as defined in Equation (5.32), $\mathbf{A} \in \mathbf{P}_k$, and $\mathbf{A}^* := \mathbb{J}_k/k$. Then the conditions of Theorem 77 are satisfied for the following integral, and we have the approximation:*

$$\begin{aligned} J(n) &= \int_{\mathbf{P}_k} \exp\left(-n\zeta(\mathbf{A})\right) d\mathbf{A} \\ &= \left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{2\pi}{n}\right)^{(k-1)^2/2} \left(\det\left(\nabla^2\zeta\right)(\mathbf{A}^*)\right)^{-1/2} \exp\left(-n\zeta\left(\mathbf{A}^*\right)\right). \end{aligned}$$

Proof. There are three conditions required for Theorem 77. Consider each in-turn.

1. To see that the integral $J(n)$ converges on $\mathbf{P}_k$, note that $\exp\left(-n\zeta(\mathbf{A})\right)$ is a continuous function and that $\mathbf{P}_k$ is a compact set.
2. To show that $\forall \epsilon \in (0,1) : d(\epsilon) > 0$, fix $\epsilon > 0$ and assume $|\mathbf{A} - \mathbf{A}^*| \geq \epsilon$ in Frobenius norm. Note that Assumption 24, gives us that ζ has a unique maximum at $\mathbf{A}^*$ and is both continuous and differentiable, $(\nabla\zeta)(\mathbf{A}^*) = \mathbf{0}$. Second, $\left(\nabla^2\zeta\right)(\mathbf{A}^*) \succ \mathbf{0}$. By applying the Fundamental Theorem of Calculus, we have that $d(\epsilon) > 0$.
3. Positive definiteness of the Hessian matrix $\left(\nabla^2\zeta\right)(\mathbf{A}^*)$ is given in Claim 81.

Since all three conditions hold, we can apply the high-dimensional Laplace method to $\zeta(\mathbf{A})$. □

Now we define Lemma 79, but defer its proof until *after* Lemma 80.

Lemma 79. *Let $G \sim \mathcal{G}_N(kn)$ as in Definition 59. Let $\rho_1 = (P_{1,1}, ..., P_{1,k})$ and $\rho_2 = (P_{2,1}, ..., P_{2,k})$ be two balanced k-partitions of $V(G)$ and let $\rho_1 \cap \rho_2 = \mathbf{A}$ be a $k \times k$ matrix where the entry in row i, column j, denoted $a_{i,j}$, equals $|P_{1,i} \cap P_{2,j}|/n$. Let $\psi(\mathbf{A})$ be defined as*

$$\psi\left(\mathbf{A}\right) \coloneqq \prod_{i,j \in [k]} f(a_{i,j})^{a_{i,j}}.$$

as was stated in Equation (5.31). Then

$$\psi\left(\mathbf{A}\right)^n \geq \mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right].$$

Lemma 80. (Second Moment Average γ-Friendly Balanced k-Partition). *With X_γ from Lemma 74 and under Assumption 24, $(\mathbb{E}X_\gamma)^2 \gtrsim \mathbb{E}X_\gamma^2$.*

Proof. For $\mathbb{E}X_\gamma^2$, we group together pairs of (ρ_1, ρ_2) by the fraction of entries upon which they agree in each part, denoted by $\rho_1 \cap \rho_2$. Note that for a fixed $\mathbf{Z}$, the number of pairs (ρ_1, ρ_2) such that $\rho_1 \cap \rho_2 = \mathbf{Z}/n$ is the product of the $\binom{kn}{n, ..., n}$ choices for ρ_1 and the $\prod_{i \in [k]} \binom{n}{z_{i,1}, ..., z_{i,k}}$ choices for ρ_2 given ρ_1. Thus,

$$\mathbb{E}X_\gamma^2 = \binom{kn}{n, ..., n} \sum_{\mathbf{Z}} \mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right] \cdot \prod_{i \in [k]} \binom{n}{z_{i,1}, ..., z_{i,k}}. \tag{5.35}$$

We will see from Lemma 79 that $\psi\left(\mathbf{Z}/n\right)^n \geq \mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right]$ where ψ is defined in Equation (5.31). Let $Q_n = \sum_{\mathbf{Z}, \mathbf{A} \coloneqq \frac{1}{n}\mathbf{Z}} \psi(\mathbf{A})^n \cdot \left(\prod_{i \in [k]} \binom{n}{a_{i,1}n, ..., a_{i,k}n}\right)$. We want to approximate Q_n by an integral, so we use function $g(\mathbf{A})$, where

$$g(\mathbf{A}) \coloneqq \frac{\psi(\mathbf{A})}{\prod_{i,j \in [k]} a_{i,j}^{a_{i,j}}} = \prod_{i,j \in [k]} \left(\frac{f(a_{i,j})}{a_{i,j}}\right)^{a_{i,j}}.$$

as defined in Equation (5.32) to simplify the terms of Q_n similar to Equation (5.24) and Equation (5.25) for $g(a)$ in the bisection case.

By Stirling's approximation, we have that when every $a_{i,j}$ of $\mathbf{A}$ is bounded away from zero and one by a constant the following is true,

$$\left(\prod_{i \in [k]} \binom{n}{a_{i,1}n, ..., a_{i,k}n}\right) \psi(\mathbf{A})^n < \frac{g(\mathbf{A})^n}{(2\pi n)^{\frac{k(k-1)}{2}} \left(\prod_{i,j \in [k]} a_{i,j}\right)^{1/2}} \left(1 + O_k\left(\frac{1}{n}\right)\right). \tag{5.36}$$

Note that this does *not* apply in the tail, when some term $a_{i,j}$ is close to 0 or 1. Here we can use the unique maximum of g from Assumption 24, to see that $\lim_{n \to \infty} \frac{g(\mathbf{A})^n}{g(\mathbf{A}^*)^n} = 0$. Thus the terms of the sum Q_n in this interval is also bounded by the maximum value at $\mathbf{A}^*$.

We approximate Q_n by an integral and then, by Lemma 78, we can use the high-dimensional Laplace Method (Theorem 77) to approximate the integral . When replacing the sum over matrices $\mathbf{Z}$ whose rows and columns sum to n with stochastic matrices $\mathbf{A} \in \mathbf{P}_n$, we first must perform a change of variables. In particular, we pull out a factor of n from each independent entry to obtain a multiple of $n^{(k-1)^2}$ which we pull out of the integral.

$$\begin{aligned}
Q_n &= n^{(k-1)^2} \int_{\mathbf{P}_k} \frac{e^{n \ln g(\mathbf{A})}}{(2\pi n)^{\frac{k(k-1)}{2}} \left(\prod_{i,j\in[k]} a_{i,j}\right)^{1/2}} d\mathbf{A} \left(1 + O_k\left(\frac{1}{n}\right)\right) \\
&\lesssim \frac{n^{(k-1)^2} \left(\frac{2\pi}{n}\right)^{\frac{(k-1)^2}{2}} e^{n \ln g\left(\frac{\mathbb{J}_k}{k}\right)} \left(\det\left(\nabla^2|_{\mathbf{A}=\mathbb{J}_k/k}\right)\right)^{-1/2}}{(2\pi n)^{\frac{k(k-1)}{2}} k^{\frac{k}{2}}} \left(1 + O_k\left(\frac{1}{n}\right)\right) \\
&= \frac{n^{(k-1)^2} \left(\frac{2\pi}{n}\right)^{\frac{(k-1)^2}{2}} k^{nk} f\left(\frac{1}{k}\right)^{kn} \left(\det\left(\nabla^2|_{\mathbf{A}=\mathbb{J}_k/k}\right)\right)^{-\frac{1}{2}}}{(2\pi n)^{\frac{k(k-1)}{2}} k^{\frac{k}{2}}} \left(1 + O_k\left(\frac{1}{n}\right)\right) \\
&\lesssim_k \frac{k^{nk} f\left(\frac{1}{k}\right)^{kn}}{(2\pi n)^{\frac{(k-1)}{2}} k^{\frac{k}{2}}} \left(1 + O_k\left(\frac{1}{n}\right)\right)
\end{aligned}$$

where $\nabla^2|_{\mathbf{A}=\mathbb{J}_k/k}$ is the Hessian of $\ln g(\mathbf{A})$ evaluated at $\mathbb{J}_k/k$. By Assumption 24, g is concave in the neighbourhood of $\frac{1}{k}\mathbb{J}_k$, so $\det\left(\nabla^2|_{\mathbf{A}=\mathbb{J}_k/k}\right)$ is a constant dependent only on k. From Lemma 79, $f\left(1/k\right)$ is the Gaussian measure of a wedge of angle $\frac{\pi}{2}$ whose apex is at distance r from the origin, where

$$r := \frac{c_k - \gamma}{\sigma_1} = (c_k - \delta)\sqrt{\frac{2(k-1)}{\left(1 + \frac{1}{k}\right)k - 2}} = (c_k - \delta)\sqrt{2}.$$

This will be important when comparing with the first moment terms arising from the denominator.

For the denominator, $(\mathbb{E}X_\gamma)^2$, we have

$$\binom{kn}{n, \ldots, n} \left(\frac{\mathbb{P}[E_\gamma(\rho_1, x_{1,1})]^{kn}}{k!}\right)^2 = \frac{k^{nk}\sqrt{k}\left(\mathbb{P}[E_\gamma(\rho_1, x_{1,1})]^2\right)^{nk}}{(2\pi n)^{\frac{(k-1)}{2}} k!} \left(1 + O_k\left(\frac{1}{n}\right)\right)$$

Recall that $\left(\mathbb{P}E_\gamma(\rho_1, x_{1,1})\right)^2 = \mathbb{P}[Z_1 \geq (c_k - \delta) \wedge Z_2 \geq (c_k - \delta)]$ for independent standard Gaussians Z_1 and Z_2. This is the Gaussian measure of a wedge of angle $\frac{\pi}{2}$ at a distance $\sqrt{2}(c_k - \delta)$ away from the origin and *it is exactly* the value of $f(1/k)$.

Putting these bounds together, $\frac{\mathbb{E}X_\gamma^2}{(\mathbb{E}X_\gamma)^2} \lesssim \frac{k!}{k^{(k+1)/2}} \lesssim_k 1$ as required. □

Proof of Lemma 79. This will be similar to the proof of Claim 66. Fix a vertex $x_{1,1}$ in part $P_{1,1}$ of ρ_1 and part $P_{2,1}$ of ρ_2 and compute $\mathbb{P}\left[E_\gamma(\rho_1; x_{1,1}) \wedge E_\gamma(\rho_2; x_{1,1})\right]$. Let $a := a_{1,1}$.

Recall from Definition 16, that in order for $x_{1,1}$ to be γ-friendly with respect to ρ_1, we required that $\sum_{x_{i,j}\in P_{1,1}} X_{i,j} \geq \frac{1}{k-1}\sum_{x_{i,j}\in P_{1,2}\cup\cdots\cup P_{1,k}} X_{i,j} + \gamma$. Similarly, in order for $x_{1,1}$ to be γ-friendly with respect to ρ_2, we require that $\sum_{x_{i,j}\in P_{2,1}} X_{i,j} \geq \frac{1}{k-1}\sum_{x_{i,j}\in P_{2,2}\cup\cdots\cup P_{2,k}} X_{i,j} + \gamma$. Thus, we define

$$R_+ = \sum_{x_{i,j}\in P_{1,1}\cap P_{2,1}} X_{i,j} - \frac{1}{k-1}\left(\sum_{x_{i,j}\in (P_{1,2}\cup\cdots\cup P_{1,k})\cap(P_{2,2}\cup\cdots\cup P_{2,k})} X_{i,j}\right) \tag{5.37}$$

$$R_-^{(1)} = \sum_{x_{i,j}\in P_{1,1}\cap(P_{2,2}\cup\cdots\cup P_{2,k})} X_{i,j} \tag{5.38}$$

$$R_-^{(2)} = \sum_{x_{i,j}\in (P_{1,2}\cup\cdots\cup P_{1,k})\cap P_{2,1}} X_{i,j} \tag{5.39}$$

where these variables are distributed like

$$\begin{aligned} R_+ &\sim \mathcal{N}\left(\frac{an}{2}, \frac{an}{4}\right) - \frac{1}{k-1}\mathcal{N}\left(\frac{(k-1)n-(1-a)n}{2}, \frac{(k-1)n-(1-a)n}{4}\right) \\ &\sim \mathcal{N}\left(-\frac{(1-a)n}{2}\left(1-\frac{1}{k-1}\right), \frac{n}{4}\left(a+\frac{1}{k-1}-\frac{1-a}{(k-1)^2}\right)\right) \\ R_-^{(i)} &\sim \mathcal{N}\left(\frac{(1-a)n}{2}, \frac{(1-a)n}{4}\right) \quad \text{for } i\in\{1,2\}. \end{aligned}$$

Further define the following random variables and extract a factor of $\sqrt{\frac{nk}{4(1-k)}}$ to obtain

$$T_1 = R_+ + \frac{R_-^{(1)}+R_-^{(2)}}{2}\left(1-\frac{1}{k-1}\right) \tag{5.40}$$

$$T_2 = \frac{R_-^{(1)}-R_-^{(2)}}{2}\left(1+\frac{1}{k-1}\right) \tag{5.41}$$

where $T_1 \sim \mathcal{N}\left(0,\sigma_1^2\right)$ for $\sigma_1^2 = \frac{(1+a)k-2}{2(k-1)}$ and $T_2 \sim \mathcal{N}\left(0,\sigma_2^2\right)$ for $\sigma_2^2 = \frac{k(1-a)}{2(k-1)}$. Since $R_-^{(1)}$ and $R_-^{(2)}$ are independent and identically distributed Gaussian random variables, $R_-^{(1)}+R_-^{(2)}$ and $R_-^{(1)}-R_-^{(2)}$ are uncorrelated jointly Gaussian random variables, and are, therefore, independent as well. Thus T_1 and T_2 are independent random variables with joint pdf

$$f_{T_1,T_2}(t_1,t_2) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{t_1^2}{2\sigma_1^2}-\frac{t_2^2}{2\sigma_2^2}\right). \tag{5.42}$$

Standardizing T_1 and T_2, it follows that

$$\mathbb{P}\left[E_\gamma(\rho_1;x_{1,1})\wedge E_\gamma(\rho_2;x_{1,1})\right] = \mathbb{P}_{Z_i\sim\mathcal{N}(0,1)}\left[Z_1\sigma_1+Z_2\sigma_2 \geq (c_k-\delta)\wedge Z_1\sigma_1-Z_2\sigma_2\geq(c_k-\delta)\right].$$

Let $\delta\in(0,\min(c_k,1))$ where c_k is the constant such that $\mathbb{P}[Z\geq c_k]=1/k$. Then the region where both inequalities holds is a wedge with angle 2θ where $\theta := \arctan\left(\frac{\sigma_1}{\sigma_2}\right)$ and whose apex is at a distance $w := \frac{c_k-\gamma}{\sigma_1}$ from the origin, denoted $\mathcal{R}$. It is important to note that *both θ and w are functions of a, the fraction of overlap between part in ρ_1 and the part in ρ_2*. See Figure 5.3.

Note that $\mathbb{P}\left[E_\gamma(\rho_1;x_{1,1})\wedge E_\gamma(\rho_2;x_{1,1})\right] = \frac{\theta}{\pi}$ when $\gamma=0$. In general, when $\gamma>0$ we have that

$$f(a) := \mu(\mathcal{R}) = \int_{\mathcal{R}} \frac{e^{-\mathbf{x}^2/2}}{2\pi}d\mathbf{x}. \tag{5.43}$$

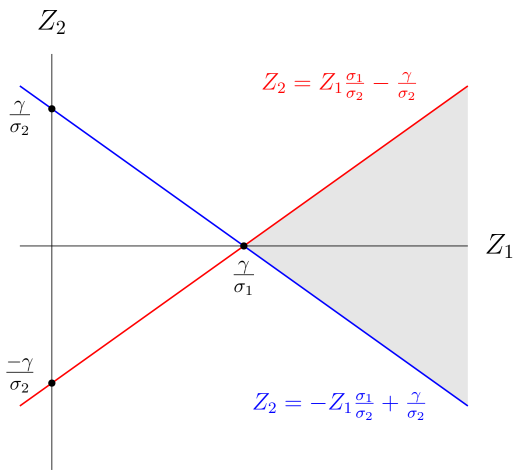
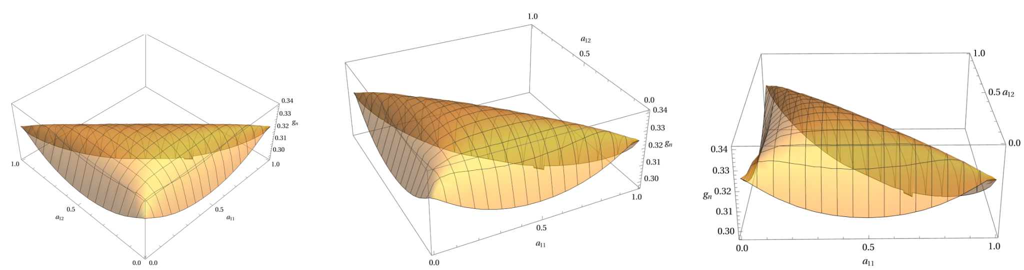


Figure 5.3: Region in gray depicts is the wedge $\mathcal{R}$ which is the domain of the two dimensional Gaussian (Z_1, Z_2) where $Z_1\sigma_1 + Z_2\sigma_2 \geq (c_k - \delta)$ and $Z_1\sigma_1 - Z_2\sigma_2 \geq (c_k - \delta)$.



(a) View 1 (b) View 2 (c) View 3

Figure 5.4: Plot of part of the function g for $k = 3$. Let $\mathbf{A}$ be the $k \times k$ stochastic matrix of the overlaps between the different parts. Recall that $g(\mathbf{A}) = \prod_{i,j}\left(\frac{f(a_{i,j})}{a_{i,j}}\right)^{a_{i,j}}$. We plotted three different views of the function $\prod_j \left(\frac{f(a_{1,j})}{a_{1,j}}\right)^{a_{1,j}}$ (one row of the $\mathbf{A}$ which is the product of three terms).

Note that when $\alpha = 1/k$, we have that $\frac{\sigma_1}{\sigma_2} = 1$ and $\theta = \frac{\pi}{4}$.

The above is an upper-bound for the probability that $x_{1,1}$ is γ-friendly with respect to the balanced k-partitions ρ_1 and ρ_2 assuming that $x_{1,1}$ is in part $P_{1,1}$ of ρ_1 and part $P_{2,1}$ of ρ_2. Over all n vertices in $P_{1,1}$, $a_{1,1}n$ will be similar to $x_{1,1}$ while the remaining $(1-a)n$ will be in part $P_{1,1}$ of ρ_1 but end up in part $P_{2,i}$ for ρ_2 where $i \in \{2, ..., k\}$. If we repeat the above computations for a vertex $x_{1,j}$ which ended up in $P_{2,i}$, we would find that $\mathbb{P}\left[E_\gamma(\rho_1; x_{1,j}) \wedge E_\gamma(\rho_2; x_{1,j})\right] \leq f(a_{1,i})$. The same is true for the $a_{i,j}n$ vertices $x_{i,j}$ in part $P_{1,i}$ of ρ_1 and part $P_{2,j}$ of ρ_2 i.e. $\mathbb{P}\left[E_\gamma(\rho_1; x_{1,j}) \wedge E_\gamma(\rho_2; x_{1,j})\right] \leq f(a_{i,j})$. Thus, if we define $\psi(\mathbf{A}) \coloneqq \prod_{i,j\in[k]} f(a_{i,j})^{a_{i,j}}$, we have that $\mathbb{P}\left[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)\right] \leq \psi(\mathbf{A})^n$. Let $g_n(\mathbf{A})$ be as defined in Equation (5.32). The plot of part of g_n is shown in Figure 5.4.

□

Note that Claim 81, stated below is related to Assumption 24. In particular, for a fixed integer $k \geq 3$, Assumption 24 says that $\ln g$ has a global optima at $\mathbb{J}_k/k$ while Claim 81 shows that $\ln g$ is concave at $\mathbb{J}_k/k$ and $\nabla^2 \ln g$ is continuous.

Claim 81. *Let $\mathbf{A}_k^*$ be the $k \times k$ matrix $\mathbb{J}_k/k$. Define the function*

$$\ln g(\mathbf{A}) = \sum_{i,j\in[k]} a_{i,j} \ln f_n(a_{i,j}) - a_{i,j} \ln a_{i,j}$$

where $g : \mathbf{P}_k \to \mathbb{R}$ was defined in Equation (5.32) as

$$g(\mathbf{A}) \coloneqq \prod_{i,j\in[k]} \left(\frac{f(a_{i,j})}{a_{i,j}}\right)^{a_{i,j}}$$

with the Birkhoff Polytope $\mathbf{P}_k$. Then $\nabla^2 \ln g$ is continuous and $\nabla^2 \ln g$ is concave and has a local maxima at $\mathbf{A}_k^*$.

Proof. In our application, we will identify the $k \times k$ matrix $\mathbf{A}_k$ with a vector $\mathbf{a}_k$ containing k^2 entries. The entries of $\mathbf{a}_k$ will be indexed just as they were in $\mathbf{A}_k$.

Let $F : \mathbb{R}^{(k-1)^2} \to \mathbb{R}^{k^2}$ be an affine transformation from the set of $(k-1)^2$ real-valued vectors to the set of k^2 real-valued vector where, given $\mathbf{a}_{k-1} \in \mathbb{R}^{(k-1)^2}$,

$$(F(\mathbf{a}_{k-1}))_{i,j} = \begin{cases} (\mathbf{a}_{k-1})_{i,j} & \text{if } i, j < k \\ 1 - \sum_{\ell=1}^{k-1} (\mathbf{a}_{k-1})_{\ell,j} & \text{if } i = k \text{ and } j < k \\ 1 - \sum_{\ell=1}^{k-1} (\mathbf{a}_{k-1})_{i,\ell} & \text{if } j = k \text{ and } i < k \\ 2 - k + \sum_{\ell_1,\ell_2\in[k-1]} (\mathbf{a}_{k-1})_{\ell_1,\ell_2} & \text{otherwise} \end{cases}.$$

Then we can write $F(\mathbf{a}) = \mathbf{M}\mathbf{a} + \mathbf{c}$ for $k^2 \times (k-1)^2$ matrix $\mathbf{M}$ and vector $\mathbf{c}$ with k^2 entries.

Further, let $\ln \tilde{g} : \mathbb{R}^{k\times k} \to \mathbb{R}$ be the same function as $\ln g$ where the domain is the set of all real vectors with k^2 entries as oppose to just the those drawn from the Birkhoff Polytope $\mathbf{P}_k$. Then, in terms of the indepenent variables, $\ln g\,(\mathbf{a}_{k-1}) = \ln \tilde{g} \circ F(\mathbf{a}_{k-1})$ and the Hessian of the $\ln g$ can be written as $\mathbf{M}^\top \nabla^2 \ln \tilde{g}(F(\mathbf{a}_{k-1}))\mathbf{M}$. For vectors $\mathbf{a}_k$ with k^2 independent entries $a_{i,j}$ for $i, j \in [k]$,

$$\ln \tilde{g}(\mathbf{a}_k) = \sum_{i,j\in[k]} a_{ij} \ln f\,(a_{i,j}) - a_{i,j} \ln a_{i,j}$$

and, in particular, for the vector $\mathbb{1}_{k^2}/k$, $\nabla \ln \tilde{g}\,(\mathbb{1}_{k^2}/k) = c \cdot \mathbb{I}_{k^2}$. Thus, to show that $\ln g$ is concave at $\mathbb{1}_{k^2}/k$, we will show that $\nabla^2 \ln g$ is negative definite, by showing that

$$\left(\frac{d}{da}\right)^2 a \ln f(a) - a \ln a < 0.$$

By logarithmic derivatives, we have that

$$\left(\frac{d}{da}\right)^2 a \ln f(a) - a \ln a = \frac{af''(a)}{f(a)} - a\left(\frac{f'(a)}{f(a)}\right)^2 + \frac{2f'(a)}{f(a)} - \frac{1}{a}. \tag{5.44}$$

Thus, we need to be able to compare the ratio of these terms involving $f(a)$, $f'(a)$, and $f''(a)$. Recall that $f:(0,1)\to\mathbb{R}$ is defined in Equation (5.30) as

$$f(a) := \mathbb{P}[\sigma_1 Z_1 + \sigma_2 Z_2 \geq (c_k - \delta) \wedge \sigma_1 Z_1 - \sigma_2 Z_2 \geq (c_k - \delta)],$$

where Z_1 and Z_2 are standard Gaussians, $c_k := \Phi^{-1}(1-1/k)$, $\sigma_1 := \sqrt{\frac{(1+a)k-2}{2(k-1)}}$, and $\sigma_2 := \sqrt{\frac{k(1-a)}{2(k-1)}}$. See Assumption 24. If we let the vector $\mathbf{x}$ be

$$\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 Z_1 + \sigma_2 Z_2 \\ \sigma_1 Z_1 - \sigma_2 Z_2 \end{bmatrix} \tag{5.45}$$

Then $\mathbf{x}$ has covariance matrix $\Sigma(a)$ with reciprocal $\Sigma^{-1}(a)$ where

$$\Sigma(a) = \begin{bmatrix} 1 & \frac{ak-1}{k-1} \\ \frac{ak-1}{k-1} & 1 \end{bmatrix} \qquad \Sigma^{-1}(a) = \frac{1}{\det(\Sigma)} \begin{bmatrix} 1 & \frac{1-ak}{k-1} \\ \frac{1-ak}{k-1} & 1 \end{bmatrix}. \tag{5.46}$$

Note that $\Sigma\left(\frac{1}{k}\right) = \Sigma^{-1}\left(\frac{1}{k}\right) = \mathbb{I}_k$. It follows that

$$f(a) = \int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty} \frac{\exp\left(\frac{-\mathbf{x}^\top \Sigma^{-1}\mathbf{x}}{2}\right)}{2\pi\sqrt{\det(\Sigma)}} dx_1 dx_2 \tag{5.47}$$

$$= \int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty} \frac{\exp\left(\frac{-\left(x_1^2 + 2x_1x_2\left(\frac{1-ak}{k-1}\right) + x_2^2\right)}{2\det(\Sigma)}\right)}{2\pi\sqrt{\det(\Sigma)}} dx_1 dx_2 \tag{5.48}$$

We want to take the derivative and second derivative of f with respect to a. To this end, we define $u(a, x_1, x_2)$ and $v(a)$ and compute their derivative and second derivative with respect to a.

$$u(a, x_1, x_2) := x_1^2 + 2x_1x_2\left(\frac{1-ak}{k-1}\right) + x_2^2 \tag{5.49}$$

$$u\left(\frac{1}{k}, x_1, x_2\right) = x_1^2 + x_2^2 \tag{5.50}$$

$$u'(a, x_1, x_2) = -2x_1x_2\left(\frac{k}{k-1}\right) \tag{5.51}$$

$$u''(a, x_1, x_2) = 0. \tag{5.52}$$

Further, we define

$$v(a) := \det(\Sigma)(a) = 1 - \left(\frac{1-ak}{k-1}\right)^2 \tag{5.53}$$

$$v\left(\frac{1}{k}\right) = 1 \tag{5.54}$$

$$v'(a) = \frac{2k(1-ak)}{(k-1)^2} \tag{5.55}$$

$$v'\left(\frac{1}{k}\right) = 0 \tag{5.56}$$

$$v''(a) = -\frac{2k^2}{(k-1)^2}. \tag{5.57}$$

Using these functions, we can write

$$f(a) = \frac{1}{2\pi}\int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty} v(a)^{-1/2}\exp\left(-\frac{u(a,x_1,x_2)}{2}v(a)^{-1}\right)dx_1dx_2 \tag{5.58}$$

$$f\left(\frac{1}{k}\right) = \frac{1}{2\pi}\int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty}\exp\left(-\frac{u\left(\frac{1}{k},x_1,x_2\right)}{2}\right)dx_1dx_2 \tag{5.59}$$

$$= \left(\int_{c_k-\delta}^{\infty}\frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}}dx\right)^2 = \left(1-\Phi\left(c_k-\delta\right)\right)^2 > \frac{1}{k^2} \tag{5.60}$$

where the last line follows by our choice of c_k. Now, take the derivative of f with respect to a.

$$f'(a) = \frac{1}{2\pi}\int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty}\left(\frac{d}{da}\right)v(a)^{-1/2}\exp\left(-\frac{1}{2}u(a,x_1,x_2)v(a)^{-1}\right)dx_1dx_2 \tag{5.61}$$

$$= \frac{1}{2\pi}\int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty} -\frac{1}{2}v(a)^{-3/2}\exp\left(-\frac{1}{2}u(a,x_1,x_2)v(a)^{-1}\right)u'(a,x_1,x_2) \tag{5.62}$$

$$+\frac{1}{2}v(a)^{-5/2}v'(a)\exp\left(-\frac{1}{2}u(a,x_1,x_2)v(a)^{-1}\right)u(a,x_1,x_2) \tag{5.63}$$

$$-\frac{1}{2}\left(v(a)\right)^{-3/2}v'(a)\exp\left(\frac{-u(a,x_1,x_2)}{2v(a)}\right)dx_1dx_2 \tag{5.64}$$

Observe that since $v'\left(\frac{1}{k}\right)=0$, the terms in Equation (5.63) and Equation (5.64) are equal to zero when we evaluate $f'(a)$ at $\frac{1}{k}$ and we have

$$f'\left(\frac{1}{k}\right) = \frac{1}{2\pi}\int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty} -\frac{1}{2}\exp\left(-\frac{u\left(\frac{1}{k},x_1,x_2\right)}{2}\right)u'\left(\frac{1}{k},x_1,x_2\right)dx_1dx_2 \tag{5.65}$$

$$= \left(\frac{k}{k-1}\right)\left(\int_{c_k-\delta}^{\infty}\frac{x\exp\left(-x^2/2\right)}{\sqrt{2\pi}}dx\right)^2 \tag{5.66}$$

$$= \left(\frac{k}{k-1}\right)\frac{\exp\left(-(c_k-\delta)^2\right)}{2\pi}. \tag{5.67}$$

Next, we take the second derivative of f with respect to a and only compute those terms which would be non-zero when $a=1/k$. For the term in Equation (5.62), if we apply the product rule on $v(a)^{3/2}$ or $u'(a,x_1,x_2)$, then the resulting terms will be equal to zero. Thus, we can only take a derivative with respect to $\exp\left(-\frac{u(a,x_1,x_2)}{2v(a)}\right)$. For the terms in Equation (5.63) and Equation (5.64), if we do not take a derivative with respect to $v'(a)$, then the result will have a multiple of $v'(a)$ and so the term will evalute to zero when we set $a=1/k$. It follows that

$$f''\left(\frac{1}{k}\right) = \frac{1}{2\pi}\int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty}\frac{1}{4}\exp\left(-\frac{u\left(\frac{1}{k},x_1,x_2\right)}{2}\right)u'\left(\frac{1}{k},x_1,x_2\right)^2 \tag{5.68}$$

$$+\frac{1}{2}v''\left(\frac{1}{k}\right)\exp\left(-\frac{u\left(\frac{1}{k},x_1,x_2\right)}{2}\right)u\left(\frac{1}{k},x_1,x_2\right) \tag{5.69}$$

$$-\frac{1}{2}v''\left(\frac{1}{k}\right)\exp\left(-\frac{u\left(\frac{1}{k},x_1,x_2\right)}{2}\right)dx_1dx_2 \tag{5.70}$$

Substituting in the definitions of $u\left(\frac{1}{k},x_1,x_2\right)$ from Equation (5.50), $u'\left(\frac{1}{k},x_1,x_2\right)$ from Equation (5.51), and $v''\left(\frac{1}{k}\right)$ from Equation (5.57), we have the following closed form for $f''\left(\frac{1}{k}\right)$,

$$f''\left(\frac{1}{k}\right) = \frac{1}{2\pi}\int_{c_k-\delta}^{\infty}\int_{c_k-\delta}^{\infty} \exp\left(-\frac{x_1^2+x_2^2}{2}\right) x_1^2x_2^2\left(\frac{k}{k-1}\right)^2 \tag{5.71}$$

$$- \left(\frac{k}{k-1}\right)^2 \exp\left(-\frac{x_1^2+x_2^2}{2}\right)\left(x_1^2+x_2^2\right) \tag{5.72}$$

$$+ \left(\frac{k}{k-1}\right)^2 \exp\left(-\frac{x_1^2+x_2^2}{2}\right) dx_1dx_2 \tag{5.73}$$

$$= \left(\left(\frac{k}{k-1}\right)\int_{c_k-\delta}^{\infty} \frac{\left(1-x^2\right)\exp\left(-x^2/2\right)}{\sqrt{2\pi}}dx\right)^2 \tag{5.74}$$

$$= \left(\frac{k}{k-1}\right)^2 \left(\frac{(c_k-\delta)^2\exp\left(-\left(c_k-\delta\right)^2\right)}{2\pi}\right) \tag{5.75}$$

Substitute the functions $f\left(\frac{1}{k}\right)$ from Equation (5.60), $f'\left(\frac{1}{k}\right)$ from Equation (5.67), and $f''\left(\frac{1}{k}\right)$ from Equation (5.75) into our original function stated in Equation (5.44) and let $w(a) := a\ln f\left(a\right) - \frac{1}{a}\ln\frac{1}{a}$ to obtain,

$$w''\left(\frac{1}{k}\right) = \frac{1}{k}\left(\frac{f''\left(\frac{1}{k}\right)}{f\left(\frac{1}{k}\right)}\right) - \frac{1}{k}\left(\frac{f'\left(\frac{1}{k}\right)}{f\left(\frac{1}{k}\right)}\right)^2 + \frac{2f'\left(\frac{1}{k}\right)}{f\left(\frac{1}{k}\right)} - k \tag{5.76}$$

$$= \frac{k}{(k-1)^2}\left(\frac{\exp\left(-\left(c_k-\delta\right)^2\right)}{2\pi\left(1-\Phi\left(c_k-\delta\right)\right)^2}\right)^2\left(\frac{\left(c_k-\delta\right)^2}{\frac{\exp(-(c_k-\delta)^2)}{2\pi(1-\Phi(c_k-\delta))^2}} - 1\right) \tag{5.77}$$

$$+ \frac{2k}{k-1}\left(\frac{\exp\left(-\left(c_k-\delta\right)^2\right)}{2\pi\left(1-\Phi\left(c_k-\delta\right)\right)^2}\right) - k \tag{5.78}$$

If we can show that the terms in Equation (5.77) and Equation (5.78) are negative, then we would have completed the proof that $\ln g$ is negative definite at $\mathbb{1}_{k^2}k$.

First, if ϕ is the PDF of the standard Gaussian, then

$$\phi(c_k-\delta) = \frac{\exp\left(-(c_k-\delta)^2/2\right)}{\sqrt{2\pi}}.$$

Further, we can write $1-\Phi(c_k-\delta)$ as the complement of the CDF of the standard Gaussian, denoted $\overline{\Phi}(c_k-\delta)$. Note that the ratio $\overline{\Phi}(x)/\phi(x)$ is the well know Mill's ratio. We denote this quantity by $m(x)$. Substituting Mill's ratio where appropriate in Equation (5.76), we have

$$w''\left(\frac{1}{k}\right) = \frac{k\left(\left(c_k-\delta\right)^2 m(c_k-\delta)^2 - 1 + 2(k-1)m(c_k-\delta)^2\right)}{(k-1)^2m(c_k-\delta)^4} - k. \tag{5.79}$$

Thus it suffices to show that

$$\frac{\left(\left(c_k-\delta\right)^2 m(c_k-\delta)^2 - 1 + 2(k-1)m(c_k-\delta)^2\right)}{(k-1)^2m(c_k-\delta)^4} < 1.$$

Recall that $m(x)$ can be bounded as

$$\frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x} < m(x) < \frac{4}{\sqrt{x^2+8}+3x}. \tag{5.80}$$

where the lower bound is due to Boyd [Boy59] and the upper bound is due to Sampford [Sam53]. Using the bounds in Equation (5.80), we have

$$m\left(x\right)^2 < \frac{16}{x^2+8+9x^2+6x^2\sqrt{1+8/x^2}} \tag{5.81}$$

$$= \frac{8}{4+5x^2+3x^2\sqrt{1+8/x^2}} \tag{5.82}$$

$$m\left(x\right)^2 > \frac{\pi^2}{2\pi+((\pi-1)^2+1)x^2+2(\pi-1)x^2\sqrt{1+(2\pi)/x^2}} \tag{5.83}$$

Plugging these into Equation (5.79) with $x = c_k - \delta$, we have in the numerator,

$$\left(x^2 m(x)^2 - 1 + 2(k-1)m(x)^2\right) \tag{5.84}$$

$$< \frac{8x^2}{4+5x^2+3x^2\sqrt{1+8/x^2}} - 1 + \frac{16(k-1)}{4+5x^2+3x^2\sqrt{1+8/x^2}} \tag{5.85}$$

$$= \frac{8x^2 - \left(4+5x^2+3x^2\sqrt{1+8/x^2}\right) + 16(k-1)}{4+5x^2+3x^2\sqrt{1+8/x^2}} \tag{5.86}$$

$$\leq \frac{3x^2\left(1-\sqrt{1+8/x^2}\right) + 16(k-1) - 4}{8x^2+4} \tag{5.87}$$

and in the denominator,

$$(k-1)^2 m(c_k-\delta)^4 = \frac{(k-1)^2\pi^4}{\left(2\pi+((\pi-1)^2+1)x^2+2(\pi-1)x^2\sqrt{1+(2\pi)/x^2}\right)^2} \tag{5.88}$$

$$\geq \frac{(k-1)^2\pi^4}{\left(2\pi+x^2\left(3.58642+4.2832\sqrt{1+(2\pi)/x^2}\right)\right)^2}. \tag{5.89}$$

Together, we have

$$\frac{\left(0.75\cdot x^2\left(1-\sqrt{1+8/x^2}\right)+4(k-1)-1\right)\cdot\left(2\pi+x^2\left(3.58642+4.2832\sqrt{1+(2\pi)/x^2}\right)\right)^2}{\left(2x^2+1\right)(k-1)^2\pi^4} \tag{5.90}$$

$$< \frac{\left(0.75\cdot x^2\left(1-\sqrt{1+8/x^2}\right)+4k-5\right)\cdot\left(2\pi+x^2\left(3.58642+4.2832\sqrt{1+(2\pi)/x^2}\right)\right)^2}{2x^2(k-1)^2\pi^4} \leq 1 \tag{5.91}$$

for integer $k \geq 7$ as $x > 1$ for small δ. When $k \in \{3,4,5,6\}$, we can explicitly evaluate Equation (5.76) to see that $w''\left(\frac{1}{k}\right) < 0$.

□

5.3 Bernoulli Edges

The general structure will again be via the second moment method though, as we will soon show, much of the groundwork has already been laid. In particular, Minzer, Sah, and Sawhney [MSS23] proved several lemmas — restated in Lemma 82, Lemma 83, and Lemma 84 for ease of use — which approximate the sum and difference of binomial random variables by the tail of a standard Gaussians random variable up to error on the order of $O(1/n)$. We will use these lemmas to translate the γ-friendly bisection problem in $G \sim \mathcal{G}_B(2n)$ to the same problem in $G \sim \mathcal{G}_N(2n)$. For the average, γ-friendly, balanced k-partitions problem, we can do the same but will require similar lemmas which we prove in the appendix, namely Lemma 85, Lemma 86, and Lemma 87 for balanced k-partitions.

Lemma 82 (A.1, [MSS23])**.** *Let $X_1 \sim \mathrm{Bin}(n, 1/2)$ and $X_2 \sim \mathrm{Bin}(n-\ell, 1/2)$ for $0 \le \ell \le n$. For n large,*

$$\mathbb{P}\left[X_1 - X_2 = t\right] = \frac{1}{\sqrt{\pi n}} \exp\left(-\left(\frac{t-\ell/2}{\sqrt{n}}\right)^2 + O\left(\frac{1}{n}\right)\right) \pm \exp\left(\Omega(\log n)^2\right) \tag{5.92}$$

Lemma 83 (A.2, [MSS23])**.**

$$\mathbb{P}\left[\mathrm{Bin}\left(n-1, 1/2\right) - \mathrm{Bin}\left(n, 1/2\right) \ge t\right] = \mathbb{P}_{Z \sim \mathcal{N}(0,1)}\left[Z \ge \frac{t\sqrt{2}}{\sqrt{n}}\right] + O\left(\frac{1}{n}\right).$$

Lemma 84 (A.4, [MSS23])**.** *Given $a \in (0,1)$, define*

$$g_\gamma(a) \coloneqq \mathbb{P}_{Z_i \sim \mathcal{N}(0,1)}\left[\sqrt{a}Z_1 + \sqrt{1-a}Z_2 \ge \gamma \wedge \sqrt{a}Z_1 - \sqrt{1-a}Z_2 \ge \gamma\right].$$

Let $X_1 \sim \mathrm{Bin}(k-1, 1/2)$, $X_2 \sim \mathrm{Bin}(n-k, 1/2)$, $X_3 \sim \mathrm{Bin}(k, 1/2)$, and $X_4 \sim \mathrm{Bin}(n-k, 1/2)$. Then for any integer Γ and $\gamma = \Gamma/\sqrt{n}$, we have the following approximation,

$$\mathbb{P}\left[X_1 - X_3 + X_2 - X_4 \ge \Gamma \wedge X_1 - X_3 - X_2 + X_4 \ge \Gamma\right] = g_\gamma(a) + O\left(\frac{1}{n}\right).$$

Lemma 85 (Binomal Local Limit)**.** *For non-negative integer $\ell \le n$,*

$$\mathbb{P}\left[\frac{\mathrm{Bin}((k-1)n, 1/2)}{k-1} - \mathrm{Bin}(n-\ell, 1/2) \in (t-1, t]\right] = \sqrt{\frac{2(k-1)}{\pi k n}} \exp\left(-\frac{2(k-1)}{nk}\left(t - \frac{\ell}{2}\right)^2 + O\left(\frac{1}{n}\right)\right) \pm \exp\left(-\Omega(\log n)^2\right) \tag{5.93}$$

Lemma 86 (Binomial Local Limit Difference)**.** *For constant non-negative integer $\ell \le n$ and constant k,*

$$\mathbb{P}\left[\mathrm{Bin}(n-1, 1/2) - \frac{\mathrm{Bin}((k-1)n, 1/2)}{k-1} \ge t\right] = \mathbb{P}\left[Z \ge t\sqrt{\frac{4(k-1)}{kn}}\right] + O_k\left(\frac{1}{n}\right).$$

Lemma 87 (Correlated Binomial Local Limit)**.** *Let an be an integer. Define $X_1 \sim \mathrm{Bin}(an, 1/2)$,*

$X_3 \sim \mathrm{Bin}((k-1)n-(1-a)n, 1/2)$, *and* $X_2, X_4 \sim \mathrm{Bin}((1-a)n, 1/2)$. *Then, for* γ,

$$\mathbb{P}\left[X_1 - \frac{X_3}{k-1} + X_2 - \frac{X_4}{k-1} \geq \gamma \wedge X_1 - \frac{X_3}{k-1} - \frac{X_2}{k-1} + X_4 \geq \gamma\right]$$
$$= \mathbb{P}_{Z_i \sim \mathcal{N}(0,1)}\left[\sigma_1 Z_1 + \sigma_2 Z_2 \geq \gamma \wedge \sigma_1 Z_1 - \sigma_2 Z_2 \geq \gamma\right] + O_k\left(\frac{1}{n}\right) \quad (5.94)$$

where $\sigma_1 = \sqrt{\frac{nk^2\left(a+\frac{k-2}{k}\right)}{8(k-1)^2}}$ *and* $\sigma_2 = \sqrt{\frac{nk^2(1-a)}{8(k-1)^2}}$.

5.3.1 γ-Friendly Bisection

Lemma 88. (First Moment γ-Friendly Bisection in $\mathcal{G}_B(2n)$). *Let* $G \sim \mathcal{G}_B(2n)$. *For any integer* $\gamma \leq -1$, *let* X_γ *be the number of* γ*-friendly bisections of* G. *Then*

$$\mathbb{E}[X_\gamma] = e^{c\sqrt{n}}. \quad (5.95)$$

Instead, when $\gamma \geq 0$, $\mathbb{E}X_\gamma = o(1)$.

Proof. Assume that $|V(G)| = 2n$. Let $V(G) = \{x_{i,j} : i \in [2], j \in [n]\}$. Let $\rho = (P_1, P_2)$ be a bisection of the vertices. Fix vertex $x_{1,1}$ and let $E_\gamma(\rho, x_{1,1})$ be the event that vertex $x_{1,1}$ is γ-friendly with respect to ρ. Further let $E_\gamma(\rho)$ be the event that *all* vertices are γ-friendly with respect to ρ. Since the edges are directed, the events $E_\gamma(\rho, x_{i,j})$ are independent and we have that $\mathbb{P}E_\gamma(\rho) = \mathbb{P}\left[E_\gamma(\rho, x_{1,1})\right]^{2n}$.

Let $X_{i,j}$ be the indicator for the edge $x_{1,1} \to x_{i,j}$ where $X_{1,1} = 0$ and $X_{i,j} = \mathrm{Bern}(1/2)$. Without loss of generality, let $P_1 = \{x_{1,j} : j \in [n]\}$ and $P_2 = \{x_{2,j} : j \in [n]\}$. Define $S_i = \sum_{j \in [n]} X_{i,j}$ and observe that $S_1 \sim \mathrm{Binom}(n-1, 1/2)$ while $S_2 \sim \mathrm{Binom}(n, 1/2)$. By Lemma 83, we have that

$$\mathbb{P}E_\gamma(\rho, x_{1,1}) = \mathbb{P}(S_1 - S_2 \geq \gamma) = \mathbb{P}_{Z \sim \mathcal{N}(0,1)}\left[Z \geq \gamma\sqrt{\frac{2}{n}}\right] + O\left(\frac{1}{n}\right).$$

If $\gamma \leq -1$, then we have that $\mathbb{P}E_\gamma(\rho, x_{1,1}) = \frac{1}{2} + \Omega\left(\frac{1}{\sqrt{n}}\right)$ and it follows that

$$\mathbb{E}X_\gamma = \sum_\rho \mathbb{P}E_\gamma(\rho) = \binom{2n}{n}\mathbb{P}\left[E_\gamma(\rho, x_{1,1})\right]^{2n} = \binom{2n}{n}\left(\frac{1}{2} + \frac{c}{\sqrt{n}}\right)^{2n} \approx e^{c\sqrt{n}} \gg 1.$$

Similarly, if $\gamma \geq 0$, then we have that $\mathbb{P}E_\gamma(\rho, x_{1,1}) = \frac{1}{2} + O\left(\frac{1}{n}\right)$ and it follows that

$$\mathbb{E}X_\gamma = \sum_\rho \mathbb{P}E_\gamma(\rho) = \binom{2n}{n}\mathbb{P}\left[E_\gamma(\rho, x_{1,1})\right]^{2n} = \binom{2n}{n}\left(\frac{1}{2} + \frac{c}{n}\right)^{2n} = o(1). \qquad \square$$

The second moment computations are similar.

Lemma 89. (Second moment for γ-Friendly Bisections in $\mathcal{G}_B(2n)$). *With* X_γ *as defined in Lemma 88,* $(\mathbb{E}X_\gamma)^2 \geq c \cdot \mathbb{E}X_\gamma^2$ *for a universal constant* c.

Proof. The proof is similar to Lemma 67. We sum over all pairs of bisections $\rho_1 = (P_{1,1}, P_{1,2})$ and $\rho_2 = (P_{2,1}, P_{2,2})$ and compute the probability that $\mathbb{P}[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)]$. Suppose that the *overlap of* ρ_1 *and* ρ_2 is $|P_{1,1} \cap P_{2,1}|$ and is equal to an integer z. Let $\alpha \coloneqq z/n$. Then, compute $\mathbb{P}[E_\gamma(\rho_1, x_{1,1}) \wedge E_\gamma(\rho_2, x_{1,1})]$ as follows.

Fix vertex $x_{1,1}$ and assume without loss of generality that it appears in $P_{1,1}$ of ρ_1 and $P_{2,1}$ of ρ_2. Let $X_{i,j}$ be the indicator for the edge $(x_{1,1}, x_{i,j})$ with $X_{1,1} = 0$. Let $S_{i,j} = \sum_{(k,\ell)\in P_{1,k}\cap P_{2,\ell}} X_{k,\ell}$ for $i, j \in \{1, 2\}$. By Lemma 84, we have

$$\begin{aligned}
&\mathbb{P}[E_\gamma(\rho_1, x_{1,1}) \wedge E_\gamma(\rho_2, x_{1,1})] \\
&\quad = \mathbb{P}[S_{1,1} + S_{1,2} - S_{2,1} - S_{2,2} \geq \gamma \wedge S_{1,1} - S_{1,2} + S_{2,1} - S_{2,2} \geq \gamma] \\
&\quad = \mathbb{P}_{Z_i\sim\mathcal{N}(0,1)}\left[\sqrt{\alpha}Z_1 + \sqrt{1-\alpha}Z_2 \geq \gamma \wedge \sqrt{\alpha}Z_1 - \sqrt{1-\alpha}Z_2 \geq \gamma\right] + O_k\left(\frac{1}{n}\right).
\end{aligned}$$

Notice that this is exactly the probability that we encountered in the proof of Claim 66 for the Gaussian distributed edges. Using the same argument, we have

$$\mathbb{P}[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)] \leq \psi_n(\alpha)^{2n},$$

where $\psi_n(\alpha)$ is defined in Equation (5.4). Plugging this probability into

$$\mathbb{E}X_\gamma^2 = \binom{2n}{n} \sum_{z=0}^{n} \binom{n}{z}^2 \mathbb{P}[E_\gamma(\rho_1) \wedge E_\gamma(\rho_2)],$$

the remainder of the argument is identical to that of Lemma 67. ☐

Proof of Theorem 21. For integer $\gamma \leq -1$, we want to show that every $G \sim \mathcal{G}_B(2n)$ contains a γ-friendly bisection with uniform positive probability. To this end, we have from Lemma 88 that $\mathbb{E}X_\gamma \gg O(1)$. Further, from Lemma 89, we have that the second moment of X_γ satisfies $(\mathbb{E}X_\gamma)^2 \geq c \cdot \mathbb{E}X_\gamma^2$ from a universal constant c. It follows from the Paley-Zygmund inequality that $\mathbb{P}[X_\gamma > 0] \geq \frac{(\mathbb{E}X_\gamma)^2}{\mathbb{E}X_\gamma^2} \geq \frac{1}{c}$ as required.

For the case where $\gamma \geq 0$, Lemma 88 also gives that $\mathbb{E}X_\gamma = o(1)$. ☐

5.3.2 γ-Friendly Balanced k-Partition

Lemma 90. (First Moment γ-Friendly Balanced k-Partition in $\mathcal{G}_N(kn)$). *Let* $G \sim \mathcal{G}_B(kn)$*. For any integer* $k > 2$ *and* $\gamma \leq \sigma(c_k - \delta)$ *where* $\delta > 0$*,* $c_k \coloneqq \Phi^{-1}(1 - 1/k)$*, and* $\sigma \coloneqq \sqrt{\frac{nk}{4(k-1)}}$*, let* X_γ *be the number of average* γ*-friendly balanced* k*-partitions of* G*. Then* $\mathbb{E}X_\gamma = e^{\Omega_k(n)}$*.*

Conversely, when $\gamma \geq \sigma c_k$*, we have* $\mathbb{E}X_\gamma = o(1)$*.*

Proof. Let X'_γ be the number of average γ-friendly balanced k-partitions of G' where $G' \sim \mathcal{G}_N(kn)$. From Lemma 86, we have that $\mathbb{E}X_\gamma = \mathbb{E}X'_\gamma + O_k(1)$. By Lemma 65, we have that $\mathbb{E}X'_\gamma = e^{\Omega_k(n)}$, then $\mathbb{E}X_\gamma = e^{\Omega_k(n)}$ as well. Conversely, when $\gamma \geq \sigma c_k$, $\mathbb{E}X'_\gamma = o(1)$ and $\mathbb{E}X_\gamma = o(1)$ as well. ☐

Lemma 91. (Second Moment γ-Friendly Balanced k-Partition in $\mathcal{G}_B(kn)$). *Suppose assumption 24 holds. With X_γ as defined in Lemma 90, $(\mathbb{E}X_\gamma)^2 \geq c \cdot \mathbb{E}X_\gamma^2$.*

Proof. Let $V(G) = \{x_{i,j} : i \in [k], j \in [n]\}$ be the set of vertices and let $\rho_1 = (P_{1,1}, ..., P_{1,k})$ and $\rho_2 = (P_{2,1}, ..., P_{2,k})$ be two balanced k-partition. Fix $x_{1,1}$ and let $E_\gamma(\rho_i, x_{1,1})$ and $E_\gamma(\rho_i)$ be the events that ρ_i is γ-friendly with respect to $x_{1,1}$ and γ-friendly respectively. As before, let $X_{i,j}$ be the indicator for the edge $x_{1,1}$ to $x_{i,j}$ where $X_{1,1} = 0$ and $X_{i,j} = \text{Bern}(1/2)$. Let $S_{i,j} = \sum_{x_{u,v} \in P_{i,j}} X_{u,v}$. Note that $S_{1,i_1}, S_{2,i_2} \sim \text{Binom}(n-1, 1/2)$ where $x_{1,1} \in P_{1,i_1}$ and $x_{1,1} \in P_{2,i_2}$. Otherwise $S_{1,i} \sim \text{Binom}(n, 1/2)$. Since our chosen γ will be much much greater than a constant, it suffices to take the distribution of S_{1,i_1}, S_{2,i_2} to be $\text{Binom}(n, 1/2)$ as well.

Suppose that $\mathbf{A}$ is the $k \times k$ stochastic which records the amount of overlap between the ρ_1 and ρ_2 i.e. entry $a_{i,j}$ in $\mathbf{A}$ is equal to $|P_{1,i} \cap P_{2,j}|/n$. Further, we define

$$X_1 = \sum_{x_{i,j} \in P_{1,1} \cap P_{2,1}} S_{i,j} \tag{5.96}$$

$$X_2 = \sum_{x_{i,j} \in P_{1,1} \cap (P_{2,2} \cup \cdots \cup P_{2,k})} S_{i,j} \tag{5.97}$$

$$X_3 = \sum_{x_{i,j} \in (P_{1,2} \cup \cdots \cup P_{1,k}) \cap (P_{2,2} \cup \cdots \cup P_{2,k})} S_{i,j} \tag{5.98}$$

$$X_4 = \sum_{x_{i,j} \in (P_{1,2} \cup \cdots \cup P_{1,k}) \cap P_{2,1}} S_{i,j} \tag{5.99}$$

It follows that

$$\begin{aligned}
&\mathbb{P}\left[E_\gamma(x_{1,1}; \rho_1) \wedge E_\gamma(x_{1,1}; \rho_2)\right] \\
&\qquad = \mathbb{P}\left[X_1 - \frac{X_3}{k-1} + X_2 - \frac{X_4}{k-1} \geq \gamma \wedge X_1 - \frac{X_3}{k-1} + X_4 - \frac{X_2}{k-1} \geq \gamma\right]
\end{aligned}$$

By Lemma 87 we have that

$$\begin{aligned}
&\mathbb{P}\left[X_1 - \frac{X_3}{k-1} + X_2 - \frac{X_4}{k-1} \geq \gamma \wedge X_1 - \frac{X_3}{k-1} + X_4 - \frac{X_2}{k-1} \geq \gamma\right] \\
&\qquad = \mathbb{P}\left[X_1' - \frac{X_3'}{k-1} + X_2' - \frac{X_4'}{k-1} \geq \gamma \wedge X_1' - \frac{X_3'}{k-1} + X_4' - \frac{X_2'}{k-1} \geq \gamma\right] + O_k\left(\frac{1}{n}\right)
\end{aligned}$$

where X_1', X_2', X_3', X_4' are Gaussian random variables with the same mean and variance as their Binomial counter part which appear in Lemma 79. The remainder on the proof follows exactly as Lemma 80. $\square$

Lemma 92. *For constant $k > 2$, let $G \sim \mathcal{G}_B(kn)$. Further, let $\gamma := \sigma(c_k - \delta)$ where $\delta > 0$, $c_k := \Phi^{-1}(1 - 1/k)$, $\sigma := \sqrt{\frac{nk}{4(k-1)}}$. Suppose that assumption 24 holds. Then G has an average γ-friendly balanced k-partition with uniform positive probability.*

Proof. We use the Paley-Zygmund inequality to show that $\mathbb{P}[X_\gamma \geq 0]$ is bounded below by a constant. From Lemma 90 we know that $\mathbb{E}X_\gamma \gg 1$. From Lemma 91 we know that there exists a constant c such that $(\mathbb{E}X_\gamma)^2 \geq c \cdot \mathbb{E}X_\gamma^2$. It follows that $\mathbb{P}\left[X_\gamma > \frac{1}{2}\right] \geq \left(1 - \frac{1}{2\mathbb{E}X_\gamma}\right)^2 \frac{1}{c} \geq \frac{1}{2c}$. $\square$

5.3.3 With High Probability

We can turn the with uniform positive probability result of Lemma 92 into the with high probability result of Theorem 25 by using a modified version of Theorem 94 from Minzer, Sah, and Sawhney [MSS23]. This result boosts constant probability events in the domain of *undirected* graphs with labeled vertices which are invariant under relabelling, i.e. $\mathcal{S}_n$-invariant, to $1-o(1)$ by reducing the friendliness *at every vertex* by $o(\sqrt{n})$. We will show that the same result applies to *directed* graphs. Since each vertex is $\Omega_k(\sqrt{n})$-friendly for the average γ-friendly balanced k-partition problem before boosting, they will remain so after.

We first define a metric between two *directed* graphs.

Definition 93. *For any two directed graphs on a fixed (labeled) vertex set V, let*

$$d(G,H) = \max_{v \in (G\Delta H)} \deg(v),$$

where $G\Delta H$ denotes the graph with edge set equal to the symmetric difference of G and H, namely, $(E(G)\backslash E(H)) \cup (E(H)\backslash E(G))$, and where $\deg(v)$ denotes the out degree of the vertex v in $G\Delta H$.

Theorem 94. (Similar to Theorem 2.2, [MSS23]). *Let $\mathcal{G}$ be a family of directed graphs on a labeled vertex set V that is invariant under permuting vertices, and let $\mu(\cdot)$ be the uniform measure on labeled directed graphs on n vertices. Let $\epsilon \in (0,1)$, possibly dependent on n. Suppose that $\mu(\mathcal{G}) \geq \epsilon$ and let*

$$\mathcal{G}_T = \left\{ H : \min_{G \in \mathcal{G}} d(G,H) \leq T \right\}.$$

Then $\mu\left(\mathcal{G}_{c\log(1/\epsilon)\sqrt{n/\log n}}\right) \geq 1-\epsilon$ for an absolute constant c.

The proof of Theorem 94 relies crucially upon a trichotomy on boolean functions f Lemma 95. Let N be the dimension of the domain of f i.e. $N = n(n-1)$ as this is the total number of edges for directed graphs. In the rest of this section, probabilities and expectations are taken with respect to the uniform measure on $\mathbb{F}_2^N$, which we identify with $\{0,1\}^N$. In Lemma 95 s_f^+ is the *positive sensitivity* of a function f defined as

$$s_f^+(\mathbf{x}) = \begin{cases} 0 & \text{if } f(\mathbf{x}) = 1 \\ \sum_{i\in[N]} f(\mathbf{x} \oplus \mathbf{e}_i) & \text{if } f(\mathbf{x}) = 0 \end{cases},$$

where $\mathbf{e}_i$ is the canonical basis vector in the i^{th} coordinate and $\oplus$ is addition in $\mathbb{F}_2^N$. Further, the *influence* of the i^{th} variable is $I_i[f] = \mathbb{P}\left[f(\mathbf{x}) \neq f(\mathbf{x}\oplus\mathbf{e}_i)\right]$, and the *total influence* of f is $I[f] = \sum_i I_i[f]$.

Lemma 95. (Similar to Lemma 2.5, [MSS23]). *Let $\epsilon \in (0,1/4]$ and $f : \{0,1\}^N \to \{0,1\}$ such that $\mathbb{E}f \in [\epsilon, 1-\epsilon]$. If f is symmetric under the natural $\mathcal{S}_n$-action of the vertices, then the following holds, where we sample $\mathbf{x}$ from the uniform measure on labeled directed graphs μ:*

(Inf 1) $\mathbb{P}\left[s_f^+(\mathbf{x}) > 0\right] \geq c\epsilon\sqrt{\frac{\log n}{n}}$.

(Inf 2) *There is an integer* $j \in [\lfloor(\log_2 n)/2\rfloor, \lfloor \log_2 N \rfloor]$ *such that*

$$2^j \mathbb{P}\left[s_f^+(\mathbf{x}) \in \left[2^j, 2^{j+1}\right]\right] \geq c\epsilon\sqrt{n}(\log n)^5.$$

(Inf 3) *There are integers* $\ell \in [2, \log^* n - 1]$, $j \in \left[\lfloor \log_2 n + 30 \log^{(\ell+1)n} \rfloor, \lfloor \log_2 n + 30 \log^{(\ell)} n \rfloor\right]$ *such that*

$$2^{j/2}\mathbb{P}\left[s_f^+(\mathbf{x}) \in \left[2^j, 2^{j+1}\right]\right] \geq \frac{c\epsilon\sqrt{n}}{\left(\log^{(\ell)} n\right)^2}.$$

Lemma 96. (Similar to Theorem 2.5 in [MSS23]). *If* $\sum_{i\in[n]} I_i[f]^2 \leq n^{-1/4}$,

$$\mathbb{E}\left[\sqrt{s_f^+(\mathbf{x})}\right] \gtrsim \mathrm{Var}(f)\sqrt{\log n}.$$

Instead of proving these theorems and lemmas which are slightly modified versions of the ones which appear in [MSS23], we will identify the necessary modifications to the proofs of the latter theorems and lemmas in order for them to apply to boolean functions over *directed* graphs as described in Theorem 94, Lemma 95, and Lemma 96.

The key component in [MSS23] to obtain a with high probability result is Lemma 2.5. In our paper, it will be the corresponding Lemma 95. Given Lemma 95, we can use it to construct an algorithm which doubles the size of $\mu(\mathcal{G})$ while possibly decreasing the friendliness at any vertex by at most $O\left(\sqrt{n/\log n}\right)$. After $\log(1/\epsilon)$ iterations we have $\mu(\mathcal{G}) \geq 1-\epsilon$.

Its proof starts by assuming that Inf 1 is false. Either $I[f] \geq \sqrt{n}(\log n)^6$, in which case

$$I[f] \asymp \sum_{j=0}^{\lfloor \log_2 N \rfloor} 2^j \mathbb{P}\left[s_f^+(\mathbf{x}) \in \left[2^j, 2^{j+1}\right]\right] \lesssim \sum_{j=\lfloor(\log_2 n)/2\rfloor}^{\lfloor \log_2 N \rfloor} 2^j \mathbb{P}\left[s_f^+(\mathbf{x}) \in \left[2^j, 2^{j+1}\right]\right] + \sqrt{n}$$

and Inf 2 is true by the pigeonhole principle, or $I[f] < \sqrt{n}(\log n)^6$ — the *only* difference between the directed and undirected case is the limit of the summations. In the latter case, we will show that Inf 3 is true.

Since $I[f] < \sqrt{n}(\log n)^6$ and f is symmetric with respect to the relabeling of vertices,

$$\sum_{(i_1,i_2)\in E(G)} I_{(i_1,i_2)}[f]^2 = \frac{I[f]^2}{n(n-1)} = \frac{(\log n)^{12}}{n-1} \leq \frac{1}{\sqrt{n}}.$$

From the statement of Lemma 95, we have $\mathbb{E}f \in [\epsilon, 1-\epsilon]$ so it follows that $\mathrm{Var}(f) \geq \epsilon$. Thus by the definition of $\mathbb{E}\left[\sqrt{s_f^+(\mathbf{x})}\right]$ and Lemma 96,

$$\sum_{j=0}^{\lfloor \log_2 N \rfloor} 2^{-j/2} w_j \asymp \mathbb{E}\left[\sqrt{s_f^+(\mathbf{x})}\right] \gtrsim \epsilon\sqrt{\log n},$$

where $w_j = 2^j \mathbb{P}\left[s_f^+(\mathbf{x}) \in \left[2^j, 2^{j+1}\right]\right]$. We bound the two parts of the sum as follows.

$$\sum_{j=0}^{\lfloor \log_2 n + 100 \rfloor} 2^{-j/2} w_j \leq 2^{(\log_2 n + 100)/2} \mathbb{P}\left[s_f^+(\mathbf{x}) > 0\right] \lesssim c\epsilon\sqrt{\log n}$$
$$\sum_{j=\lfloor \log_2 n + 30 \log\log n \rfloor}^{\lfloor \log_2 n(n-1) \rfloor} 2^{-j/2} w_j \leq 2^{-\lfloor \log_2 n + 30 \log\log n \rfloor/2} c\epsilon\sqrt{n}(\log n)^5 \lesssim \frac{c\epsilon}{\log n}$$

which implies that the summation from $\lfloor \log_2 n + 100 \rfloor$ to $\lfloor \log_2 n + 30 \log\log n \rfloor$ is bounded from below by $\Omega(\epsilon\sqrt{n})$. Conversely, Inf 3 shows that this sum is bounded above by $c\epsilon\sqrt{\log n}$ and we can arrive at a contradiction by choosing a sufficiently small value for c. This part of the proof is identical to that of Lemma 95 which appears in [MSS23]. We now prove Theorem 25.

Proof of Theorem 25. From Lemma 92 we know that for any $\delta > 0$, a random digraph drawn from $\mathcal{G}_B(kn)$ has an average $(c_k - \delta/2)\sqrt{\frac{nk}{(k-1)}}$-friendly balanced k-partition with uniform positive probability for a constant c_k dependent on k. In particular, if $\mathcal{P}$ is the family of graphs which have an average $(c_k - \delta/2)\sqrt{\frac{nk}{(k-1)}}$-friendly balanced k partition, then $\mu(\mathcal{P}) = \Omega_\delta(1)$. By choosing ϵ such that $\epsilon \geq \exp\left(-(\log n)^{1/4}\right)$, we can apply Theorem 94 to obtain $\mu\left(\mathcal{P}_{\sqrt{n}(\log n)^{-1/4}}\right) = 1 - \epsilon = 1 - o(1)$. $\mathcal{P}_{\sqrt{n}(\log n)^{-1/4}}$ is the family of graphs whose metric as stated in Definition 93 differs from $\mathcal{P}$ by at most $O(\sqrt{n}(\log n)^{-1/4})$. Since $\sqrt{n}(\log n)^{-1/4} = o\left(\sqrt{\frac{nk}{(k-1)}}\right)$, with high probability, a digraph drawn from $\mathcal{G}_B(kn)$ has an average $(c_k - \delta)\sqrt{\frac{nk}{(k-1)}}$-friendly balanced k partition. □

5.4 Open Problems

We summarize the results covered in this and prior works in Table 5.1. Of particular note, for the bisection case in $\mathcal{G}_B(2n)$, we were only able to show a positive result w.u.p.p. as opposed to w.h.p. as was done for the average balanced k-partition case and the undirected bisection case. Unlike those two cases where random graphs are likely to have $\Omega_k(\sqrt{n})$-friendly balanced partitions and can sacrifice $o_k(\sqrt{n})$ friends per vertex to use Theorem 94, random *directed* graphs typically only have $-O(1)$-friendly bisections and cannot make a similar sacrifice. Generally, it seems difficult to adapt sensitivity/influence based approaches on boolean functions such as those seen in Theorem 94 to cases with small margins of error.

We conjecture the following of maximum $-O_k(1)$-friendly balanced k-partitions in directed graphs.

Conjecture 97. *For $\gamma = O_k(1)$, graphs drawn from $\mathcal{G}_N(kn)$ and $\mathcal{G}_B(kn)$, have maximum γ-friendly balanced k-partitions w.h.p.*

Similar to the directed bisection case, even *if* it is possible to prove conjecture 97 w.u.p.p., it will probably require a new approach to turn such a result into a w.h.p. one.

As per Table 5.1, there are many open problems in the undirected case. Among these the average γ-friendly balanced k-partition is probably the most tractable as it is the most similar to the original

	Directed			Undirected		
Bisection in $\mathcal{G}_N(2n)$	Lemma 60			Implied by Theorem 1.3 [MSS23]		
Balanced k-Partition in $\mathcal{G}_N(kn)$	Average Lemma 61	Max Lemma 62	Sum Lemma 63	Average	Max	Sum
Bisection in $\mathcal{G}_B(2n)$	Theorem 21			Theorem 1.3 [MSS23]		
Balanced k-Partition in $\mathcal{G}_B(kn)$	Average Theorem 25	Max Theorem 26	Sum Theorem 27	Average	Max	Sum

Table 5.1: A survey of the current state of the γ-friendly balanced partition problem. Blue cells represent a positive result w.h.p. Yellow cells represent a positive result w.u.p.p. Orange cells represent a negative result which some hope of obtaining a positive result for smaller values of γ. And red cells represent a negative result which cannot be improved. All uncolored cells represent avenues for further exploration.

result of Minzer, Sah, Sawhney [MSS23]. A negative result for the sum γ-friendly balanced k-partition might also be easy to prove similar to the directed case though with a smaller threshold e.g. on the order of $-\Omega_k(\sqrt{n})$ as opposed to $-\Omega_k(n)$. A positive result for the maximum γ-friendly balanced k-partition problem in undirected graphs however, would be tricky as standard second moment techniques would be difficult to apply; unlike in the average case where overlapping parts are essentially independent, computing the overlap probability for maximum γ-friendly balanced k-partitions would be rather complex.

Further, we can ask: what happens when $k = \omega(1)$? Currently, k is a constant so it is negligible with respect to n. It might also be possible to prove results similar to Theorem 25 for k where $k/n \to 0$.

5.5 Proofs from the Bipartite Case

Proof of Claim 70. We show that $g_n'(\alpha) < 0$ for $\alpha \in [0, 0.01)$ using the original definition of $g_n(\alpha)$,

$$g_n(\alpha) = \frac{f_n(\alpha)^\alpha f_n(1-\alpha)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \text{ where } f_n(\alpha) = \frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{\delta^2}{\pi n}.$$

Using logarithmic derivatives, we have that $g_n' = (\ln g_n)' g_n$. By inspection, we note that $g_n(\alpha) > 0$ for all $\alpha \in [0, 0.01)$ so it suffices to show that $(\ln g_n)' < 0$ in order to show that $g_n' < 0$ and that g_n is monotonically decreasing on this interval.

By Equation (5.16), we have that

$$(\ln g_n)' = \ln\left(\frac{(1-\alpha) f_n(\alpha)}{\alpha f_n(1-\alpha)}\right) + \frac{\alpha f_n'(\alpha)}{f_n(\alpha)} - \frac{(1-\alpha) f_n'(1-\alpha)}{f_n(1-\alpha)}.$$

Note that $f_n(\alpha)' = \frac{1}{2\pi\sqrt{\alpha(1-\alpha)}} - \frac{\delta^2}{2\pi n\sqrt{\alpha(1-\alpha)}}$ and we have the following Taylor expansions:

$$\begin{aligned}
\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right) &= \sqrt{\alpha} + \frac{\alpha^{3/2}}{6} + O\left(\alpha^{5/2}\right) && \text{about } \alpha = 0\\
\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right) &= \frac{\pi}{2} - \sqrt{\alpha} - \frac{\alpha^{3/2}}{6} + O\left(\alpha^{5/2}\right) && \text{about } \alpha = 1\\
\sqrt{\frac{1-\alpha}{\alpha}} &= \frac{1}{\sqrt{\alpha}} - \frac{\sqrt{\alpha}}{2} - \frac{\alpha^{3/2}}{8} + O\left(\alpha^{5/2}\right) && \text{about } \alpha = 0\\
\sqrt{\frac{\alpha}{1-\alpha}} &= \sqrt{\alpha} + \frac{\alpha^{3/2}}{2} + O\left(\alpha^{5/2}\right) && \text{about } \alpha = 1.
\end{aligned}$$

Plugging these into the following expression, we have

$$\ln\left(\frac{(1-\alpha)f_n(\alpha)}{\alpha f_n(1-\alpha)}\right) = \ln\frac{(1-\alpha)\left(\frac{\sqrt{\alpha}+\frac{\alpha^{3/2}}{6}+O\left(\alpha^{5/2}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \left(\frac{1}{\sqrt{\alpha}} - \frac{\sqrt{\alpha}}{2} - \frac{\alpha^{3/2}}{8} + O\left(\alpha^{5/2}\right)\right)\frac{\delta^2}{\pi n}\right)}{\alpha\left(\frac{\frac{\pi}{2}-\sqrt{\alpha}-\frac{\alpha^{3/2}}{6}+O\left(\alpha^{5/2}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \left(\sqrt{\alpha} + \frac{\alpha^{3/2}}{2} + O\left(\alpha^{5/2}\right)\right)\frac{\delta^2}{\pi n}\right)}$$

$$\begin{aligned}
\frac{\alpha}{f_n(\alpha)} - \frac{(1-\alpha)}{f_n(1-\alpha)} &= \frac{\alpha}{\left(\frac{\sqrt{\alpha}+\frac{\alpha^{3/2}}{6}+O\left(\alpha^{5/2}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \left(\frac{1}{\sqrt{\alpha}} - \frac{\sqrt{\alpha}}{2} - \frac{\alpha^{3/2}}{8} + O\left(\alpha^{5/2}\right)\right)\frac{\delta^2}{\pi n}\right)}\\
&\quad - \frac{(1-\alpha)}{\left(\frac{\frac{\pi}{2}-\sqrt{\alpha}-\frac{\alpha^{3/2}}{6}+O\left(\alpha^{5/2}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}} + \left(\sqrt{\alpha} + \frac{\alpha^{3/2}}{2} + O\left(\alpha^{5/2}\right)\right)\frac{\delta^2}{\pi n}\right)}
\end{aligned}$$

Note that $f_n'(\alpha) = f_n'(1-\alpha)$ and $f_n'(\alpha) > 0$ when $\alpha \in [0, 0.01]$ so it suffices to show that $\frac{\alpha}{f_n(\alpha)} - \frac{(1-\alpha)}{f_n(1-\alpha)} \geq 0$. By performing the above computations, we see that $(\ln g_n)' < 0$. □

Proof of Claim 71. We show that $g_n'' > 0$. On this interval we can use the simplified definition of g_n which incorporates the simplified definition of f_n as seen in Equation (5.19). Using logarithmic differentials,

$$g_n'' = g_n\left(\ln g_n\right)'' + \left(\frac{g_n'}{g_n}\right)^2 g_n.$$

Since $g_n > 0$ on this interval, it suffices to show that $(\ln g_n)'' > 0$. By plugging in the values of f_n' and f_n'' from Equations (5.20) and (5.21) into Equation (5.17) and clearing the denominator, we have

$$\begin{aligned}
f_n(\alpha)^2 f_n(1-\alpha)^2\left(\ln g_n\right)'' &= \frac{f_n(\alpha)f_n(1-\alpha)}{\pi\sqrt{\alpha(1-\alpha)}}\left(f_n(1-\alpha) + f_n(\alpha)\right) && (5.100)\\
&\quad + \frac{(2\alpha-1)f_n(\alpha)f_n(1-\alpha)}{4\pi(\alpha(1-\alpha))^{3/2}}\left(\alpha f_n(1-\alpha) + (1-\alpha)f_n(\alpha)\right) && (5.101)\\
&\quad - \frac{1}{4\pi^2\alpha(1-\alpha)}\left(\alpha f_n(1-\alpha)^2 + (1-\alpha)f_n(\alpha)^2\right) - \frac{f_n(\alpha)^2 f_n(1-\alpha)^2}{\alpha(1-\alpha)}. && (5.102)
\end{aligned}$$

Note that the Taylor approximation of various components of the above function are

$$\begin{aligned}
\frac{1}{\alpha(1-\alpha)} &= \frac{1}{\alpha} + 1 + \alpha + \alpha^2 + O(\alpha^2) && \text{approximation at } \alpha = 0 \\
f_n(\alpha) &= \frac{\delta}{\sqrt{\pi n}} + 0.0318843 + 159957(\alpha - 0.01) \\
&\quad - 39.5853(\alpha - 0.01)^2 + O(\alpha - 0.01)^3 && \text{approximation at } \alpha = 0.01 \\
f_n(1-\alpha) &= \frac{\delta}{\sqrt{\pi n}} + 0.468116 - 1.59957(0.99 - \alpha) \\
&\quad - 39.5853(0.99 - \alpha)^2 + O(\alpha - 0.01)^3 && \text{approximation at } \alpha = 0.99
\end{aligned}$$

By plugging in the above approximations into the right-hand side of Equation (5.100), we see that it is positive for $\alpha \in [0.01, 0.1]$. □

Proof of Claim 72. To show $g_n(\alpha)' > 0$, consider the logarithmic derivative $g_n' = g_n(\ln g_n)'$. Since g_n positive on this interval, it suffices to show that $(\ln g_n)' > 0$ on $[0.1, \alpha_{\max} - \xi)$. From Equation (5.16), we have that

$$\begin{aligned}
(\ln g_n)' &= \ln \frac{f_n(\alpha)}{f_n(1-\alpha)} + \frac{\alpha f_n'(\alpha)}{f_n(\alpha)} - \frac{(1-\alpha) f_n'(1-\alpha)}{f_n(1-\alpha)} + \ln \frac{(1-\alpha)}{\alpha} \\
&\geq \ln \frac{f_n(\alpha)}{f_n(1-\alpha)} + \frac{\alpha f_n'(\alpha)}{f_n(\alpha)} - \frac{(1-\alpha) f_n'(1-\alpha)}{f_n(1-\alpha)} \\
&\geq \frac{2\left(\frac{(1-\alpha)f_n(\alpha)}{\alpha f_n(1-\alpha)} - 1\right)}{\left(\frac{(1-\alpha)f_n(\alpha)}{\alpha f_n(1-\alpha)} + 1\right)} + \frac{\alpha f_n'(\alpha)}{f_n(\alpha)} - \frac{(1-\alpha) f_n'(1-\alpha)}{f_n(1-\alpha)} \\
&\geq \frac{((1-\alpha)f_n(\alpha) - \alpha f_n(1-\alpha)) \cdot (2 f_n(\alpha) f_n(1-\alpha) - f_n'(\alpha)\left((1-\alpha)f_n(\alpha) + \alpha f_n(1-\alpha)\right))}{f_n(\alpha) f_n(1-\alpha)\left((1-\alpha)f_n(\alpha) + \alpha f_n(1-\alpha)\right)}.
\end{aligned}$$

where the first inequality follows from the lower-bound $\ln(x) \geq 2(x-1)/(x+1)$ for $x \geq 1$ in Equation (3) of [Top04] and the second follows as $f_n'(\alpha) = f_n'(1-\alpha)$. It remains to show $2f_n(\alpha)f_n(1-\alpha) \geq f_n'(\alpha)\left((1-\alpha)f_n(\alpha) + \alpha f_n(1-\alpha)\right)$. Observe that

$$\begin{aligned}
2f_n(\alpha)f_n(1-\alpha) &= 2\left(\frac{\arctan\sqrt{\frac{\alpha}{1-\alpha}}}{\pi} + \frac{\delta}{\sqrt{\pi n}}\right) \cdot \left(\frac{\arctan\sqrt{\frac{1-\alpha}{\alpha}}}{\pi} + \frac{\delta}{\sqrt{\pi n}}\right) \\
&= 2\left(\frac{\arctan\sqrt{\frac{\alpha}{1-\alpha}} \cdot \arctan\sqrt{\frac{1-\alpha}{\alpha}}}{\pi^2} + \delta\sqrt{\frac{\pi}{n}} + \frac{\delta^2}{\pi n}\right) \\
&\geq \frac{2}{\pi^2} \arctan\sqrt{\frac{\alpha}{1-\alpha}} \cdot \arctan\sqrt{\frac{1-\alpha}{\alpha}} + \frac{\delta}{\sqrt{n}} \\
f_n'(\alpha)\left((1-\alpha)f_n(\alpha) + \alpha f_n(1-\alpha)\right) &= f_n'(\alpha)\left(\frac{(1-\alpha)\arctan\sqrt{\frac{\alpha}{1-\alpha}}}{\pi} + \frac{\alpha \arctan\sqrt{\frac{1-\alpha}{\alpha}}}{\pi} + \frac{\delta}{\sqrt{\pi n}}\right) \\
&\leq \frac{(1-\alpha)\arctan\sqrt{\frac{\alpha}{1-\alpha}} + \alpha\arctan\sqrt{\frac{1-\alpha}{\alpha}}}{2\pi^2\sqrt{\alpha(1-\alpha)}} + \frac{\delta}{\sqrt{\pi n}}
\end{aligned}$$

since $\frac{1}{2\pi\sqrt{\alpha(1-\alpha)}} < 1$ for $\alpha \in (0.1, \alpha_{\max} - \xi]$. Thus by observing that the function

$$E(\alpha) = \frac{2}{\pi^2} \arctan \sqrt{\frac{\alpha}{1-\alpha}} \cdot \arctan \sqrt{\frac{1-\alpha}{\alpha}} - \frac{(1-\alpha)\arctan\sqrt{\frac{\alpha}{1-\alpha}} + \alpha \arctan \sqrt{\frac{1-\alpha}{\alpha}}}{2\pi^2\sqrt{\alpha(1-\alpha)}}$$

is positive for $\alpha \in (0.1, \alpha_{\max} - \xi]$, we have that g_n' is monotonically increasing. See Figure 5.5.

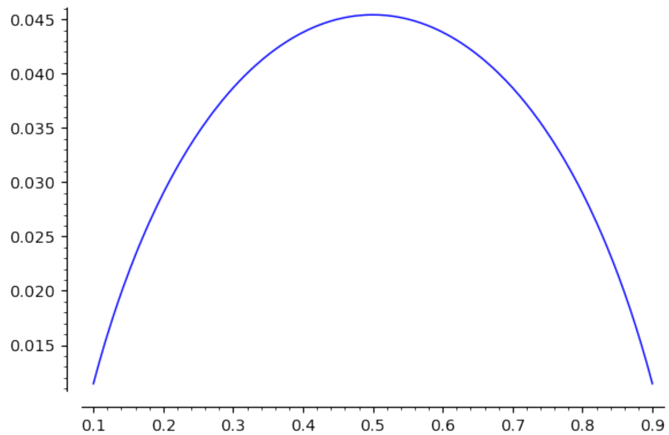


Figure 5.5: Graph of $E(\alpha) = \frac{2}{\pi^2} \arctan \sqrt{\frac{\alpha}{1-\alpha}} \cdot \arctan \sqrt{\frac{1-\alpha}{\alpha}} - \frac{(1-\alpha)\arctan\sqrt{\frac{\alpha}{1-\alpha}} + \alpha \arctan \sqrt{\frac{1-\alpha}{\alpha}}}{2\pi^2\sqrt{\alpha(1-\alpha)}}$.

To see that $(1-\alpha)f_n(\alpha) \geq \alpha f_n(1-\alpha)$, define $t(\alpha) = \alpha / f_n(\alpha)$ and show that $t(\alpha)$ is monotonically increasing by considering its derivative.

$$\begin{aligned}
t'(\alpha) &= \frac{1}{f_n(\alpha)}\left(1 - \frac{\alpha f_n'(\alpha)}{f_n(\alpha)}\right) \\
&= \frac{1}{f_n(\alpha)}\left(1 - \left(\frac{1}{2\pi}\sqrt{\frac{\alpha}{1-\alpha}}\right) \cdot \left(\frac{\arctan\left(\sqrt{\frac{\alpha}{1-\alpha}}\right)}{\pi} + \frac{\delta}{\sqrt{\pi n}}\right)^{-1}\right) \\
&\geq \frac{1}{f_n(\alpha)}\left(1 - \frac{1}{2}\left(1 - \frac{a}{3(1-\alpha)}\right)^{-1}\right) \geq 0.
\end{aligned}$$

for $\alpha \in (0.1, \alpha_{\max} - \xi]$, since $\arctan(x) \geq x - x^3/3$. □

5.6 Binomial Distribution Computations

5.6.1 Bisection

The follows are corollaries of lemmas of [MSS23] which approximate the difference of binomials by standard Gaussians.

Lemma 82 (A.1, [MSS23])**.** *Let* $X_1 \sim \mathrm{Bin}(n, 1/2)$ *and* $X_2 \sim \mathrm{Bin}(n-\ell, 1/2)$ *for* $0 \leq \ell \leq n$. *For* n

large,

$$\mathbb{P}\left[X_1 - X_2 = t\right] = \frac{1}{\sqrt{\pi n}} \exp\left(-\left(\frac{t-\ell/2}{\sqrt{n}}\right)^2 + O\left(\frac{1}{n}\right)\right) \pm \exp\left(\Omega(\log n)^2\right) \tag{5.92}$$

Lemma 83 (A.2, [MSS23])**.**

$$\mathbb{P}\left[\mathrm{Bin}\left(n-1,1/2\right) - \mathrm{Bin}\left(n,1/2\right) \geq t\right] = \mathbb{P}_{Z\sim\mathcal{N}(0,1)}\left[Z \geq \frac{t\sqrt{2}}{\sqrt{n}}\right] + O\left(\frac{1}{n}\right).$$

Lemma 84 (A.4, [MSS23])**.** *Given $a \in (0,1)$, define*

$$g_\gamma(a) \coloneqq \mathbb{P}_{Z_i\sim\mathcal{N}(0,1)}\left[\sqrt{a}Z_1 + \sqrt{1-a}Z_2 \geq \gamma \wedge \sqrt{a}Z_1 - \sqrt{1-a}Z_2 \geq \gamma\right].$$

Let $X_1 \sim \mathrm{Bin}(k-1,1/2)$, $X_2 \sim \mathrm{Bin}(n-k,1/2)$, $X_3 \sim \mathrm{Bin}(k,1/2)$, and $X_4 \sim \mathrm{Bin}(n-k,1/2)$. Then for any integer Γ and $\gamma = \Gamma/\sqrt{n}$, we have the following approximation,

$$\mathbb{P}\left[X_1 - X_3 + X_2 - X_4 \geq \Gamma \wedge X_1 - X_3 - X_2 + X_4 \geq \Gamma\right] = g_\gamma(a) + O\left(\frac{1}{n}\right).$$

5.6.2 Balanced k-Partition

We need probabilities similar to the above in order to translate our balanced k-part γ friendly results for $\mathcal{G}_\mathcal{N}$ into the same results for $\mathcal{G}_\mathcal{B}$.

Lemma 85 (Binomal Local Limit)**.** *For non-negative integer $\ell \leq n$,*

$$\mathbb{P}\left[\frac{\mathrm{Bin}((k-1)n,1/2)}{k-1} - \mathrm{Bin}(n-\ell,1/2) \in (t-1,t]\right] = \\ \sqrt{\frac{2(k-1)}{\pi k n}} \exp\left(-\frac{2(k-1)}{nk}\left(t-\frac{\ell}{2}\right)^2 + O\left(\frac{1}{n}\right)\right) \pm \exp\left(-\Omega(\log n)^2\right) \tag{5.93}$$

Proof. It suffices to consider the case where $|t| \geq \sqrt{n}\log n/4$. In the following, let $\tau = \frac{i}{k-1} - \frac{n}{2}$,

$$
\begin{aligned}
&\mathbb{P}\left[\frac{\mathrm{Bin}((k-1)n,1/2)}{k-1} - \mathrm{Bin}(n-\ell,1/2) \in (t-1,t]\right] \\
&= \frac{1}{2^{nk-\ell}} \sum_{|i/(k-1)-n/2| \leq \sqrt{n}\log n} \binom{n(k-1)}{i}\binom{n-\ell}{\lfloor \frac{i}{k-1}\rfloor - t} \pm \exp\left(-\Omega(\log n)^2\right) \\
&= \frac{1}{2^{nk-\ell}} \sum_{|\tau| \leq \sqrt{n}\log n} \binom{n(k-1)}{\frac{n(k-1)}{2} + \tau(k-1)}\binom{n-\ell}{\frac{n}{2} + (\lfloor \frac{i}{k-1}\rfloor - \frac{n}{2}) - t} \pm \exp\left(-\Omega(\log n)^2\right) \\
&= \frac{2}{\sqrt{\pi^2(k-1)n^2}} \sum_{|\tau| \leq \sqrt{n}\log n} \exp\left(-\frac{2\tau^2(k-1)}{n}\right) \exp\left(-\frac{2\left(\lfloor \frac{i}{k-1}\rfloor - n/2 - t + \ell/2\right)^2}{n}\right) \\
&\qquad \cdot \left(1 + O\left(\frac{1}{n}\right)\right) \pm \exp\left(-\Omega(\log n)^2\right) \\
&\approx \frac{\exp\left(-\frac{2(k-1)}{nk}\left(t - \frac{\ell}{2}\right)^2\right)}{\sqrt{\pi k n/(2(k-1))}} \\
&\qquad \cdot \sum_{|\tau| \leq \sqrt{n}\log n} \sqrt{\frac{2k}{\pi n (k-1)^2}} \exp\left(-\frac{2k}{n(k-1)^2}\left(\tau(k-1) - \frac{t(k-1)}{k} + \frac{\ell(k-1)}{2k}\right)^2\right) \\
&\qquad \cdot \left(1 + O\left(\frac{1}{n}\right)\right) \pm \exp\left(-\Omega(\log n)^2\right) \\
&= \sqrt{\frac{2(k-1)}{\pi k n}} \exp\left(-\frac{2(k-1)}{nk}\left(t - \frac{\ell}{2}\right)^2 + O\left(\frac{1}{n}\right)\right) \pm \exp\left(-\Omega(\log n)^2\right)
\end{aligned}
$$

where we use the approximation $i - (k-1) \leq \lfloor \frac{i}{k-1}\rfloor (k-1) \leq i$ for the second to last inequality, and the sum via a Riemann integral to obtain the last equality. ☐

Lemma 86 (Binomial Local Limit Difference)**.** *For constant non-negative integer $\ell \leq n$ and constant k,*

$$
\mathbb{P}\left[\mathrm{Bin}(n-1,1/2) - \frac{\mathrm{Bin}((k-1)n,1/2)}{k-1} \geq t\right] = \mathbb{P}\left[Z \geq t\sqrt{\frac{4(k-1)}{kn}}\right] + O_k\left(\frac{1}{n}\right).
$$

Proof. Using Lemma 85, we have that

$$\mathbb{P}\left[\mathrm{Bin}(n-1,1/2)-\frac{\mathrm{Bin}((k-1)n,1/2)}{k-1}\geq t\right]=\mathbb{P}\left[\frac{\mathrm{Bin}((k-1)n,1/2)}{k-1}-\mathrm{Bin}(n-1,1/2)\leq -t\right]$$
$$=\sum_{-\sqrt{n}\log n\leq x\leq -t}\sqrt{\frac{2(k-1)}{\pi kn}}\exp\left(-\frac{2(k-1)}{nk}\left(x-\frac{1}{2}\right)^2+O_k\left(\frac{1}{n}\right)\right)\pm\exp\left(-\Omega(\log n)^2\right).$$
$$=\sum_{-\sqrt{n}\log n\leq x\leq -t}\sqrt{\frac{2(k-1)}{\pi kn}}\exp\left(-\frac{2(k-1)}{nk}\left(x-\frac{1}{2}\right)^2\right)+O_k\left(\frac{1}{n}\right)$$
$$=\int_{-\infty}^{-t+1/2}\sqrt{\frac{2(k-1)}{\pi kn}}\exp\left(-\frac{2(k-1)}{nk}\left(x-\frac{1}{2}\right)^2\right)dx+O_k\left(\frac{1}{n}\right)$$
$$=\int_{-\infty}^{-t}\sqrt{\frac{2(k-1)}{\pi kn}}\exp\left(-\left(x\sqrt{\frac{2(k-1)}{nk}}\right)^2\right)dx+O_k\left(\frac{1}{n}\right)$$
$$\approx\int_{-\infty}^{-t\sqrt{\frac{4(k-1)}{kn}}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx+O_k\left(\frac{1}{n}\right)$$
$$=\mathbb{P}_{Z\sim\mathcal{N}(0,1)}\left[Z\geq t\sqrt{\frac{4(k-1)}{kn}}\right]+O_k\left(\frac{1}{n}\right)$$

Using the midpoint rule when approximating the sum with the integral. □

Lemma 87 (Correlated Binomial Local Limit)**.** *Let an be an integer. Define* $X_1\sim\mathrm{Bin}(an,1/2)$, $X_3\sim\mathrm{Bin}((k-1)n-(1-a)n,1/2)$, *and* $X_2,X_4\sim\mathrm{Bin}((1-a)n,1/2)$. *Then, for* γ,

$$\mathbb{P}\left[X_1-\frac{X_3}{k-1}+X_2-\frac{X_4}{k-1}\geq\gamma\wedge X_1-\frac{X_3}{k-1}-\frac{X_2}{k-1}+X_4\geq\gamma\right]$$
$$=\mathbb{P}_{Z_i\sim\mathcal{N}(0,1)}\left[\sigma_1Z_1+\sigma_2Z_2\geq\gamma\wedge\sigma_1Z_1-\sigma_2Z_2\geq\gamma\right]+O_k\left(\frac{1}{n}\right)\quad(5.94)$$

where $\sigma_1=\sqrt{\frac{nk^2\left(a+\frac{k-2}{k}\right)}{8(k-1)^2}}$ *and* $\sigma_2=\sqrt{\frac{nk^2(1-a)}{8(k-1)^2}}$.

Proof. We sum over all possible values of $X_1-\frac{X_3}{k-1}+X_2-\frac{X_4}{k-1}$ and $X_1-\frac{X_3}{k-1}-\frac{X_2}{k-1}+X_4$ which are greater than or equal to γ. In particular, let $X_1-\frac{X_3}{k-1}=\gamma-t$, $X_2-\frac{X_4}{k-1}=t_1$ and $X_4-\frac{X_2}{k-1}=t_2$

such that $t_1, t_2 \geq t$. We will sum over all possible values of t, t_1, and t_2.

$$\begin{aligned}
&\mathbb{P}\left[X_1 - \frac{X_3}{k-1} + X_2 - \frac{X_4}{k-1} \geq \gamma \wedge X_1 - \frac{X_3}{k-1} - \frac{X_2}{k-1} + X_4 \geq \gamma\right] \\
&= \sum_t \mathbb{P}\left[X_1 - \frac{X_3}{k-1} = \gamma - t\right] \sum_{t_1 \geq t,\ t_2 \geq t} \mathbb{P}\left[X_2 - \frac{X_4}{k-1} = t_1 \wedge X_4 - \frac{X_2}{k-1} = t_2\right] \\
&= \sum_t \mathbb{P}\left[X_1 - \frac{X_3}{k-1} = \gamma - t\right] \\
&\quad \cdot \sum_{t_1 \geq t,\ t_2 \geq t} \mathbb{P}\left[X_2 = \frac{(t_1(k-1)+t_2)(k-1)}{k(k-2)} \wedge X_4 = \frac{(t_2(k-1)+t_1)(k-1)}{k(k-2)}\right] \\
&= \sum_t \mathbb{P}\left[X_1 - \frac{X_3}{k-1} = \gamma - t\right] \\
&\quad \cdot \sum_{t_1 \geq t,\ t_2 \geq t} \mathbb{P}\left[X_2 = \frac{(t_1(k-1)+t_2)(k-1)}{k(k-2)}\right] \cdot \mathbb{P}\left[X_4 = \frac{(t_2(k-1)+t_1)(k-1)}{k(k-2)}\right]
\end{aligned}$$

We approximate the inner sum up to a multiplicative error of $1 + O_k(1/n)$ using Stirling's approximation. Let $N = (1-a)n$, $r_1 = \frac{(t_1(k-1)+t_2)(k-1)}{k(k-2)}$, $r_2 = \frac{(t_2(k-1)+t_1)(k-1)}{k(k-2)}$, $\tau_1 = \frac{N}{2} - r_1$, and $\tau_2 = \frac{N}{2} - r_2$.

$$\begin{aligned}
\sum_{t_1 \geq t,\ t_2 \geq t} \mathbb{P}[X_2 = r_1] \cdot \mathbb{P}[X_4 = r_2] &= \sum_{t_1 \geq t,\ t_2 \geq t} \binom{N}{\frac{N}{2} - \tau_1} \binom{N}{\frac{N}{2} - \tau_2} \frac{1}{2^{2N}} \\
&= \left(1 + O_k\left(\frac{1}{n}\right)\right) \sum_{t_1 \geq t, t_2 \geq t} \frac{2}{\pi N} \exp\left(\frac{-2\left(\left(\frac{N}{2} - r_1\right)^2 + \left(\frac{N}{2} - r_2\right)^2\right)}{N}\right) \\
&= \left(1 + O_k\left(\frac{1}{n}\right)\right) \frac{2}{\pi N} \int_t^\infty \int_t^\infty \exp\left(\frac{-2\left(\left(\frac{N}{2} - r_1\right)^2 + \left(\frac{N}{2} - r_2\right)^2\right)}{N}\right) dt_1 dt_2 \\
&= \left(1 + O_k\left(\frac{1}{n}\right)\right) \frac{2k(k-2)}{\pi N (k-1)^2} \\
&\quad \cdot \int_{\frac{t(k-1)^2}{k(k-2)}}^\infty \int_{\frac{t(k-1)^2}{k(k-2)}}^\infty \exp\left(\frac{-2\left(\left(\frac{N}{2} - r_1\right)^2 + \left(\frac{N}{2} - r_2\right)^2\right)}{N}\right) dr_1 dr_2
\end{aligned}$$

where the last line follows by transforming the equation in-terms of t_1 and t_2 — remember r_1 and r_2 are functions of t_1 and t_2 — into functions of r_1 and r_2. Notice that the Jacobian of the transformation is

$$\begin{vmatrix} \frac{\partial t_1}{\partial r_1} & \frac{\partial t_1}{\partial r_2} \\ \frac{\partial t_2}{\partial r_1} & \frac{\partial t_2}{\partial r_2} \end{vmatrix} = \begin{vmatrix} 1 & \frac{-1}{k-1} \\ \frac{-1}{k-1} & 1 \end{vmatrix} = \frac{k(k-2)}{(k-1)^2}.$$

Next consider $\mathbb{P}\left[X_1 - \frac{X_3}{k-1} = \gamma - t\right]$ for some fixed t. Again, using Stirling's approximation, we let

$\tau_1 = \frac{N_1}{2} - \ell$ and $\tau_2 = \frac{N_3}{2} - (\ell - t)(k-1)$ in the sum to obtain,

$$
\begin{aligned}
\mathbb{P}\left[X_1 - \frac{X_3}{k-1} = \gamma - t\right] &= \sum_{\ell} \binom{N_1}{\frac{N_1}{2} - \tau_1}\binom{N_3}{\frac{N_3}{2} - \tau_2} 2^{-(N_1+N_3)} \\
&= \sum_{\ell} \frac{2}{\pi\sqrt{N_1 N_3}} \exp\left(-\frac{2\tau_1^2}{N_1} - \frac{2\tau_2^2}{N_3}\right)\left(1 + O_k\left(\frac{1}{n}\right)\right) \\
&= \frac{2}{\pi\sqrt{N_1 N_3}}\left(1 + O_k\left(\frac{1}{n}\right)\right) \int \exp\left(-\frac{2\left(\frac{N_1}{2} - \ell\right)^2}{N_1} - \frac{2\left(\frac{N_3}{2} - (\ell - t)(k-1)\right)^2}{N_3}\right) d\ell \\
&= \frac{2}{\pi\sqrt{N_1 N_3}}\left(1 + O_k\left(\frac{1}{n}\right)\right) \exp\left(-\frac{N_1 N_3 + (N_3 + 2(k-1)t)^2 + \frac{4\left((k-1)^2 N_1 + N_3\right)^2}{N_1^2(kN_3 + 2(k-1)^2 t)}}{2N_3}\right)
\end{aligned}
$$

where the last line follows by completing the square with respect to ℓ.

$$
\begin{aligned}
\mathbb{P}&\left[X_1 - \frac{X_3}{k-1} + X_2 - \frac{X_4}{k-1} \geq \gamma \wedge X_1 - \frac{X_3}{k-1} - \frac{X_2}{k-1} + X_4 \geq \gamma\right] \\
&= \left(1 + O_k\left(\frac{1}{n}\right)\right) \cdot \left(\frac{4k(k-2)}{\pi^2\sqrt{N_1 N_3} N (k-1)^2}\right) \\
&\cdot \int \exp\left(-\frac{N_1 N_3 + (N_3 + 2(k-1)t)^2 + \frac{4\left((k-1)^2 N_1 + N_3\right)^2}{N_1^2(kN_3 + 2(k-1)^2 t)}}{2N_3}\right) \\
&\cdot \left(\int_{\frac{t(k-1)^2}{k(k-2)}}^{\infty} \int_{\frac{t(k-1)^2}{k(k-2)}}^{\infty} \exp\left(\frac{-2\left(\left(\frac{N}{2} - r_1\right)^2 + \left(\frac{N}{2} - r_2\right)^2\right)}{N}\right) dr_1 dr_2\right) dt
\end{aligned}
$$

Compare the above integral with one which we would have gotten if instead of the Binomial random variables, we instead had Gaussian random variables with the same mean and variance. In particular, let Y_1, Y_2, Y_3, and Y_4 correspond to X_1, X_2, X_3, and X_4 respectively. Then we have

$$
\begin{aligned}
&\mathbb{P}\left[Y_1 - \frac{Y_3}{k-1} + Y_2 - \frac{Y_4}{k-1} \geq \gamma \wedge Y_1 - \frac{Y_3}{k-1} - \frac{Y_2}{k-1} + Y_4 \geq \gamma\right] \\
&= \int \mathbb{P}\left[Y_1 - \frac{Y_3}{k-1} = \gamma - t\right]\left(\int_t^{\infty}\int_t^{\infty} \mathbb{P}\left[Y_2 - \frac{Y_4}{k-1} \geq \gamma \wedge Y_2 - \frac{Y_4}{k-1} \geq \gamma\right] dt_1 dt_2\right) dt.
\end{aligned}
$$

Again, we consider two parts. In particular, for the inner integral, we have

$$
\begin{aligned}
\int_t^{\infty}\int_t^{\infty} &\mathbb{P}\left[Y_2 - \frac{Y_4}{k-1} \geq t_1 \wedge Y_2 - \frac{Y_4}{k-1} \geq t_2\right] dt_1 dt_2 \\
&= \int_t^{\infty}\int_t^{\infty} \mathbb{P}\left[Y_2 = \frac{(t_1(k-1) + t_2)(k-1)}{k(k-2)}\right] \cdot \mathbb{P}\left[Y_4 = \frac{(t_2(k-1) + t_1)(k-1)}{k(k-2)}\right] dt_1 dt_2 \\
&= \frac{1}{2\pi\sigma^2}\int_t^{\infty}\int_t^{\infty} \exp\left(-\frac{(\mu - r_1)^2}{2\sigma^2} - \frac{(\mu - r_2)^2}{2\sigma^2}\right) dt_1 dt_2 \\
&= \frac{2k(k-2)}{\pi N (k-1)^2}\int_{\frac{t(k-1)^2}{k(k-2)}}^{\infty}\int_{\frac{t(k-1)^2}{k(k-2)}}^{\infty} \exp\left(\frac{-2\left(\left(\frac{N}{2} - r_1\right)^2 + \left(\frac{N}{2} - r_2\right)^2\right)}{N}\right) dr_1 dr_2
\end{aligned}
$$

by a change of variables where $\mu = \frac{N}{2}$ and $\sigma = \frac{\sqrt{N}}{2}$ are the mean and standard deviation of both Y_2 and Y_4. For the outer probability we have,

$$\begin{aligned}
\mathbb{P}\left[Y_1 - \frac{Y_3}{k-1} = \gamma - t\right] &= \int \mathbb{P}\left[Y_1 = \ell\right] \mathbb{P}\left[Y_3 = (t-\ell)(k-1)\right] dt \\
&= \int \frac{1}{2\pi\sqrt{\sigma_{Y_1}\sigma_{Y_3}}} \exp\left(-\frac{(\mu_{Y_1} - \ell)^2}{2\sigma_{Y_3}^2} - \frac{(\mu_{Y_3} - (t-\ell)(k-1))^2}{2\sigma_{Y_3}^2}\right) \\
&= \frac{2}{\pi\sqrt{N_1 N_3}} \exp\left(-\frac{N_1 N_3 + (N_3 + 2(k-1)t)^2 + \frac{4\left((k-1)^2 N_1 + N_3\right)^2}{N_1^2 (k N_3 + 2(k-1)^2 t)}}{2N_3}\right)
\end{aligned}$$

by completing the square as before where μ_{Y_1}, μ_{Y_3}, σ_{Y_1}, and σ_{Y_3} are the mean and variance of Y_1 and Y_3 respectively. In both cases, we see that the bound on the Binomial random variables is the same as the bound on Gaussian random variables up to $O_k\left(\frac{1}{n}\right)$ error. We know from the proof of lemma 79, that

$$\begin{aligned}
&\mathbb{P}\left[Y_1 - \frac{Y_3}{k-1} + Y_2 - \frac{Y_4}{k-1} \geq \gamma \wedge Y_1 - \frac{Y_3}{k-1} - \frac{Y_2}{k-1} + Y_4 \geq \gamma\right] \\
&\qquad = \mathbb{P}_{Z_i \sim \mathcal{N}(0,1)}\left[\sigma_1 Z_1 + \sigma_2 Z_2 \geq \gamma \wedge \sigma_1 Z_1 - \sigma_2 Z_2 \geq \gamma\right] + O_k\left(\frac{1}{n}\right),
\end{aligned}$$

so our desired equality must hold. $\square$

Bibliography

[AAL23] Jamil Arbas, Hassan Ashtiani, and Christopher Liaw. "Polynomial Time and Private Learning of Unbounded Gaussian Mixture Models". In: *International Conference on Machine Learning, ICML 2023, 23-29 July 2023, Honolulu, Hawaii, USA*. Vol. 202. Proceedings of Machine Learning Research. PMLR, 2023, pp. 1018–1040 (cit. on p. 76).

[ABN18] Christoph Aistleitner, Dmitriy Bilyk, and Aleksandar Nikolov. "Tusnády's problem, the transference principle, and non-uniform QMC sampling". In: *Monte Carlo and Quasi-Monte Carlo Methods: MCQMC 2016, Stanford, CA, August 14-19 12*. Springer. 2018, pp. 169–180 (cit. on p. 2).

[ABPW17] Omer Angel, Sébastien Bubeck, Yuval Peres, and Fan Wei. "Local max-cut in smoothed polynomial time". In: *Proceedings of the 49th annual acm sigact symposium on theory of computing*. 2017, pp. 429–437 (cit. on p. 16).

[ACIW19] Haris Aziz, Ioannis Caragiannis, Ayumi Igarashi, and Toby Walsh. "Fair Allocation of Indivisible Goods and Chores". In: *Proceedings of the 28th IJCAI*. 2019, pp. 53–59 (cit. on p. 53).

[ACKK23] Michael Anastos, Oliver Cooley, Mihyun Kang, and Matthew Kwan. "Partitioning problems via random processes". In: *arXiv preprint arXiv:2307.06453* (2023) (cit. on pp. 15, 16, 18).

[ADLO19] Louigi Addario-Berry, Luc Devroye, Gábor Lugosi, and Roberto I. Oliveira. "Local optima of the Sherrington-Kirkpatrick Hamiltonian". In: *J. Math. Phys.* 60.4 (2019), pp. 043301, 13 (cit. on p. 15).

[AGM15] Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D. Milman. *Asymptotic geometric analysis. Part I*. Vol. 202. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015, pp. xx+451 (cit. on p. 33).

[AM06] Dimitris Achlioptas and Cristopher Moore. "Random k-SAT: Two moments suffice to cross a sharp threshold". In: *SIAM Journal on Computing* 36.3 (2006), pp. 740–762 (cit. on pp. 68, 73).

[AMP90] Ron Aharoni, Eric C Milner, and Karel Prikry. "Unfriendly partitions of a graph". In: *Journal of Combinatorial Theory, Series B* 50.1 (1990), pp. 1–10 (cit. on p. 53).

[AN21] Dylan J Altschuler and Jonathan Niles-Weed. "The discrepancy of random rectangular matrices". In: *Random Structures & Algorithms* (2021) (cit. on p. 8).

[APZ19] Benjamin Aubin, Will Perkins, and Lenka Zdeborová. "Storage capacity in symmetric binary perceptrons". In: *J. Phys. A* 52.29 (2019), pp. 294003, 32 (cit. on pp. 17–19).

[BAKZ22] Freya Behrens, Gabriel Arpino, Yaroslav Kivva, and Lenka Zdeborová. "(Dis) assortative partitions on random regular graphs". In: *Journal of Physics A: Mathematical and Theoretical* 55.39 (2022), p. 395004 (cit. on p. 14).

[Bal89] Keith Ball. "Volumes of sections of cubes and related problems". In: *Geometric aspects of functional analysis (1987–88)*. Vol. 1376. Lecture Notes in Math. Springer, Berlin, 1989, pp. 251–260 (cit. on p. 33).

[Ban10] Nikhil Bansal. "Constructive algorithms for discrepancy minimization". In: *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*. IEEE. 2010, pp. 3–10 (cit. on pp. 7, 9).

[Ban93] Wojciech Banaszczyk. "Balancing vectors and convex bodies". In: *Studia Mathematica* 106.1 (1993), pp. 93–100 (cit. on p. 32).

[Ban98] Wojciech Banaszczyk. "Balancing vectors and Gaussian measures of n-dimensional convex bodies". In: *Random Structures & Algorithms* 12.4 (1998), pp. 351–360 (cit. on p. 29).

[BC89] Ole E Barndorff-Nielsen and David Roxbee Cox. *Asymptotic techniques for use in statistics*. Vol. 11. Springer, 1989 (cit. on p. 74).

[BDG16] Nikhil Bansal, Daniel Dadush, and Shashwat Garg. "An algorithm for komlos conjecture matching Banaszczyk's bound". In: *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*. IEEE. 2016, pp. 788–799 (cit. on p. 29).

[BDM23] Sander Borst, Daniel Dadush, and Dan Mikulincer. "Integrality gaps for random integer programs via discrepancy". In: *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM. 2023, pp. 1692–1733 (cit. on p. 2).

[BDP20] Valentin Bouquet, François Delbot, and Christophe Picouleau. "Partition of graphs with maximum degree ratio". In: *arXiv preprint arXiv:2007.12434* (2020) (cit. on p. 16).

[BJ02] A. Bogomolnaia and M.O. Jackson. "Stability of Hedonic Coalition Structures". In: *Games and Economic Behavior* 38.2 (2002), pp. 201–230 (cit. on p. 53).

[BKN14] Nikhil Bansal, Ravishankar Krishnaswamy, and Viswanath Nagarajan. "Better Scalable Algorithms for Broadcast Scheduling". In: *ACM Trans. Algorithms* 11.1 (2014), 3:1–3:24 (cit. on p. 2).

[BL16] Amir Ban and Nati Linial. "Internal partitions of regular graphs". In: *Journal of Graph Theory* 83.1 (2016), pp. 5–18 (cit. on p. 14).

[BMM22] V. Bilò, G. Monaco, and L. Moscardelli. "Hedonic Games with Fixed-Size Coalitions". In: *proceedings of the 35th AAAI*. 9287–9295. 2022 (cit. on p. 53).

[Boh90] Géza Bohus. "On the discrepancy of 3 permutations". In: *Random Structures & Algorithms* 1.2 (1990), pp. 215–220 (cit. on p. 21).

[Boy59] A.V. Boyd. "Inequalities for Mill's Ratio". In: *Reports of Statistical Applications Research* 6 (1959), pp. 44–46 (cit. on p. 87).

[BS02] Béla Bollobás and Alex D Scott. "Problems and results on judicious partitions". In: *Random Structures & Algorithms* 21.3-4 (2002), pp. 414–430 (cit. on p. 14).

[BS96] József Beck and Vera T Sós. "Discrepancy theory". In: *Handbook of combinatorics (vol. 2)*. MIT Press. 1996, pp. 1405–1446 (cit. on p. 1).

[BSTY98] Jean-Daniel Boissonnat, Micha Sharir, Boaz Tagansky, and Mariette Yvinec. "Voronoi Diagrams in Higher Dimensions under Certain Polyhedral Distance Functions". In: *Discrete & Computational Geometry* 19.4 (1998), pp. 485–519 (cit. on p. 50).

[BTV06] Cristina Bazgan, Zsolt Tuza, and Daniel Vanderpooten. "The satisfactory partition problem". In: *Discrete applied mathematics* 154.8 (2006), pp. 1236–1245 (cit. on pp. 15, 57).

[BTV07] Cristina Bazgan, Zsolt Tuza, and Daniel Vanderpooten. "Efficient algorithms for decomposing graphs under degree constraints". In: *Discrete Applied Mathematics* 155.8 (2007), pp. 979–988 (cit. on pp. 14, 55, 56).

[BTV10] Cristina Bazgan, Zsolt Tuza, and Daniel Vanderpooten. "Satisfactory graph partition, variants, and generalizations". In: *European Journal of Operational Research* 206.2 (2010), pp. 271–280 (cit. on pp. 14, 53, 55).

[BY19] Nathanaël Barrot and Makoto Yokoo. "Stable and Envy-free Partitions in Hedonic Games." In: *IJCAI*. 2019, pp. 67–73 (cit. on p. 53).

[CFH19] Á. Cseh, T. Fleiner, and P. Harján. "Pareto Optimal Coalitions of Fixed Size." In: *Mechanism and Institution Design* 4.13 (2019), pp. 87–108 (cit. on p. 53).

[CGVYZ20] Xi Chen, Chenghao Guo, Emmanouil V Vlatakis-Gkaragkounis, Mihalis Yannakakis, and Xinzhi Zhang. "Smoothed complexity of local Max-Cut and binary Max-CSP". In: *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*. 2020, pp. 1052–1065 (cit. on pp. 14, 16).

[Cha01] Bernard Chazelle. *The discrepancy method: randomness and complexity*. Cambridge University Press, 2001 (cit. on p. 1).

[Cha93] Bernard Chazelle. "An optimal convex hull algorithm in any fixed dimension". In: *Discrete & Computational Geometry* 10.4 (1993), pp. 377–409 (cit. on p. 50).

[CKMPSW19] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang. "The unreasonable fairness of maximum Nash welfare". In: *ACM Transactions on Economics and Computation (TEAC)* 7.3 (2019), pp. 1–32 (cit. on p. 53).

[CST14] William Chen, Anand Srivastav, and Giancarlo Travaglini (Editors). *A panorama of discrepancy theory*. Vol. 2107. Springer, 2014 (cit. on p. 53).

[DGZ23] Yatin Dandi, David Gamarnik, and Lenka Zdeborová. "Maximally-stable Local Optima in Random Graphs and Spin Glasses: Phase Transitions and Universality". In: *arXiv preprint arXiv:2305.03591* (2023) (cit. on pp. 14, 18).

[DMR23] Luc Devroye, Abbas Mehrabian, and Tommy Reddad. *The total variation distance between high-dimensional Gaussians with the same mean*. 2023. arXiv: `1810.08693 [math.ST]` (cit. on p. 76).

[DNTT18] Daniel Dadush, Aleksandar Nikolov, Kunal Talwar, and Nicole Tomczak-Jaegermann. "Balancing vectors in any norm". In: *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2018, pp. 1–10 (cit. on pp. 10, 32, 34).

[DR50] A. Dvoretzky and C. A. Rogers. "Absolute and unconditional convergence in normed linear spaces". In: *Proc. Nat. Acad. Sci. U.S.A.* 36 (1950), pp. 192–197 (cit. on p. 33).

[DS19] Jian Ding and Nike Sun. "Capacity lower bound for the Ising perceptron". In: *STOC'19—Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*. ACM, New York, 2019, pp. 816–827 (cit. on pp. 18, 19).

[FFGHHKLS22] Odile Favaron, Gerd Fricke, Wayne Goddard, Sandra M Hedetniemi, Stephen T Hedetniemi, Petter Kristiansen, Renu C Laskar, and Duane Skaggs. "Offensive alliances in graphs". In: *International Symposium on Computer and Information Sciences*. CRC Press. 2022, pp. 298–302 (cit. on p. 14).

[FKNSS22] Asaf Ferber, Matthew Kwan, Bhargav Narayanan, Ashwin Sah, and Mehtaab Sawhney. "Friendly bisections of random graphs". In: *Communications of the American Mathematical Society* 2.10 (2022), pp. 380–416 (cit. on p. 14).

[Fra21] Cole Franks. "A simplified disproof of Beck's three permutations conjecture and an application to root-mean-squared discrepancy". In: *Combinatorics, Probability and Computing* 30.3 (2021), pp. 398–411 (cit. on pp. 9, 21).

[FS20] Cole Franks and Michael Saks. "On the discrepancy of random matrices with many columns". In: *Random Structures & Algorithms* 57.1 (2020), pp. 64–96 (cit. on p. 8).

[FTT04] Gary William Flake, Robert E Tarjan, and Kostas Tsioutsiouliklis. "Graph clustering and minimum cut trees". In: *Internet Mathematics* 1.4 (2004), pp. 385–408 (cit. on p. 55).

[Gho62] Alain Ghouila-Houri. "Caractérisation des matrices totalement unimodulaires". In: *Comptes Redus Hebdomadaires des Séances de l'Académie des Sciences (Paris)* 254 (1962), pp. 1192–1194 (cit. on p. 24).

[GK00] Michael U Gerber and Daniel Kobler. "Algorithmic approach to the satisfactory graph partitioning problem". In: *European Journal of Operational Research* 125.2 (2000), pp. 283–291 (cit. on p. 55).

[GL18] David Gamarnik and Quan Li. "On the max-cut of sparse random graphs". In: *Random Structures Algorithms* 52.2 (2018), pp. 219–262 (cit. on p. 14).

[Glu89] Efim Davydovich Gluskin. “Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces”. In: *Mathematics of the USSR-Sbornik* 64.1 (1989), p. 85 (cit. on pp. 6, 34).

[GMR05] Venkatesan Guruswami, Daniele Micciancio, and Oded Regev. “The complexity of the covering radius problem”. In: *Computational Complexity* 14.2 (2005), pp. 90–121 (cit. on p. 39).

[Gol78] E Mark Gold. “Complexity of automaton identification from given data”. In: *Information and control* 37.3 (1978), pp. 302–320 (cit. on p. 39).

[Gre] Ben Green. “100 open problems”. In: *manuscript, available on request to Professor Green* () (cit. on p. 14).

[HR06] Ishay Haviv and Oded Regev. “Hardness of the covering radius problem on lattices”. In: *Computational Complexity, 2006. CCC 2006. Twenty-First Annual IEEE Conference on*. IEEE. 2006, 14–pp (cit. on p. 39).

[HR17] Rebecca Hoberg and Thomas Rothvoss. “A logarithmic additive integrality gap for bin packing”. In: *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM. 2017, pp. 2616–2625 (cit. on p. 1).

[HR19] Rebecca Hoberg and Thomas Rothvoss. “A fourier-analytic approach for the discrepancy of random set systems”. In: *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM. 2019, pp. 2547–2556 (cit. on p. 8).

[HRRY17] Rebecca Hoberg, Harishchandra Ramadas, Thomas Rothvoss, and Xin Yang. “Number balancing is as hard as Minkowski's theorem and shortest vector”. In: *International Conference on Integer Programming and Combinatorial Optimization*. Springer. 2017, pp. 254–266 (cit. on p. 44).

[HS14] Nils Hebbinghaus and Anand Srivastav. “Multicolor discrepancy of arithmetic structures”. In: *A Panorama of Discrepancy Theory*. Springer, 2014, pp. 319–424 (cit. on pp. 16, 55).

[IKLM95] Christian Icking, Rolf Klein, Ngoc-Minh Lé, and Lihong Ma. “Convex distance functions in 3-space are different”. In: *Fundamenta Informaticae* 22.4 (1995), pp. 331–352 (cit. on p. 51).

[JR22] Haotian Jiang and Victor Reis. “A Tighter Relation Between Hereditary Discrepancy and Determinant Lower Bound”. In: *Symposium on Simplicity in Algorithms (SOSA)*. SIAM. 2022, pp. 308–313 (cit. on pp. ii, 9, 10, 29).

[JRT24] Arun Jambulapati, Victor Reis, and Kevin Tian. “Linear-sized sparsifiers via near-linear time discrepancy theory”. In: *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM. 2024, pp. 5169–5208 (cit. on p. 2).

[Kun23] Dmitriy Kunisky. “The discrepancy of unsatisfiable matrices and a lower bound for the Komlós conjecture constant”. In: *SIAM Journal on Discrete Mathematics* 37.2 (2023), pp. 586–603 (cit. on pp. 20, 24, 29).

[Lar14] Kasper Green Larsen. “On range searching in the group model and combinatorial discrepancy”. In: *SIAM Journal on Computing* 43.2 (2014), pp. 673–686 (cit. on p. 2).

[Lav18] Misha Lavrov. *Finding sum of k from zero to n of the absolute value of twice k minus n times n choose k.* 2018 (cit. on p. 27).

[LM15] Shachar Lovett and Raghu Meka. “Constructive discrepancy minimization by walking on the edges”. In: *SIAM Journal on Computing* 44.5 (2015), pp. 1573–1582 (cit. on p. 7).

[LMMS04] Richard J Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. “On approximately fair allocations of indivisible goods”. In: *Proceedings of the 5th ACM Conference on Electronic Commerce.* 2004, pp. 125–131 (cit. on p. 53).

[LMNS23] Lily Li, Evi Micha, Aleksandar Nikolov, and Nisarg Shah. “Partitioning friends fairly”. In: *Proceedings of the AAAI Conference on Artificial Intelligence.* Vol. 37. 5. 2023, pp. 5747–5754 (cit. on pp. 2, 53).

[LN20] Lily Li and Aleksandar Nikolov. “On the Computational Complexity of Linear Discrepancy”. In: *28th Annual European Symposium on Algorithms (ESA 2020).* Schloss-Dagstuhl-Leibniz Zentrum für Informatik. 2020 (cit. on pp. 2, 38).

[LN24] Lily Li and Aleksandar Nikolov. “On the Gap between Hereditary Discrepancy and the Determinant Lower Bound”. In: *SIAM Journal on Discrete Mathematics* 38.2 (2024), pp. 1222–1238 (cit. on pp. 2, 20).

[LSV86] László Lovász, Joel Spencer, and Katalin Vesztergombi. “Discrepancy of set-systems and matrices”. In: *European Journal of Combinatorics* 7.2 (1986), pp. 151–160 (cit. on pp. ii, 4, 6, 9, 21, 32).

[Man21] Pasin Manurangsi. “Linear discrepancy is Π2-hard to approximate”. In: *Information Processing Letters* 172 (2021), p. 106164 (cit. on pp. 11, 51).

[Mat09] Jiří Matoušek. *Geometric discrepancy: An illustrated guide.* Vol. 18. Springer Science & Business Media, 2009 (cit. on pp. 1, 6, 21, 48).

[Mat13] Jiří Matoušek. “The determinant bound for discrepancy is almost tight”. In: *Proceedings of the American Mathematical Society* 141.2 (2013), pp. 451–460 (cit. on pp. ii, 8–10, 21, 28–31).

[Mat95] Jiří Matoušek. “Tight Upper Bounds for the Discrepancy of Halfspaces”. In: *Discrete and Computational Geometry* 13.1 (1995), pp. 593–601 (cit. on p. 37).

[MMPP23] Calum MacRury, Tomáš Masařík, Leilani Pai, and Xavier Pérez-Giménez. “The phase transition of discrepancy in random hypergraphs”. In: *SIAM Journal on Discrete Mathematics* 37.3 (2023), pp. 1818–1841 (cit. on p. 8).

[MN12] Shanmugavelayutham Muthukrishnan and Aleksandar Nikolov. “Optimal private halfspace counting via discrepancy”. In: *Proceedings of the forty-fourth annual ACM symposium on Theory of computing.* 2012, pp. 1285–1292 (cit. on p. 2).

[MS22] Pasin Manurangsi and Warut Suksompong. "Almost envy-freeness for groups: Improved bounds via discrepancy theory". In: *Theoretical Computer Science* 930 (2022), pp. 179–195 (cit. on p. 53).

[MSS23] Dor Minzer, Ashwin Sah, and Mehtaab Sawhney. "On Perfectly Friendly Bisections of Random Graphs". In: *arXiv preprint arXiv:2305.03543* (2023) (cit. on pp. 14, 17, 18, 63, 88, 92–95, 98, 99).

[Nik15] Aleksandar Nikolov. "Randomized rounding for the largest simplex problem". In: *Proceedings of the forty-seventh annual ACM symposium on Theory of computing*. 2015, pp. 861–870 (cit. on pp. 33, 34).

[Nik17] A. Nikolov. "Tighter bounds for the discrepancy of boxes and polytopes". In: *Mathematika* 63.3 (2017), pp. 1091–1113 (cit. on pp. 2, 29).

[NNN12] Alantha Newman, Ofer Neiman, and Aleksandar Nikolov. "Beck's three permutations conjecture: A counterexample and some consequences". In: *2012 IEEE 53rd Annual Symposium on Foundations of Computer Science*. IEEE. 2012, pp. 253–262 (cit. on pp. 9, 21, 32).

[OBT12] Martin Olsen, Lars Bækgaard, and Torben Tambo. "On non-trivial Nash stable partitions in additive hedonic games with symmetric 0/1-utilities". In: *Information Processing Letters* 112.23 (2012), pp. 903–907 (cit. on p. 53).

[Pál10] Dömötör Pálvölgyi. "Indecomposable coverings with concave polygons". In: *Discrete & Computational Geometry* 44.3 (2010), pp. 577–588 (cit. on pp. 9, 21, 32).

[Pet16] Dominik Peters. "Graphical hedonic games of bounded treewidth". In: *Proceedings of the AAAI Conference on Artificial Intelligence*. Vol. 30. 1. 2016 (cit. on p. 53).

[PX21a] Will Perkins and Changji Xu. "Frozen 1-RSB structure of the symmetric Ising perceptron". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. 2021, pp. 1579–1588 (cit. on p. 18).

[PX21b] Will Perkins and Changji Xu. "Frozen 1-RSB structure of the symmetric Ising perceptron". In: *STOC '21—Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. ACM, New York, 2021, pp. 1579–1588 (cit. on p. 19).

[Roc70] R. Tyrrell Rockafellar. *Convex analysis*. Vol. No. 28. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1970, pp. xviii+451 (cit. on p. 33).

[Rot13] Thomas Rothvoß. "Approximating bin packing within o (log OPT* log log OPT) bins". In: *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*. IEEE. 2013, pp. 20–29 (cit. on p. 1).

[Rot17] Thomas Rothvoss. "Constructive discrepancy minimization for convex sets". In: *SIAM Journal on Computing* 46.1 (2017), pp. 224–234 (cit. on pp. 7, 8).

[Sam53] Michael R Sampford. "Some inequalities on Mill's ratio and related functions". In: *The Annals of Mathematical Statistics* 24.1 (1953), pp. 130–132 (cit. on p. 87).

[Sch98] Alexander Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998 (cit. on pp. 24, 40).

[Sha04] Khurram Hassan Shafique. "Partitioning a graph in alliances and its application to data clustering". PhD thesis. University of Central Florida, 2004 (cit. on p. 55).

[Spe85] Joel Spencer. "Six standard deviations suffice". In: *Transactions of the American mathematical society* 289.2 (1985), pp. 679–706 (cit. on p. 6).

[Spe94] Joel Spencer. *Ten lectures on the probabilistic method*. SIAM, 1994 (cit. on p. 6).

[SST01] Joel H Spencer, Aravind Srinivasan, and Prasad Tetali. "The discrepancy of permutation families". In: *Unpublished manuscript* (2001) (cit. on p. 21).

[Ste48] Hugo Steinhaus. "The problem of fair division". In: *Econometrica* 16 (1948), pp. 101–104 (cit. on p. 56).

[Sti96] Michael Stiebitz. "Decomposing graphs under degree constraints". In: *Journal of Graph Theory* 23.3 (1996), pp. 321–324 (cit. on pp. 14, 55).

[TMR20] Paxton Turner, Raghu Meka, and Philippe Rigollet. "Balancing Gaussian vectors in high dimension". In: *Conference on Learning Theory*. PMLR. 2020, pp. 3455–3486 (cit. on p. 8).

[Top04] Flemming Topsøe. "Some bounds for the logarithmic function". In: *RGMIA Res. Rep. Collection* 7.2 (2004), pp. 1–20 (cit. on p. 97).

[Tou83] Godfried T Toussaint. "Computing largest empty circles with location constraints". In: *International journal of computer & information sciences* 12.5 (1983), pp. 347–358 (cit. on p. 49).

[Ver18] Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*. Vol. 47. Cambridge university press, 2018 (cit. on pp. 36, 37).

[Won01] Roderick Wong. *Asymptotic approximations of integrals*. SIAM, 2001 (cit. on p. 78).

[WY92] Gerhard J Woeginger and Zhongliang Yu. "On the equal-subset-sum problem". In: *Information Processing Letters* 42.6 (1992), pp. 299–302 (cit. on p. 44).

Glossary

covering radius The covering radius of a lattice Λ with L_p norm is

$$\rho(\Lambda) = \max_{\mathbf{w} \in \mathbb{R}^n} \min_{\mathbf{x} \in \Lambda} \|\mathbf{w} - \mathbf{x}\|_p$$

. 38

determinant lower bound For a matrix $\mathbf{A}$, let its — be defined as

$$\operatorname{detlb}(\mathbf{A}) \coloneqq \max_{k \in \min(m,n)} \max_{\mathbf{B}} |\det(\mathbf{B})|^{1/k}$$

where $\mathbf{B}$ is a $k \times k$ submatrix of $\mathbf{A}$. 6

discrepancy Let $\mathcal{S}$ be a subset of the power set of X. The discrepancy of a set $S \in \mathcal{S}$ with respect to a colouring $\chi : X \to \{-1, 1\}$ is defined as

$$\operatorname{disc}(S, \chi) = \left| \sum_{s \in S} \chi(s) \right|.$$

The discrepancy of set system $(X, \mathcal{S})$ is

$$\operatorname{disc}(\mathcal{S}) = \min_{\chi} \max_{S \in \mathcal{S}} \operatorname{disc}(S, \chi)$$

. 1

envy-free For $r \geq 0$, a k-partition X of a graph G is — up to r, denoted EF-r, if, for every pair of agents $i, i' \in V$, $u_i(X(i)) \geq u_i(X(i') \cup \{i\} \setminus \{i'\}) - r$. When $r = 0$, we simply refer to this as envy-freeness (EF). 13

envy-free up to one For each agent a_i, let $v_i : \mathcal{P}(\mathcal{R}) \to \mathbb{R}$ be their valuation function for the utility of each subset of the resources. Let $\pi : \mathcal{R} \to \mathcal{A}$ be a partition of the resources into bundles for each agent. We say that π is — if for every agent a_i with bundle b_i, after removing the most valuable items from the bundle of some agent a_j to obtain bundle b'_j, $v_i(b_i) \geq v_i(b'_j)$. 13

fundamental parallelepiped Given a matrix $\mathbf{A}$ with n columns, its — is $\mathcal{P}_{\mathbf{A}} = \{\mathbf{x} \in [-1, 1]^n : \|\mathbf{A}\mathbf{x}\|_\infty \leq 1\}$. 5, 11

hedonic games Hedonic games are coalition formation games with hedonic preferences. Here, *coalition formation games* is a partition of agents into disjoint coalitions and *hedonic preferences* are those which an agent derives value only from the other agents in its own coalition and not on how agents in other coalitions are grouped. 13, 53

hereditary discrepancy Given $\mathbf{A} \in \mathbb{R}^{m\times n}$, the hereditary discrepancy of $\mathbf{A}$ is the maximum discrepancy over all sub-matrices of $\mathbf{A}$. Formally, if $\mathbf{B}$ is a sub-matrix of $\mathbf{A}$, then

$$\text{herdisc}(\mathbf{A}) = \max_{\mathbf{B}} \min_{\mathbf{x}\in\{-1,1\}^n} \|\mathbf{Bx}\|_\infty$$

. 4

hereditary partial vector discrepancy The — for matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is $\text{hpartvdisc}(\mathbf{A}) = \min \lambda$ such that for every $S \subseteq [n]$ there exist vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{R}^n$ satisfying $\|\sum_{j\in S} a_{i,j}\mathbf{v}_j\|_2^2 \le \lambda^2$ for all $i \in [m]$ and the following SDP constraints:

$$\begin{aligned} \sum_{j\in[n]} \|\mathbf{v}_j\|_2^2 &\ge \frac{|S|}{2} \\ \|\mathbf{v}_j\|_2^2 &\le 1 \qquad \forall j \in S, \\ \|\mathbf{v}_j\|_2^2 &= 0 \qquad \forall j \in [n]\backslash S \end{aligned}$$

. 9

incidence matrix For a set system $(\mathcal{S}, X)$ where $|S| = m$ and $|X| = n$, its — is the matrix $\mathbf{A} \in \{0,1\}^{m\times n}$ where each row cooresponds to a $S \in \mathcal{S}$ and for that row, the entry in column i is equal to one if and only if $i \in S$. 3

linear discrepancy For $\mathbf{A} \in \mathbb{R}^{m\times n}$, the — of $\mathbf{A}$ is $\text{lindisc}(\mathbf{A}) = \max_{\mathbf{w}\in[0,1]^n} \min_{\mathbf{x}\in\{0,1\}^n} \|\mathbf{A}(\mathbf{w} - \mathbf{x})\|_\infty$. 5

one-sided discrepancy For set-system $\mathcal{S}$ on universe X, the — of $\mathcal{S}$ is

$$\text{sdisc}(\mathcal{S}, X) \coloneqq \min_{\chi: X\to\{-1,1\}} \max_{S_i\in\mathcal{S}} \chi_i \cdot \chi(S_i).$$

Using matrix notation with incidence matrix $\mathbf{A}$ of $\mathcal{S}$ — can also be defined as

$$\text{sdisc}(\mathbf{A}) = \min_t \exists \chi \in \{-1,1\}^n : \chi_i \cdot \langle \mathbf{a}_i, \chi\rangle \le t \text{ for all } i \in [m]$$

where $\mathbf{a}_1^\top, ..., \mathbf{a}_m^\top$ are the rows of $\mathbf{A}$. 14

totally unimodular An integer valued matrix where every sub-determinant is in $\{-1,0,1\}$. 24, 25

vector discrepancy For $\mathbf{A} \in \mathbb{R}^{m\times n}$, the — of $\mathbf{A}$ is $\text{vecdisc}(\mathbf{A}) = \min \lambda$ such that there exists unit vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ which satisfies the SDP

$$\left\| \sum_{i \in [n]} \mathbf{v}_i A_{j,i} \right\|_2 \leq \lambda \text{ for each set } j \in [m].$$

Note that $\text{vecdisc}(\mathbf{A}) \leq \text{disc}(\mathbf{A}) \leq \text{herdisc}(\mathbf{A})$. 9

Acronyms

CRP covering radius problem. 51

LEB largest empty ball. 49

LEC largest empty circle. 49

TUM totally unimodular. 9, 24, 25