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Dijkstra's shortest paths algorithm

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Shown below is pseudocode for Dijkstra's algorithm. The input is a directed graph $G = (V, E)$ with non-negative edge weights $\mathbf{wt}(u, v)$ for every edge $(u, v) \in E$, and a distinguished node s, called the ***source*** (or ***start***) node. The algorithm computes, for each node $u \in V$, the weight of a minimum-weight path from s to u. (It can be easily modified to compute, for each node u, the predecessor of u in a minimum-weight path from s to u, in addition to the weight of such a path.)

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1  R := ∅
2  d(s) := 0
3  for each v ∈ V − {s} do d(v) := ∞
4  while R ≠ V do
5      let u be a node not in R with minimum d-value (i.e., u ∈ V − R and ∀ u′ ∈ V − R, d(u) ≤ d(u′))
6      R := R ∪ {u}
7      for each v ∈ V such that (u, v) ∈ E do
8          if d(u) + wt(u, v) < d(v) then d(v) := d(u) + wt(u, v)
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Intuitively Dijkstra's algorithm works as follows. It maintains a set R (the "explored" part of the graph, consisting of nodes to which it has determined the weight of shortest paths). We say that an $s \to u$ path is an ***R-path*** (to u) if every node on the path except (possibly) u is in the set R. The algorithm also maintains, for every node u, a label $d(u)$, which is the minimum weight of R-paths to u. The algorithm starts with an empty R, and it greedily expands the set R with the node u that is not presently in R and has minimum d-value. The algorithm then updates the d-values of the nodes adjacent to u to account for fact that there may now exist shorter R-paths to these nodes, going through u. When R contains all nodes, ***any*** $s \to u$ path is an R-path to u and so $d(u)$ is the minimum weight of $s \to u$ paths, which is what we want to compute.

We now prove the correctness of the algorithm. X_i denotes the value of a variable X at the end of iteration i of the while loop. For every node u, we define $\delta(u)$ to be the minimum weight of $s \to u$ paths (∞ if there is no $s \to u$ path). The first claim states that the d-value of every node does not increase in time.

Claim 1 *For every node v and iterations i, j, if $i \leq j$ then $d_i(v) \geq d_j(v)$.*

PROOF. After being initialized (in line 1 or 2), the value of $d(v)$ is changed only in line 8, where it is obviously assigned a smaller value than before. □

Claim 2 *If node u is added to R in iteration i the value of $d(u)$ does not change in iteration i.*

PROOF. Suppose for contradiction that u is added to R in iteration i and $d(u)$ changes in iteration i. Thus, by the algorithm, (u, u) is an edge and $d_{i-1}(u) + \mathbf{wt}(u, u) < d_{i-1}(u)$ (where $d_0(u)$ — i.e., if $i = 1$ — refers to the initial value of $d(u)$). But then $\mathbf{wt}(u, u) < 0$, contradicting the assumption that weight of every edge is non-negative. □

Claim 3 *For every node v and iteration i, if $d_i(v) = k \neq \infty$ then there is an R_i-path to v of weight k.*

PROOF. By induction on the iteration number $i \geq 0$. For the basis $i = 0$, i.e., just before we start the while loop, we have $R_0 = \varnothing$, $d_0(s) = 0$, and $d_0(u) = \infty$ for every node $u \neq s$. It is true that there is an $s \to s$ R_0-path of weight 0.

For the induction step, let $i > 0$ and suppose the claim holds at the end of iteration $i - 1$. Let u be the node added to R in iteration i and consider any node v. If $d(v)$ does not change in iteration i, the claim holds after iteration i by induction hypothesis. If $d(v)$ changes in iteration i and since (by Claim 2) $d_{i-1}(u) = d_i(u)$, by the algorithm $d_i(v) = d_{i-1}(u) + \mathbf{wt}(u, v)$ and $d_{i-1}(u) \neq \infty$. By the induction hypothesis, there is an R_{i-1}-path to u of weight $d_{i-1}(u)$, say path p. Since $R_i = R_{i-1} \cup \{u\}$, path p followed by the edge (u, v) is an R_i-path to v, whose weight is $\mathbf{wt}(p) + \mathbf{wt}(u, v) = d_{i-1}(u) + \mathbf{wt}(u, v) = d_i(u) + \mathbf{wt}(u, v)$. So, the claim holds after iteration i. ☐

Claim 4 *For every node u, if u is added to R in iteration i and $d_i(u) = \infty$ then there is no $s \to u$ path.*

PROOF. Suppose, for contradiction, that some node u is added to R in iteration i and $d_i(u) = \infty$ but there is an $s \to u$ path. Without loss of generality, assume that i is the earliest iteration in which this happens. Since s is added to R in iteration 1 and $d_1(s) \neq \infty$, $i > 1$. So $s \in R_{i-1}$ and $u \notin R_{i-1}$ (since u is added to R in iteration i). Thus, there is an edge (x, y) on the $s \to u$ path with $x \in R_{i-1}$ and $y \notin R_{i-1}$. Let $j < i$ be the iteration in which x was added to R; by the definition of i, $d_j(x) \neq \infty$ and so by the algorithm $d_j(y) \neq \infty$. By Claim 1, $d_{i-1}(y) \neq \infty$. The facts that (a) $y \notin R_{i-1}$ and (b) $d_{i-1}(y) \neq \infty$ contradict that in iteration i the algorithm added to R the node u with $d_{i-1}(u) = \infty$. ☐

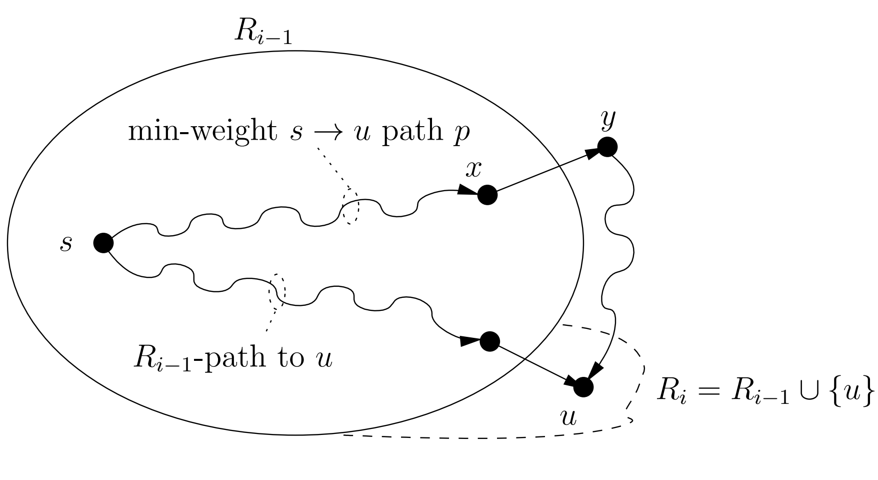
Claim 5 *For every node u and every iteration $i \geq 1$, if u is added to R in iteration i, then $d_i(u) = \delta(u)$.*

PROOF. By complete induction on i. Suppose u is the node added to R in iteration i, and suppose the claim holds for all nodes added to R before iteration i.

If $i = 1$, the claim holds since (by the initialization in lines 2–3) the node added to R in iteration 1 is s and $d_1(s) = 0 = \delta(s)$.

If $i > 1$, the claim holds by Claim 4 if $d_i(u) = \infty$. So, suppose $d_i(u) \neq \infty$. Then by Claim 3 there is an R_i-path of weight $d_i(u)$ from s to u; therefore $d_i(u) \geq \delta(u)$. We will now show that $d_i(u) \leq \delta(u)$, proving that $d_i(u) = \delta(u)$, as wanted.

Since there is an R_{i-1}-path to u, there is also a minimum weight $s \to u$ path, say p (refer to Figure 1).



Since u is added to R in iteration i, $u \notin R_{i-1}$ (recall that $i > 1$, so iteration $i-1$ exists, and R_{i-1} is well defined). Since $s \in R_{i-1}$ and $u \notin R_{i-1}$, p contains an edge (x, y) such that $x \in R_{i-1}$ and $y \notin R_{i-1}$ (it is possible that $y = u$). Let j be the iteration is which x was added to R, so $j \leq i-1$. Since p is a minimum-weight $s \to u$ path, the prefix p_x of p up to node x is a minimum-weight $s \to x$ path, i.e., $\mathbf{wt}(p_x) = \delta(x)$. We have:

$$
\begin{aligned}
d_i(u) &= d_{i-1}(u) && \text{by Claim 2} \\
&\leq d_{i-1}(y) && \text{by definition of } u\text{, since } u, y \notin R_{i-1} \\
&\leq d_j(y) && \text{by Claim 1, since } j \leq i-1 \\
&\leq d_j(x) + \mathbf{wt}(x, y) && \text{by Claim 2 and line 8, since } x \text{ is added to } R \text{ in iteration } j \\
&= \delta(x) + \mathbf{wt}(x, y) && \text{by the induction hypothesis, since } j < i \\
&= \mathbf{wt}(p_x) + \mathbf{wt}(x, y) && \text{since } \mathbf{wt}(p_x) = \delta(x)\text{, as argued above} \\
&\leq \mathbf{wt}(p) && \text{since all edges have non-negative weight} \\
&= \delta(u) && \text{by definition of } p
\end{aligned}
$$

So, $d_i(u) \leq \delta(u)$, as needed to complete the proof that $d_i(u) = \delta(u)$. □

The algorithm terminates, since one node is added to R in each iteration. The next theorem states that when the algorithm terminates, it has computed the weight of a minimum-weight $s \to u$ path, for every node u.

Theorem 6 *When the algorithm terminates, for every node u, $d(u) = \delta(u)$.*

PROOF. By Claim 5, when u is added to R, $d(u) = \delta(u)$. By Claim 1, $d(u)$ cannot later be assigned a larger value, and by Claim 3 it cannot later be assigned a smaller value. So when the algorithm terminates, $d(u) = \delta(u)$. □

Running time of Dijkstra's algorithm. Let n be the number of nodes and m be the number of edges in the graph. The running time of Dijkstra's algorithm depends on the data structure used to store $d(u)$.

In the simplest implementation, we store d in an array of n elements, one per node, in no particular order. The initialization of R and d takes $O(n)$ time. The while loop is executed $n-1$ times, because initially R has one node, we add one node to it in each iteration, and the loop ends when all n nodes are in R. Each iteration of the loop takes $O(n)$ time (to find the minimum element in array d of a node that is not in R, and to update the relevant entries of d). So the loop in total takes $O(n^2)$ time. This implementation then takes $O(n) + O(n^2) = O(n^2)$ time.

We can also use a heap to store the d-values of the nodes that are not in R. Thus we can find a node not in R with the minimum d-value by performing an ExtractMin operation, which takes $O(\log n)$ time; and we can update the value of d for a node by performing a ChangeKey operation, which also takes $O(\log n)$ time. We perform n ExtractMin operations, one in each iteration of the while loop. We perform at most m ChangeKey operations: at most once for each edge (u, v), in the iteration of the while loop in which u is added to R. In the initialization we must also perform a BuildHeap operation, to create the initial heap; this takes $O(n)$ time. Thus, the total time required to process all these operations is $O(n) + O(n \log n) + O(m \log n) = O\big((m+n)\log n\big)$. If we assume that there is a path from s to each node, then $m \geq n-1$, and so the above expression simplifies to $O(m \log n)$.

If the graph is "dense", i.e., it has (roughly) an edge between every two nodes, then $m = \Theta(n^2)$, and in that case the simple array implementation is actually faster! However, in practice often the graph is "sparse" — typically, each node has a constant or perhaps a logarithmic number of neighbours, so $m = \Theta(n)$ or $m = \Theta(n \log n)$. In this case, the heap implementation of Dijkstra's algorithm is substantially faster.