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Proof of "Flow Theorem"

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Let $\mathcal{F} = (G, s, t, c)$ be a flow network, where $G = (V, E)$. The theorem below relates the value of an arbitrary flow f in F to the traffic on the edges connecting the two components of an arbitrary (s,t)-cut (S, T) of F. This result is key in the proof of correctness of the Ford-Fulkerson algorithm, as we saw in class.

Notation. If $u \in V$, $out(u)$ is the set of edges out of u, i.e., $out(u) = \{(u, v) : \exists v \in V \text{ such that } (u, v) \in E\}$; similarly, $in(u)$ is the set of edges into u, i.e., $in(u) = \{(v, u) : \exists v \in V \text{ such that } (v, u) \in E\}$. We generalize this for $X \subseteq V$ in the obvious way: $out(X)$ is the set of edges out of nodes in X, i.e., $out(X) = \cup_{u \in X} out(u)$; similarly, $in(X)$ is the set of edges into nodes in X, i.e., $in(X) = \cup_{u \in X} in(u)$. Note that if both endpoints of an edge are in X, then the edge is in $out(X)$ as well as in $in(X)$. The theorem below states that the value of any flow can be determined by looking at the traffic on the edges that cross any cut (S, T): Add up the traffic on the edges entering T from S and subtract the traffic on the edges entering S from T.

Flow Theorem. *For any flow f and any (s,t)-cut (S, T) of the flow network (G, s, t, c),*

$$\mathcal{V}(f) = \sum_{e \in out(S) \cap in(T)} f(e) \quad - \sum_{e \in out(T) \cap in(S)} f(e).$$

PROOF. Let f be an arbitrary flow and (S, T) be an arbitrary (s,t)-cut of (G, s, t, c). By definition of $\mathcal{V}(f)$ and the fact that there are no edges into s we have that $\mathcal{V}(f) = \sum_{e \in out(s)} f(e) - \sum_{e \in in(s)} f(e)$; and for all nodes $v \in S - \{s\}$ by the conservation property we have that $\sum_{e \in out(v)} f(e) - \sum_{e \in in(v)} f(e) = 0$. Therefore,

$$\begin{aligned}
\mathcal{V}(f) &= \sum_{v \in S} \Big(\sum_{e \in out(v)} f(e) \; - \sum_{e \in in(v)} f(e) \Big) \\
&= \sum_{v \in S} \sum_{e \in out(v)} f(e) \; - \sum_{v \in S} \sum_{e \in in(v)} f(e) && \text{[rearrange terms]} \\
&= \sum_{e \in out(S)} f(e) \; - \sum_{e \in in(S)} f(e) && \text{[def. of } out,\ in\text{]} \\
&= \Big(\cancel{\sum_{e \in out(S) \cap in(S)} f(e)} + \sum_{e \in out(S) \cap in(T)} f(e) \Big) - \Big(\cancel{\sum_{e \in in(S) \cap out(S)} f(e)} + \sum_{e \in in(S) \cap out(T)} f(e) \Big) \\
&= \sum_{e \in out(S) \cap in(T)} f(e) \; - \sum_{e \in out(T) \cap in(S)} f(e).
\end{aligned}$$

In going from the pre-penultimate to the penultimate line in the above derivation, we use the fact that an edge out of a node in S goes either into a node in S or to a node in T but not both (because S and T are disjoint); and, similarly, an edge into a node in S goes out of either a node in S or a node in T but not both. Thus, $out(S) = \big(out(S) \cap in(S)\big) \cup \big(out(S) \cap in(T)\big)$; and, similarly, $in(S) = \big(in(S) \cap out(S)\big) \cup \big(in(S) \cap out(T)\big)$, where both unions are disjoint. $\square$