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Fitting Selected Random Planetary Systems to Titius–Bode Laws

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Simple "solar systems" are generated with planetary orbital radii r distributed uniformly random in log r between 0.2 and 50 AU, with masses and order identical to our own Solar System. A conservative stability criterion is imposed by requiring that adjacent planets are separated by a minimum distance of k times the sum of their Hill radii for values of k ranging from 0 to 8. Least-squares fits of these systems to generalized Bode laws are performed and compared to the fit of our own Solar System. We find that this stability criterion and other "radius-exclusion" laws generally produce approximately geometrically spaced planets that fit a Titius–Bode law about as well as our own Solar System. We then allow the random systems the same exceptions that have historically been applied to our own Solar System. Namely, one gap may be inserted, similar to the gap between Mars and Jupiter, and up to 3 planets may be "ignored," similar to how some forms of Bode's law ignore Mercury, Neptune, and Pluto. With these particular exceptions, we find that our Solar System fits significantly better than the random ones. However, we believe that this choice of exceptions, designed specifically to give our own Solar System a better fit, gives it an unfair advantage that would be lost if other exception rules were used. We compare our results to previous work that uses a "law of increasing differences" as a basis for judging the significance of Bode's law. We note that the law of increasing differences is not physically based and is probably too stringent a constraint for judging the significance of Bode's law. We conclude that the significance of Bode's law is simply that stable planetary systems tend to be regularly spaced and conjecture that this conclusion could be strengthened by the use of more rigorous methods of rejecting unstable planetary systems, such as long-term orbit integrations.

"For a statistician, fitting a three-parameter curve of uncertain form to ten points with three exceptions certainly brings one to the far edge of the known world."

— Bradley Efron (1971)

Key Words: **planetary dynamics; orbits; planetary formation; celestial mechanics; computer techniques.**

1. INTRODUCTION

The Titius–Bode "law,"

$$r_i = 0.4 + 0.15 \times 2^i, \qquad i = -\infty, 1, \ldots, 8, \tag{1}$$

roughly describes the planetary semi-major axes in astronomical units (AU), with Mercury assigned $i = -\infty$, Venus $i = 1$, Earth $i = 2$, etc. Usually the asteroid belt is counted as $i = 4$. The law fits the planets Venus through Uranus quite well, and successfully predicted the existence and locations of Uranus and the asteroids. However, (i) the law breaks down badly for Neptune and Pluto; (ii) there is no reason why Mercury should have $i = -\infty$ rather than $i = 0$, except that it fits better that way; (iii) the total mass of the asteroid belt is far smaller than the mass of any planet, so it is not clear that it should be counted as one. The question of the significance of Bode's law has taken on increased interest with discoveries of extra-solar planets and is also worth reexamination because computer speeds now permit more powerful statistical tests than were previously possible.

A partial history of the law and attempts to explain it up to the year 1971 can be found in Nieto (1972). Most modern arguments concerning the validity of Bode's law can be assigned to one of three broad classes:

1. Attempts to elucidate the physical processes leading to Bode's law. These are based on a variety of mechanisms, including dynamical instabilities in the protoplanetary disk (Graner and Dubrulle 1994, Dubrulle and Graner 1994, Li *et al.* 1995), gravitational interactions between planetesi-

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mals (Lecar 1973), or long-term instabilities of the planetary orbits (Hills 1970, Llibre and Piñol 1987, Conway and Elsner 1988). We shall not comment on these explanations, except to say that we find none of them entirely convincing.

2. Discussions that ignore physics but try to assess whether the success of Bode's law is statistically significant: Good (1969) performs a likelihood test under the null hypothesis that the planet distances should be distributed uniformly random in log r. He includes the asteroid belt, but ignores Mercury, Neptune, and Pluto, subjectively assigning (i.e., guessing) a factor of 5 penalty to his likelihood ratio for ignoring these planets. He concludes that there is a likelihood ratio of 300–700 in favor of Bode's law being "real" rather than artifactual. Efron (1971) attacks Good's analysis, in particular his choice of null hypothesis. (Good's and Efron's articles are followed by over a dozen extended "comments" from other statisticians.) Efron notes that the difference between semi-major axes of adjacent planets is an increasing function of distance for all adjacent planet pairs except Neptune–Pluto. He proposes, without physical basis, that this *law of increasing differences* is a better null hypothesis, the only reason cited being that Bode's law "predicts" increasing differences. Duplicating Good's analysis with this new null hypothesis, he computes a likelihood ratio in favor of Bode's law of only 8:5 and concludes that "there is no compelling evidence for believing that Bode's law is not artifactual." Conway and Zelenka (1988) repeat Efron's analysis using the law of increasing differences, this time ignoring only Pluto, and computing a more realistic penalty for doing so. They compute a likelihood ratio of approximately unity, also concluding that Bode's law is artifactual. We believe that these analyses are flawed because there is no physical basis for the law of increasing differences; in fact later we will show that systems that are stable according to our criteria only rarely satisfy the law of increasing differences.

3. Discussions of other laws that may influence the spacing of the planets. Many of these involve resonances between the mean motions of the planets, such as Molchanov (1968; but see Hénon 1969), Birn (1973), and Patterson (1987). A promising development is the recognition that planets are capable of migrating significant distances after their formation (Fernández and Ip 1984, Wetherill 1988, Ipatov 1993, Lin *et al.* 1996, Trilling *et al.* 1998). For example, this process can explain the resonant relationship between Neptune and Pluto (Malhotra 1993, 1995) and may explain the spacing of the terrestrial planets (Laskar 1997).

The present paper combines the first two of these approaches: we generate a broad range of possible model planetary systems and exclude those that are known to be dynamically unstable. We then ask which of the remaining ones satisfy laws similar to Bode's.

2. METHOD

2.1. Radius-Exclusion Laws

A necessary, but not sufficient, condition for the stability of a planetary system is that its planets never get "too close to each other" (Lecar 1973). This can be formalized into several "radius-exclusion laws."

1. Simple scaling arguments for near-circular, coplanar orbits suggest that a test particle on a stable orbit cannot approach a planet more closely than k Hill radii for some k, using the Hill radius h as defined by Lissauer (1987, 1993),

$$h = H_M r, \qquad H_M = \left(\frac{M}{3M_\odot}\right)^{\frac{1}{3}}, \tag{2}$$

for a planet of mass M, semi-major axis r, and *fractional Hill radius* H_{M}. We shall extend this criterion to two adjacent planets with nonzero mass by summing their respective Hill radii. There are other plausible ways of combining adjacent planets: it might be more physically reasonable to use the sum of the masses to define a single combined Hill radius, although it is not clear that this is preferred when more than two planets are involved. Exponents other than 1/3 may also be reasonable (Wisdom 1980, Chambers *et al.* 1996). However, the difference between these approaches is probably unimportant given the uncertainty in k, as discussed below.

2. For noncircular orbits, we also expect that the aphelion distance of the inner planet is less than the perihelion distance of the outer one. In other words, if the ith planet has semi-major axis r_i and eccentricity e_i, we expect that $r_i(1 + e_i) < r_{i+1}(1 - e_{i+1})$. Taking this further, we may demand (very conservatively) that the planets are separated by a Hill radius even at their closest possible approach, giving

$$r_i(1 + e_i + H_{M_i}) < r_{i+1}(1 - e_{i+1} - H_{M_{i+1}}).$$

3. Several authors have argued that boundaries between stable and unstable orbits occur at resonances of the form j:(j + 1) (Birn 1973, Wisdom 1980, Weidenschilling and Davis 1985, Patterson 1987, Holman and Murray 1996). Weidenschilling and Davis (1985) argue that two planets are unlikely to form closer than their mutual 2:3 resonance (because small solid bodies are trapped in the outer j:(j + 1) resonances of a protoplanet due to gas-induced drag; once trapped, their eccentricities are pumped up, causing orbit crossing). For the 2:3 resonance, we can define $H_{2:3}$ by using Kepler's third law to define $R_{2:3} = (3/2)^{2/3}$, and splitting the distance between two adjacent planets by solving $R_{2:3} = 1 + H_{2:3} + R_{2:3}\, H_{2:3}$.

Combining all three of Hill radii, eccentricities, and the 2:3 resonance, we obtain

$$r_i(1 + V_i) < r_{i+1}(1 - V_{i+1}), \qquad V_i = \max(H_{2:3}, e_i + H_{M_i}).$$

Some planetary systems in this paper were generated using planetary masses and ordering of these masses identical to our own Solar System, while others used equal fractional radius exclusion for all planets.

Clearly our results will be highly dependent upon the extent of radius exclusion. For the Hill radius of Eq. (2), which was derived for the case of two small planets orbiting a massive central object, a value of $k = 2$–4 is believed to leave the two planets in permanently stable orbits (Wetherill and Cox 1984, 1985, Lissauer 1987, Wetherill 1988, Gladman 1993). For more than two planets, recent work by Chambers *et al.* (1996) suggests that *no* value of k gives permanent stability. Instead, the stability time scale grows exponentially with increasing orbit separation, with billion-year stability for our Solar System requiring $k \gtrsim 13$. Furthermore, simulations of the stability of test particles in the current Solar System (Holman 1997) seem to show that there remain few stable orbits in the outer Solar System other than those near Trojan points. This provides circumstantial evidence that a small value of k is not enough to separate stable orbits, since the outer planets are separated from each other by more than 15 Hill radii. For these reasons, our experiments use several radius-exclusion laws, including various combinations of Hill radii, 2:3 resonances, eccentricities, and k.

It is easy to see why radius-exclusion laws tend to produce planetary distances that approximately follow a geometric progression. If a fixed fractional radius exclusion V is used for every planet, and planets are packed as tightly as possible according to the radius-exclusion law, then the physical extent of radius exclusion at distance r is rV, and the resulting planetary separations would follow an exact geometric progression with semi-major axis ratio $(1 + V)/(1 - V)$. If the planets are packed less tightly, then noise is added to the fit.

2.2. *Generating and Fitting Planetary Systems*

Assume that planet i has a fractional radius exclusion of V_i. To construct a sample system that satisfies the radius-exclusion law, we generate a list of nine planet distances distributed uniformly random in $\log r$ between 0.2 and 50 AU and then sort them into increasing order $\{r_i < r_{i+1}\}_{i=0}^{7}$. If the list does not satisfy

$$r_{i+1} - r_i > V_{i+1} + V_i, \qquad i = 0, \ldots, 7,$$

then the entire list is discarded and we start over. The "trials" column of Table I lists the average number of such trials that are required to generate a sample $\{r_i\}_{i=0}^{8}$ satisfying the described radius-exclusion law. We then generate 4096 such samples for each of the radius-exclusion laws described in Table I. For e_i, we use the maximum eccentricity for each planet in our Solar System over the past 3 million years (Quinn *et al.* 1991). Not surprisingly, the mean number of trials needed to build a "valid" system that satisfies the radius-exclusion criterion increases as the exclusion radii get larger. Our most stringent exclusion criterion is $8H_i$, which demands that adjacent planets are separated by 8 times the sum of their respective Hill radii. We did not test more stringent cases because they would be prohibitively expensive in terms of computer time (cf. the "trials" column).

For each sample that satisfies the relevant radius-exclusion, we perform a nonlinear least-squares fit of the distances r_i to

$$a + bc^i, \tag{3}$$

which we call a "generalized Bode law." The fit is performed by minimizing the objective function

$$\chi^2 = \sum_{i=0}^{8} \left(\frac{\log(a + bc^i) - \log r_i}{\sigma_i} \right)^2, \tag{4}$$

constrained so that $a, b > 0$ and $c > 1$. We fit on $\log r$ rather than r because we want the fractional error of each planet to be weighted equally. In the Appendix, we analytically derive approximations to the standard deviations σ_i. The initial guess for the parameters in the objective function is

$$c_0 = \frac{1}{8} \sum_{i=0}^{7} \frac{r_{i+1}}{r_i},$$

$$b_0 = r_8 / c_0^8,$$

$$a_0 = \max(0, r_0 - b_0).$$

To more accurately reflect the various forms of Bode's law, we also attempt fits that "ignore" 1, 2, and 3 planets. We do this by performing fits on all $\binom{9}{j}$, $j = 0, 1, 2, 3$, possible combinations of ignoring j out of 9 planets, and choosing the best fit for each j. We then repeat the entire procedure, allowing one gap to be inserted between the two adjacent planets with the largest r_{i+1}/r_i ratio, to mimic the gap between Mars and Jupiter. This gives us 8 fits out of $2\sum_{j=0}^{3}\binom{9}{j}$, = 260 combinations of ways to ignore planets for each sample planetary system.

3. RESULTS

3.1. χ^2 *Fits*

Results of all the fits for each type of system are presented in Figs. 1 and 2. Not surprisingly, the best fit for

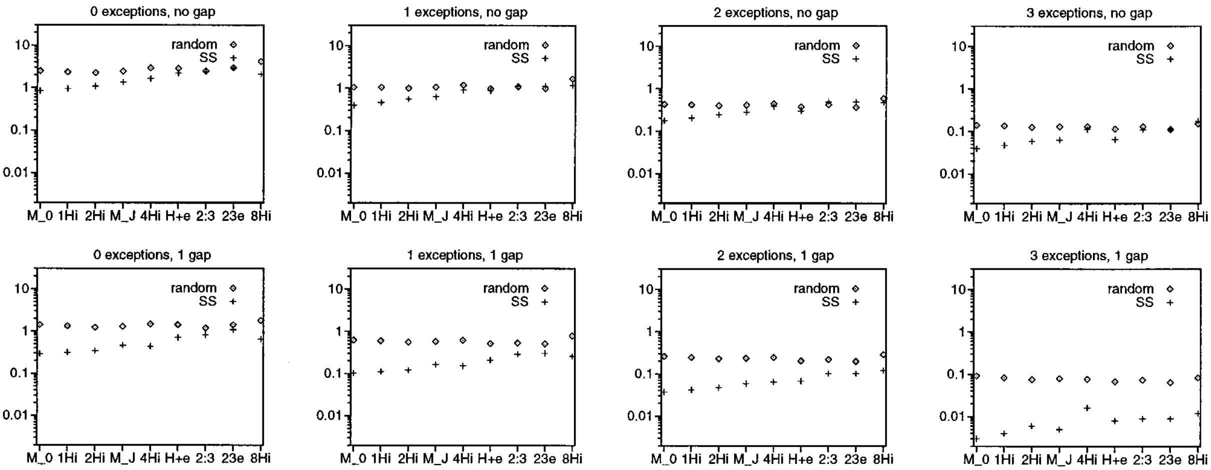

FIG. 1. χ^2 fits, using Eq. (4). A system with a "gap" has had a virtual planet inserted between the two real planets that are most widely spaced in log r, to mimic the gap between Mars and Jupiter in our Solar System. The χ^2 of our own Solar System is labeled with "+" symbols, and the means of 4096 random ones are labeled with diamonds. The labels on the horizontal axis correspond to the names described in Table I.

our own Solar System always occurs when a gap is added between Mars and Jupiter, while Mercury, Neptune, and Pluto are ignored. Even for our own Solar System, the fit depends slightly on the radius-exclusion law, since this affects the denominator σ_i (see Appendix); for the three cases in which the radius exclusion for each planet is identical in log r (M_0, M_J, 2:3), the best fit for our Solar System is identically

$$0.450 + 0.132 \times 2.032^i, \qquad i = 0, 1, \dots, 8,$$

with χ^2 values of 0.003, 0.005, and 0.009, respectively. This result can be compared to the original Bode's law, Eq. (1).

Consider the case where no gaps are allowed, and up to three planets may be ignored (top row of Fig. 1). When there is no radius-exclusion law (left edge of each figure), the χ^2 for the Solar System is always substantially less (by a factor 3–6) than the mean of the random systems with the same number of ignored planets; this is consistent with the conclusion that the Solar System satisfies a generalized Bode law significantly better than ones that are uniformly random in log r. However, as we apply more stringent radius-exclusion laws (moving right in each figure), the χ^2 values for the Solar System become quite similar to the mean of the random systems, indicating that the Solar System is no closer to a generalized Bode law than random ones that satisfy radius exclusion. This situation changes in the bottom row of Fig. 1, which shows the case where a gap is allowed. In particular, in the case with three planets ignored and one gap, our Solar System's best χ^2 value is

TABLE I
The Various Radius-Exclusion Laws Used

Name	Description	Trials	Ex → LID(%)	LID → Ex(%)
M_0	Each planet has zero mass	1	0.4	100
$1H_i$	Hill radii corresponding to planets of our Solar System	1.92	0.8	93.3
$2H_i$	Like $1H_i$, except radius exclusion of 2 Hill radii	3.94	1.6	87.2
M_J	All planets have Jupiter's fractional Hill radius	7.88	1.9	49.3
$4H_i$	Like $1H_i$, except radius exclusion of 4 Hill radii	19.3	4.5	57.6
H_{ei}	Adjacent planets no closer than $H_i + e_i$	44.1	3.4	23.1
2:3	Adjacent planets no closer than the 2:3 resonance	230	10.5	24.8
$23e$	Adjacent planets no closer than $\max(H_i + e_i, H_{2:3})$	479	9.7	16.9
$8H_i$	Like $1H_i$, except radius exclusion of 8 Hill radii	2820	11.7	1.7

Note. The table is ordered by the "trials" column, which is the 4096-sample average number of Monte-Carlo trials required from a log-uniform distribution to find a sample that satisfies the corresponding radius exclusion criterion. The last two columns (see Section 4) compare agreement between exclusion laws and the law of increasing differences (LID) for a nine-planet system: specifically, how often the listed exclusion law produces a system satisfying LID (second-last column), and vice versa (last column).

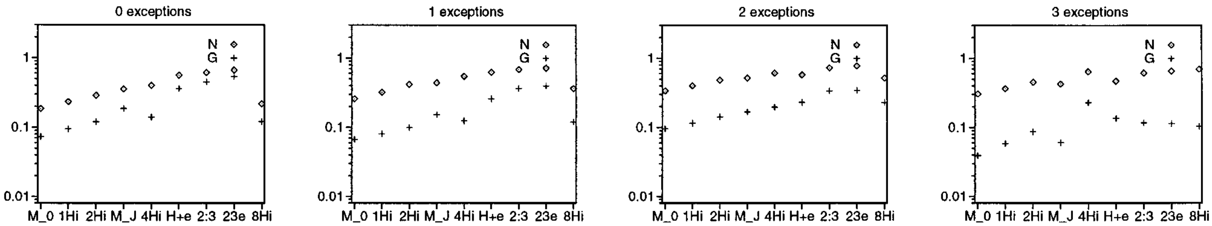

FIG. 2. The quantile of the Solar System's χ^2, i.e., the fraction of the 4096 random systems that have a χ^2 better (i.e., smaller) than that of the Solar System. We include the cases both with and without a gap ("G" and "N", respectively). The labels on the horizontal axis correspond to the names described in Table I.

consistently 1–1.5 orders of magnitude smaller than the mean of the random systems. We suggest that this is because the particular exceptions we investigated were historically designed specifically to make our Solar System fit better. In a different planetary system, other exceptions or even entirely different (and arbitrary) exception rules could be envisaged; for example, the planets could be split into two groups, one group satisfying one rule, while the other group satisfies another. The possibilities for inventing exception rules are essentially endless.

Figure 2 shows the Solar System's quantile—the fraction of random systems with the same exceptions that have a χ^2 better (i.e., smaller) than that of the Solar System. A small quantile would indicate that the Solar System fits Bode's law better than most random systems. Our Solar System's quantile is not exceptional if no gap is allowed, ranging from about 0.15 to 0.7, and only mildly exceptional if a gap is allowed ($\gtrsim$0.04), and generally becomes less exceptional as the random planetary systems are chosen with increasingly stringent radius-exclusion laws (moving right in each figure). The rightmost case, labeled $8H_i$, gives anomalously low quantiles in several cases. This probably reflects the fact that our evaluation of the normalization of χ^2 (see Appendix) is only approximate when the planets have different radius exclusions, and the approximation worsens with increasing exclusion.

One could also argue that the main asteroid belt should "count" as an object. Figure 3 shows the Solar System's quantile for this case, with no exceptions allowed. In this case, the Solar System's χ^2 value shows that it fits Bode's law better than 90%–98% of the random systems. However, this result is not particularly significant, given that the case we are examining is still, to some extent, tailored to the properties of our Solar System.

3.2. *Distributions of a, b, and c*

Although our chief concern in this paper is with how *well* random planetary systems fit Eq. (3), we can also discuss the values of the fitting parameters a, b, and c that we obtained.

The distribution of c is shown in Fig. 4. In general, increasing the strength of the radius exclusion shifts the distribution to the right, because we are forcing more space between planets. The peak of the histogram is sharpest with the strongest radius exclusion ($8H_i$) and no exceptions, because the regions allowed to contain planets are tightly squeezed by the exclusion. If we allow a gap (cf. the two bottom figures compared to the two top ones), Eq. (3) effectively models an extra planet, so the distribution shifts slightly toward smaller c values. Finally, increasing the number of exceptions (cf. the two right figures compared to the two left ones) causes the distribution to spread out and flatten (because there are fewer constraints), and shift to the right (because the planets that get ignored most often are the inner and outer ones—cf. Fig. 6). One should not read too much into the fact that the values of c for stringent radius exclusion laws tend to cluster around the value of 2.0, which is the value for our own Solar System. This value is biased by our decision to fit 9 planets between 0.2 and 50 AU, because $(50/0.2)^{1/8} = 1.99$.

Scatter plots of b *vs* a are presented in Fig. 5. The most obvious feature is that most systems appear above the line $a + b \approx 0.2$; this is expected because 0.2 is the smallest orbital radius that we allow. As we increase the exclusion from H_i to $8H_i$ (cf. the two bottom figures compared to the two top ones), the scatter decreases for the same reason

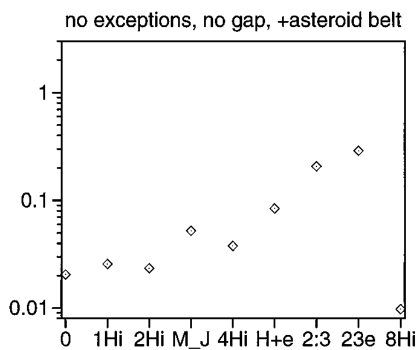

FIG. 3. The quantile of the Solar System, compared to 4096 random ones, when no exceptions or gaps are allowed, but the main asteroid belt is included as a "planet" at radius 2.8 AU.

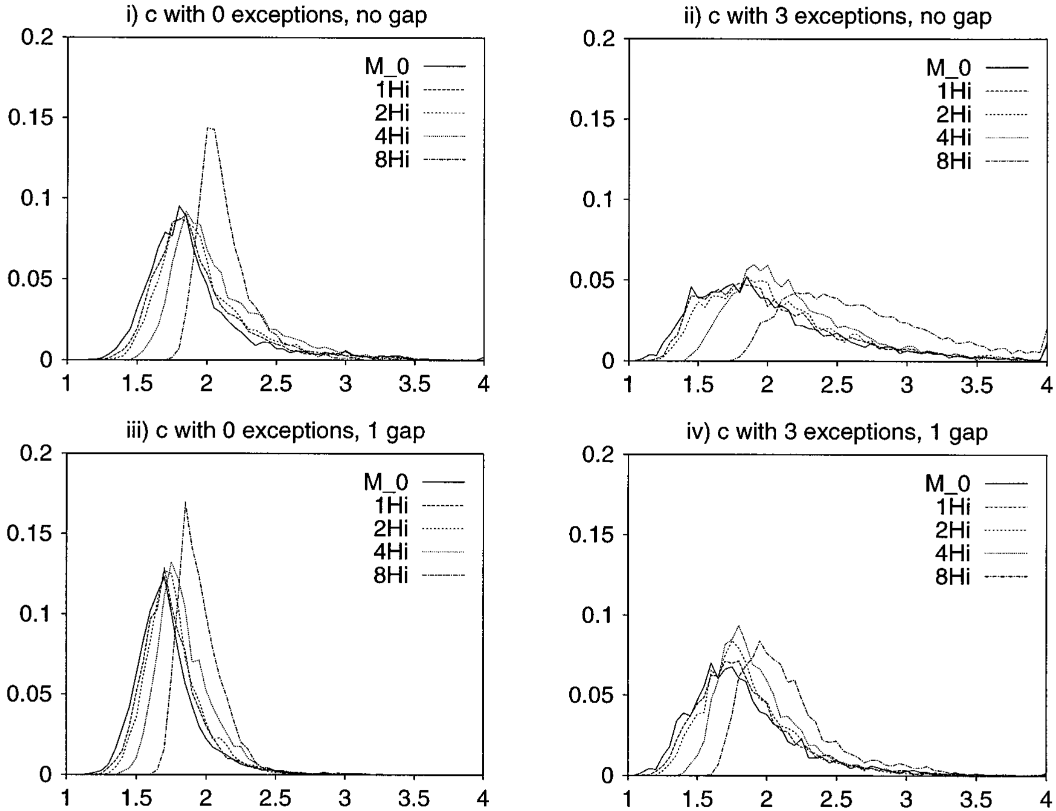

FIG. 4. Normalized histogram of the value of the fitting parameter c from Eq. (3). In each of these figures, the number of exceptions (i.e., the number of planets that are ignored in the fit) and the number of gaps (always 0 or 1) are held constant while we plot the distribution for various radius-exclusion laws as discussed in Section 2.1. The graphs that have 1 and 2 exceptions can be well approximated by interpolating between the left and the right figures.

the distribution of c becomes more peaked in Fig. 4: the systems are more tightly constrained by the exclusion law. The value of the parameter a in Eq. (3) is zero in about 25% of the samples, in which case Bode's law corresponds to a geometric sequence.

Finally, there was no observable correlation of a, b, or c with χ^2.

3.3. Other Observations

A histogram of which planet gets ignored when there is one exception and no radius exclusion is shown in Fig. 6. The innermost and outermost planets are ignored most often, which is expected since a planet with only one neighbor is less constrained than those with two.

The placement of the gap is approximately uniform between all planet pairs, for all types of systems. Adding a gap produced a better fit in about 65% of all cases.

It is prudent to show that our results are not strongly dependent upon the assumption that the underlying distribution is uniform in log r. If instead we assume a disk surface density that scales as $r^{-3/2}$, which roughly corresponds to the expected density in the protoplanetary disk (Lissauer 1993), then if all the planets are equally massive, they should be uniformly distributed in $\sqrt{r}$. We therefore performed our entire suite of experiments again, this time trying to fit an $a + bc^i$ law to planetary systems with an underlying random distribution that is uniform in $\sqrt{r}$. We find that most of the above results are qualitatively unchanged. For example, in the case with no exceptions and no gap, the fit of the random systems worsens, so that our Solar System's quantile gets better, but only by about 0.05 to 0.15. In the case of three exceptions and a gap, the Solar System's quantile is almost the same in the $\sqrt{r}$ distribution as in the log r distribution. Furthermore, as radius exclusion increases, the effect of the underlying distribution is suppressed because radius exclusion is biased toward accepting planetary systems that follow a roughly geometric progression. We conclude that our comparisons are not strongly affected by the assumption that the underlying distribution is uniform in log r.

4. THE LAW OF INCREASING DIFFERENCES

Efron (1971) and Conway and Zelenka (1988) have noted that the distance between planets in the Solar System

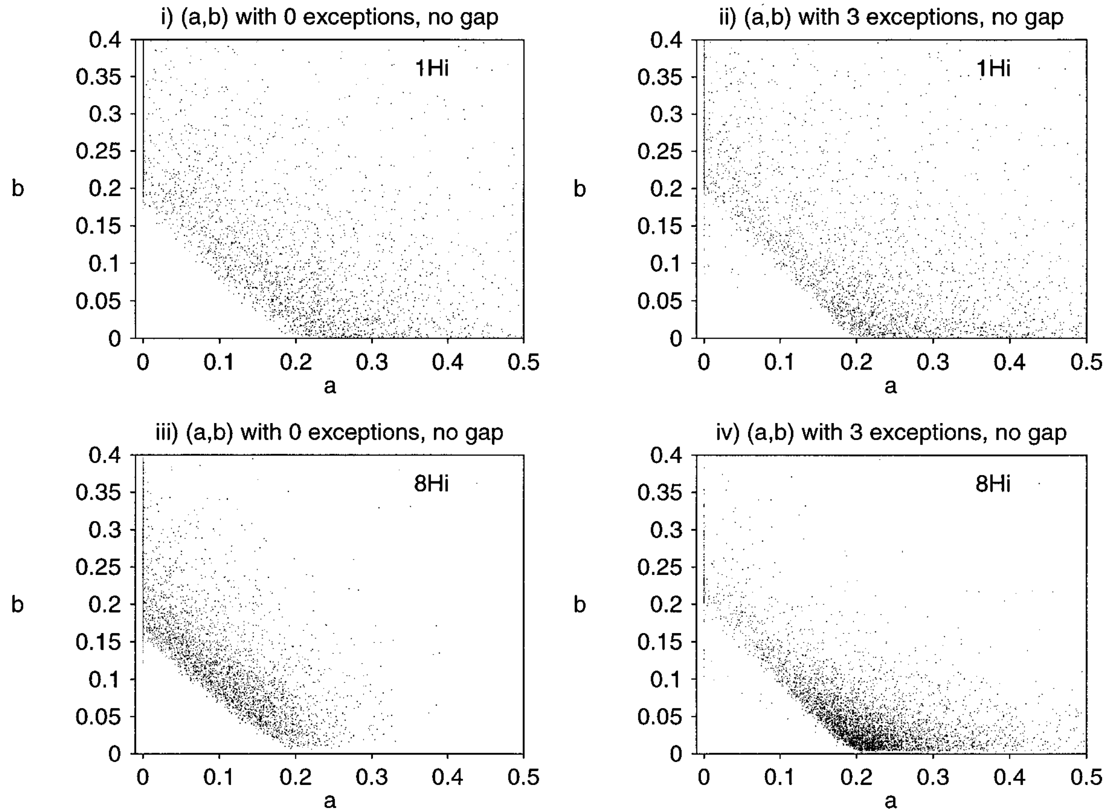

FIG. 5. Scatter plots of the fitting parameters b and a from Eq. (3). In Figs. i–iv, a is exactly zero in 47, 26, 47, and 5% of the cases, respectively.

is an increasing function of distance for all adjacent pairs except Neptune–Pluto; the "major" satellites of Jupiter, Saturn and Uranus also satisfy this relation (Conway and Zelenka 1988). They propose that this "law of increasing differences" is a reasonable null hypothesis to use when testing the statistical significance of Bode's law. They note that a pure log-uniform distribution produces increasing differences only a small percentage of the time and that if we assume the law of increasing differences then the success of Bode's law is unsurprising.

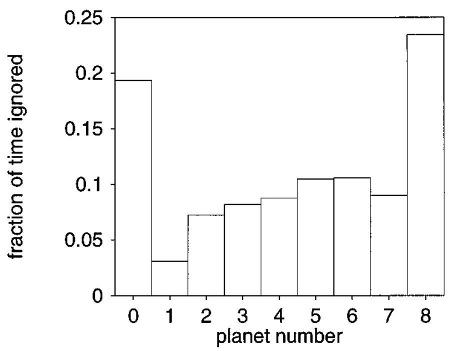

FIG. 6. A histogram showing the fraction of systems in which planet i is ignored, as a function of i, when one exception is allowed and there is no radius exclusion. The distribution does not change significantly in other systems.

We are uncomfortable with this law because it has no physical basis. For example, there is no dependence on planetary mass, which is unrealistic. We also note that the law does not apply to all satellites around any planet and that there is no natural way to define what constitutes a "major" satellite. This concern has prompted us to examine the relation between radius-exclusion laws and the law of increasing differences. Table I shows the occurrences of agreement between the law of increasing differences and radius exclusions. As Table I shows, (i) a system that satisfies radius exclusion rarely satisfies the law of increasing differences; (ii) one that satisfies the law of increasing differences will often satisfy all but the most stringent exclusion laws. We also observed that (iii) the number of trials required to find a random log-uniform sample that satisfies the law of increasing differences is 1 to 2 orders of magnitude larger than that for radius exclusion; (iv) random planetary systems that satisfy the law of increasing differences have χ^2 values that are 1.3 to 3 times smaller than ones generated using radius exclusion.

For these reasons, we believe that the law of increasing differences is a much more restrictive assumption than radius-exclusion laws, and in the absence of any physical justification, it does not form a sound basis from which to judge the validity of Bode's law.

5. DISCUSSION AND CONCLUSIONS

We have measured the deviation from Bode's law of planetary systems whose distances are distributed uniformly random in log r, subject to radius exclusion constraints. We find that, as radius exclusion becomes more stringent, the systems tend to fit Bode's law better. We compare these fits to that of our own Solar System. We find that, when no exceptions or gaps are allowed, our Solar System fits marginally better than random systems that follow weak radius-exclusion laws, but fits no better, or even worse, than those that satisfy more stringent but still reasonable radius exclusions. If one gap is allowed to be added, and up to three planets are ignored, then our Solar System fits significantly better than random ones built with weak radius exclusions and marginally better than ones with strong radius exclusions; however, this modest success for Bode's law probably arises because these rules (three planets removed, one gap added) were designed specifically in earlier centuries to make our Solar System fit better.

Even though our underlying distance distribution, uniform in log r, is scale invariant, the analysis of Graner and Dubrulle (1994) and Dubrulle and Graner (1994) does not apply to most of our cases, since the distribution of planetary masses and Hill radii is not scale-invariant. Further, we found that, even in the cases where the radius-exclusion law *is* scale-invariant (M_0, M_J, 2:3), the best-fitting generalized Bode law has $a \neq 0$ in 36% of the cases and hence is *not* scale-invariant.

Our approach to Bode's law is very simplistic. We ignore all of the details of planet formation (secular resonances, planet migration, resonance capture, chaos, *etc.*) and use simple stability criteria to discard unstable systems. The natural next step is to replace the approximate stability criteria we have used by actual orbit integrations. We conjecture that if we repeat the experiments of the present paper using a direct integration of several Gyr (e.g., Wisdom and Holman 1991) to discard unstable systems, we would find that the surviving systems fit a generalized Bode law better than the random ones in this paper and approximately as well as our own Solar System. This would strengthen our conclusion that the significance of Bode's law is simply that stable planetary systems tend to be regularly spaced.

APPENDIX: MEAN, VARIANCE, COMPRESSION OF UNIFORM DISTRIBUTIONS

If n numbers are chosen from the uniform random interval $U[0, 1]$ and sorted into nondecreasing order, then the mean and variance of the ith one X_i are (Rice 1988, problem 4.15)

$$\overline{X}_i = i/(n+1), \qquad \sigma^2_{X_i} = \frac{i(n-i+1)}{(n+1)^2(n+2)}, \qquad i = 1, \ldots, n, \tag{5}$$

which we then scale to the interval [log(0.2), log(50)] used in this paper. Radius exclusion makes the allowed intervals between planets smaller, thus compressing the uniform standard deviations, Eq. (5), by a factor C_i. To compute C_i, assume $r_i = 1$ and that all the planets have the same fractional Hill radius H and perfectly fit a Bode's law with exponent $c = \beta$ and $a = 0$. It is easy to show that

$$C_i = \frac{\log(\beta)}{\log(\beta) + \log(1-H) - \log(1+H)}, \tag{6}$$

finally giving

$$\sigma_i^2 = (\sigma_{x_i}/C_i)^2, \tag{7}$$

which we substitute into Eq. (4). When distance is measured in log r and all planets have the same radius exclusion, Eq. (6) is exact; otherwise it is only approximate.

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