# Notes on complex numbers

This is a brief review of complex numbers, to provide the relevant background that students need to follow the lectures on the Fast Fourier Transform (FFT). Students have previously encountered this material in their linear algebra courses.

- **The "imaginary unit"** $i = \sqrt{-1}$**.** This is a quantity which, multiplied by itself, yields $-1$; i.e., $i^2 = -1$. It is called "imaginary" because no real number has this property.

- **Standard representation of complex numbers.** A complex number has the form $z = a + bi$, where $a$ and $b$ are real numbers; $a$ is the "real part" of $z$ and $b$ is the "imaginary part" of $z$. We can think of $z$ as the pair $(a, b)$, and so complex numbers can be mapped to the Cartesian plane. Just as we can geometrically view the set of real numbers as the set of points along a straight line (the "real line"), we can view complex numbers as the set of points on a plane (the "complex plane"). So in a sense complex numbers are a way of "packaging" a pair of real numbers into a single quantity.

- **Equality between complex numbers.** We define two complex numbers to be equal to each other if and only if they have the same real parts and the same imaginary parts. In other words, if viewed as points on the plane, the two numbers coincide.

- **Operations on complex numbers.** We can perform addition and multiplication on compex numbers. Let $z = a + bi$ and $z' = a' + b'i$.

  - *Addition:* The sum of $z$ and $z'$ is, by definition, $z + z' = (a + a') + (b + b')i$. This can be "justified" algebraically simply by noting that $z + z' = (a + bi) + (a' + b'i) = (a + a') + (b + b')i$. (The word "justified" is in quotes above, because here we are abusing notation horribly, but very usefully. We use the symbol $+$ in three different ways: to separate the real and imaginary part of the complex numbers, to represent the addition operation on real numbers, and to represent the addition operation on complex numbers that we are defining!) The geometric intuition for this operation is clear if we view $z$ and $z'$ as points on the plain; or, equivalently, as two-dimensional position vectors. Then the sum of the two complex numbers is the sum of the two vectors.

  - *Multiplication:* The product of $z$ and $z'$ is, by definition, the complex number $z \cdot z' = (aa' - bb') + (ab' + a'b)i$. This can also be "justified" algebraically (through the same abuse of notation): $z \cdot z' = (a + bi) \cdot (a' + b'i) = aa' + ab'i + a'bi + bb'(i^2) = (aa' - bb') + (ab' + a'b)i$. There is no obvious geometric interpretation of this operation in this form, but we will see a nice interpretation below (see Euler's formula).

- **The field of complex numbers.** Addition and multiplication of complex numbers defined in the above manner behave like addition and multiplication of real numbers in the sense that they satisfy the following properties (where $z, z', z''$ are complex numbers):

  - *Commutativity of addition and multiplication:* $z + z' = z' + z$, and $z \cdot z' = z' \cdot z$.

  - *Associativity of addition and multiplication:* $z + (z' + z'') = (z + z') + z''$, and $z \cdot (z' \cdot z'') = (z \cdot z') \cdot z''$.

  - *Additive and multiplicative identities:* There is a "zero" complex number, namely $0 + 0i$, which added to any complex number $z$ yields $z$; and a "one" complex number, $1 + 0i$, which muliplied by any complex number $z$ yields $z$. Verify.

– *Additive inverse:* For each complex number $z$ there is a complex number $z'$, called the ***additive inverse*** of $z$ (and denoted $-z$), such that $z + z'$ is the complex "zero". The additive inverse of $a + bi$ is $(-a) + (-b)i$. <span style="color:red">Verify.</span>

– *Multiplicative inverse:* For each complex number $z$ ***other than*** the complex number "zero" $(0 + 0i)$, there is a complex number $z'$, called the ***multiplicative inverse*** of $z$, and denoted $z^{-1}$, such that $z \cdot z'$ is the complex "one". The multiplicative inverse of $a + bi$ (when at least one of $a, b$ is non-zero) is the complex number $a/(a^2 + b^2) + \left(-b/(a^2 + b^2)\right)i$. <span style="color:red">Verify.</span>

Any algebraic structure, i.e., any set of objects with two operators $+$ ("plus") and $\cdot$ ("times"), that satisfy the above properties is called a ***field***. The rational numbers, the real numbers, and the complex numbers are examples of fields. <span style="color:red">(Is the set of natural numbers with ordinary addition and multiplication a field? Is the set of all integers with ordinary addition and multiplication a field?)</span> Such structures in their abstract form are one of the topics discussed in MATD01. In general, we can define a vector space — and therefore do linear algebra — over any field, not just the field of real numbers. So we can do linear algebra over the field of complex numbers: We can define matrices of complex numbers and add or multiply such matrices as we do matrices of reals. We can define vectors of complex numbers, spaces of such vectors, linear transformations between such spaces, and so on.

- ***Polar coordinate representation of complex numbers:*** Consider the complex number $z = a + ib$ as the point $(a, b)$ in two-dimensional space. Recall that a point in two-dimensional space can be fixed either by its Cartesian coordinates (in our case the pair $(a, b)$), or by its ***polar coordinates*** $(r, \theta)$, where $r$ is the distance of $(a, b)$ from the origin $(0, 0)$, i.e., $r = \sqrt{a^2 + b^2}$; and $\theta$ is the angle between the horizontal axis and the line segment that starts at the origin and ends at point $(a, b)$. (The distance $r$ from the origin determines a circle centered at the origin with radius $r$, and the angle $\theta$ determines the point on that circle where $(a, b)$ lies.) By its definition the angle $\theta$ satisfies the property that $\cos\theta = a/r$ and $\sin\theta = b/r$. Thus, the complex number $z = a + bi$ can also be written as $z = r(\cos\theta + i\sin\theta)$; this is sometimes called the ***polar form*** of $z$. So, the complex number $z$ can be viewed as the pair $(a, b)$ (in Cartesian coordinates), or as the pair $(r, \theta)$ (in polar coordinates).

- ***Euler's formula:*** Euler's formula states that for any real number $\theta$, $e^{i\theta} = \cos\theta + i\sin\theta$, where the arguments to sin and cos are expressed in radians. It describes a fundamental relationship between the basic trigonometric functions and the exponential function. <span style="color:red">We will take the formula for granted, without proof. It can be proved by considering the McLaurin expansions of $e^{i\theta}$, $\cos\theta$, and $\sin\theta$. Formally speaking, $e^{i\theta}$ — an exponential to a complex number! — is **defined** as the McLaurin expansion.</span>

Plugging $\theta = \pi$ into Euler's formula we get $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0 = -1$, i.e. $e^{i\pi} + 1 = 0$. <span style="color:red">Pause here to marvel at the awesomeness of this: The four most ubiquitous constants in mathematics $(0, 1, \pi, e)$ all tied up in one amazing equality.</span>

As we have seen, the complex number $z = a + bi$, in polar form, is $z = r(\cos\theta + i\sin\theta)$; by Euler's formula, $z = re^{i\theta}$. This representation is called the ***exponential form*** for complex numbers and it is extremely useful. (Recall that $r = \sqrt{a^2 + b^2}$; this quantity is called the ***magnitude*** of $z$. The angle $\theta$ is such that $\cos\theta = a/r$ and $\sin\theta = b/r$; this quantity is called the ***argument*** of $z$ or, if you are an electrical engineer, the ***phase*** of $z$.)

The exponential form gives a useful geometric interpretation for multiplication of complex numbers. Let $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$ be two complex numbers. Then $z \cdot z' = (re^{i\theta}) \cdot (r'e^{i\theta'}) = rr'e^{i(\theta + \theta')}$. In other words, to multiply two complex numbers in exponential form, we multiply their magnitudes and add their angles!

- **The unit circle:** Consider the complex numbers on the complex plane. The **unit circle** consists of all complex numbers with magnitude 1, i.e., the numbers that are at distance 1 from the origin. Obviously these are precisely the points along the circle whose centre is the origin and whose radius is 1. That is, these are the complex numbers of the form $e^{i\theta}$, where $0 \leq \theta < 2\pi$.

  Consider any two such numbers, $z = e^{i\theta}$ and $z' = e^{i\theta'}$. Their product is $z \cdot z' = e^{i(\theta+\theta')}$, which is another point on the unit circle. For any $n \in \mathbb{N}$, the $n$th power of $z$ is $z^n = (e^{i\theta})^n = e^{in\theta}$ — i.e., the point on the unit circle whose angle is $n$ times the angle of $z$.

- **The $n$-th roots of unity:** The complex $n$-th roots of unity are the solutions to the equation $z^n = 1$ over the complex numbers. (More precisely, the solutions to the equation $z^n = 1 + 0i$.) Note that the number 1, as a complex number, has magnitude 1 and angle that is any integer multiple of $2\pi$, i.e., is of the form $2\pi k$, where $k$ is an integer; all these are simply the angle 0. Thus, if $z = re^{i\theta}$ is a complex root of unity, we must have $z^n = r^n e^{in\theta} = e^{i2\pi k}$, for some integer $k$. Thus, it must be the case that $r = 1$ and $n\theta = 2\pi k$, i.e., $\theta = (2\pi/n)k$, for some integer $k$. Note that there are exactly $n$ distinct angles $\theta$ in the range $0 \leq \theta < 2\pi$ that satisfy $\theta = (2\pi/n)k$, for some integer $k$, namely those obtained by setting $k = 0, 1, \ldots n-1$. (Setting $k$ to other integer values yields one of these $n$ angles, when reduced to the range between 0 and $2\pi$.) Therefore, for each $n$, there are exactly $n$ distinct comlpex $n$-th roots of unity. They all lie of the unit circle and they go around the circle, starting from the point $(1,0)$ (the compex number 1 that has mantitude 1 and angle 0), and proceeding counterclockwise in angle increments of $2\pi/n$.