# Menger's Theorem

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(This document describes optional material for this course.) In our discussion of the problem of finding a maximum cardinality set of edge-disjoint $s \to t$ paths in a graph we developed results that we can leverage to prove Menger's Theorem, a well-known result in graph theory. It is one of those max-this-equals-min-that results, like max-flow-equals-min-cut and bipartite-max-matching-equals-min-vertex-cover. All these results are instances of a general phenomenon known as "linear programming duality". I didn't have time to discuss Menger's Theorem in class, so here it is, for those who are interested.

**Menger's Theorem.** *Let $G = (V, E)$ be a digraph and $s, t \in V$. The maximum number of edge-disjoint $s \to t$ paths in $G$ is equal to the minimum number of edges whose removal from $G$ disconnects $t$ from $s$.*

Proof. Let $P$ be any set of edge-disjoint $s \to t$ paths, and $D$ be any set of edges whose removal from $G$ disconnects $t$ from $s$. By the pigeonhole principle, $|P| \leq |D|$: Every path in $P$ (pigeon) must use an edge in $D$ (pigeonhole); otherwise, the removal of $D$ from $G$ does not disconnect $t$ from $s$. So, there is a function $\phi : P \to D$ such that path $p \in P$ uses edge $\phi(p) \in D$. Since $P$ is edge-disjoint, $\phi$ must be one-to-one. So, by the pigeonhole principle $|P| \leq |D|$. This immediately implies the following:

**Fact.** *If $|P| = |D|$ then (a) $P$ is a* maximum *cardinality set of edge-disjoint $s \to t$ paths, and (b) $D$ is a* minimum *cardinality set of edges whose removal disconnects $t$ from $s$.*

Let $P$ be a maximum cardinality set of edge-disjoint $s \to t$ paths. Let $\mathcal{F}$ be the flow network obtained from $G$ by removing all edges into $s$ and from $t$, and assigning capacity 1 to every edge. Let $f$ be a maximum flow of $\mathcal{F}$, $(S, T)$ be a minimum $(s, t)$-cut of $\mathcal{F}$, and $D = out(S) \cap in(T)$; i.e., $D$ is the set of edges that cross the cut from $S$ to $T$. By definition, the removal of $D$ from the graph of $\mathcal{F}$ disconnects $t$ from $s$, and so the removal of $D$ from $G$ also disconnects $t$ from $s$ (why?). We have:

$$\begin{aligned} |P| &= \mathcal{V}(f) && \text{[proved in class in discussion of max edge-dispoint path problem]} \\ &= c(S, T) && \text{[by max-flow-min-cut]} \\ &= |D| && \text{[by definition of } D \text{ and the fact that all edges have capacity 1]} \end{aligned}$$

By part (b) of the above Fact, $D$ is a minimum cardinality set of edges whose removal from $G$ disconnects $t$ from $s$.

So, we proved that the maximum number of edge-disjoint $s \to t$ paths in $G$ is equal to the minimum number of edges whose deletion from $G$ disconnects $t$ from $s$. □

This proof of Menger's theorem immediately suggests an algorithm that, given a directed graph $G$ and nodes $s, t$, finds a minimum cardinality set of edges whose deletion from $G$ disconnects $t$ from $s$:

(1) Construct $\mathcal{F}$ from $G$, $s$, and $t$
(2) Find a maximum flow $f$ of $\mathcal{F}$
(3) Using $f$, find a minimum cut $(S, T)$ of $\mathcal{F}$
(4) Return the set of edges $out(S) \cap in(T)$

This takes $O(mn)$ time, where $m$ is the number of edges of $G$ and $n$ is the number of nodes of $G$.