Question 2 (cont'd)

Now we will define the subproblems differently.

Assume the v_is are positive integers.

For $i = 0,1, \ldots, n$, let $V_i = \sum_{t=1}^{i} v_t$. ($V_0 = 0$.)

Total value of items $1, \ldots, i$

For $i = 0,1, \ldots, n$, and $v = 0,1, \ldots, V_i$,

$W(i, v) =$ the **minimum weight** of a subset of items $\{1,2, \ldots, i\}$ whose **value** is $\geq v$.

Compare to the subproblems we defined before:

For $i = 0,1, \ldots, n$, and $c = 0,1, \ldots, C$,

$K(i, c) =$ the **maximum value** of a subset of items $\{1,2, \ldots, i\}$ whose **weight** is $\leq c$.

Question 2 (cont'd)

For $i = 0,1, \dots, n$, and $v = 0,1, \dots, V_i$,

$W(i, v) =$ the minimum weight of a subset of items $\{1,2, \dots, i\}$ whose value is $\geq v$.

- Give a recursive formula to compute the subproblems.
- Describe your DP algorithm in pseudocode.
- Analyze the running time of your algorithm.
- Modify the algorithm to find the actual set of items of maximum value whose weight does not exceed the knapsack capacity C.

Question 2 — answer (cont'd)

- Recursive formula to compute the subproblems.

Case 1: $v > V_{i-1}$. (Lightest set of items of value $\geq v$ must use item i.)

$$W(i, v) = W(i-1, \max(0, v - v_i)) + w_i$$

Case 2: $v \leq V_{i-1}$. (Lightest set of items of value $\geq v$ may or may not use item i.)

$$W(i, v) = \min(W(i-1, v), \\ W(i-1, \max(0, v - v_i)) + w_i)$$

Question 2 — answer (cont'd)

- Describe your DP algorithm in pseudocode.

$V[0] := 0$; **for** $i := 1$ **to** n **do** $V[i] := V[i-1] + v_i$
for $i := 0$ **to** n **do** $W[i, 0] := 0$
for $v := 1$ **to** $V[n]$ **do** $W[0, v] := 0$
for $i := 1$ **to** n **do**
  for $v := 1$ **to** $V[i]$ **do**
    if $v > V[i-1]$ **then**
      $W[i, v] := W[i-1, \max(0, v - v_i)] + w_i$
    else
      $W[i, v] := \min(W[i-1, v],$
                $W[i-1, \max(0, v - v_i)] + w_i)$
return $\max\{v : W[n, v] \leq C\}$

Question 2 — answer (cont'd)

- Analyze the running time of your algorithm.

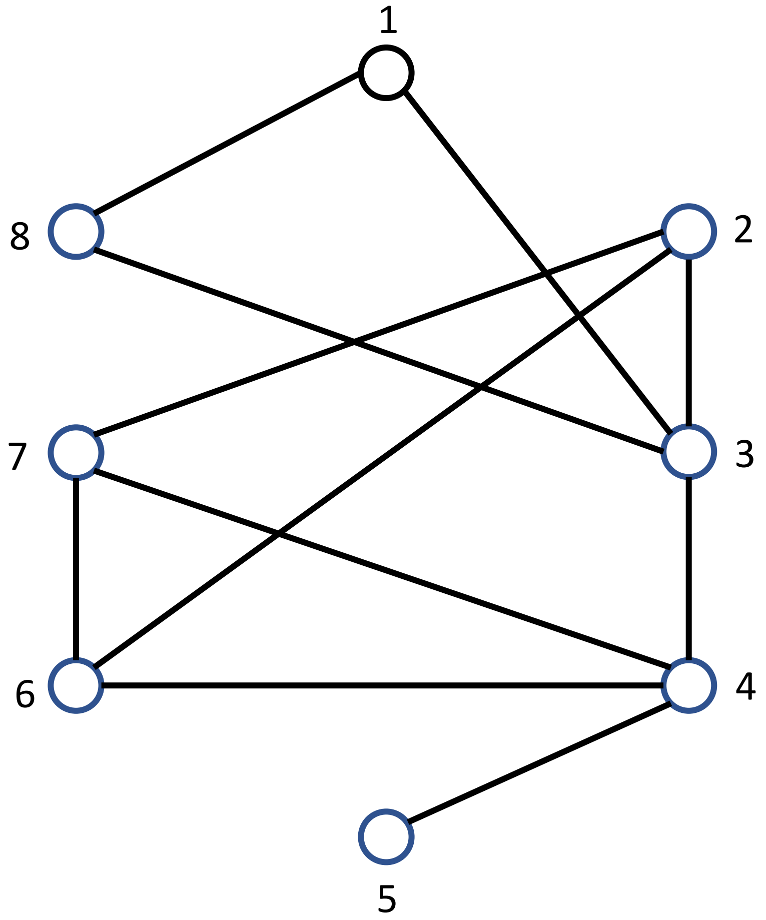
$$\Theta(n \cdot \sum_{i=1}^{n} v_i)$$

This is pseudopolynomial.

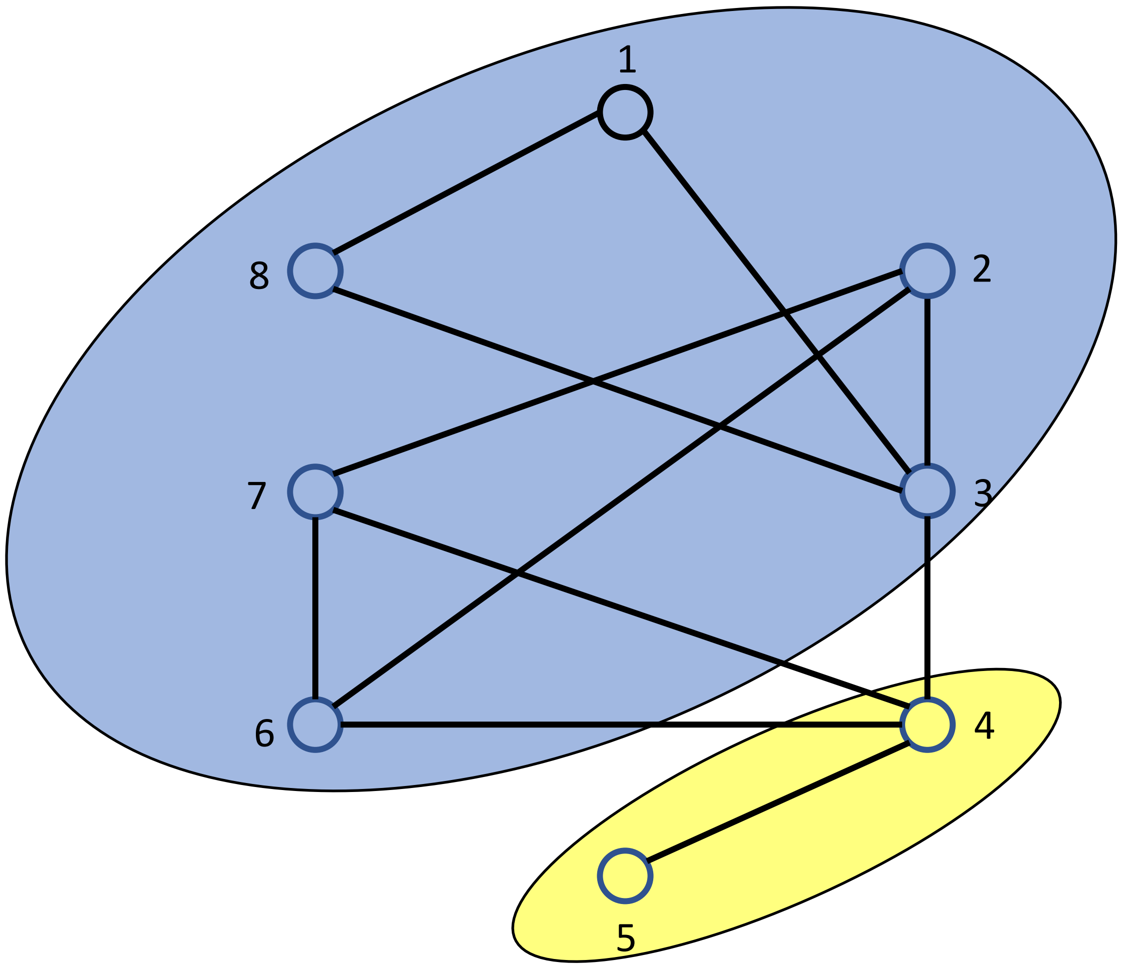
This version of the algorithm (with subproblems based on value rather than weight) is the basis for a **polynomial-time approximation algorithm** for knapsack that we will see at the end of the course.

Approximation local search algorithm for max cut

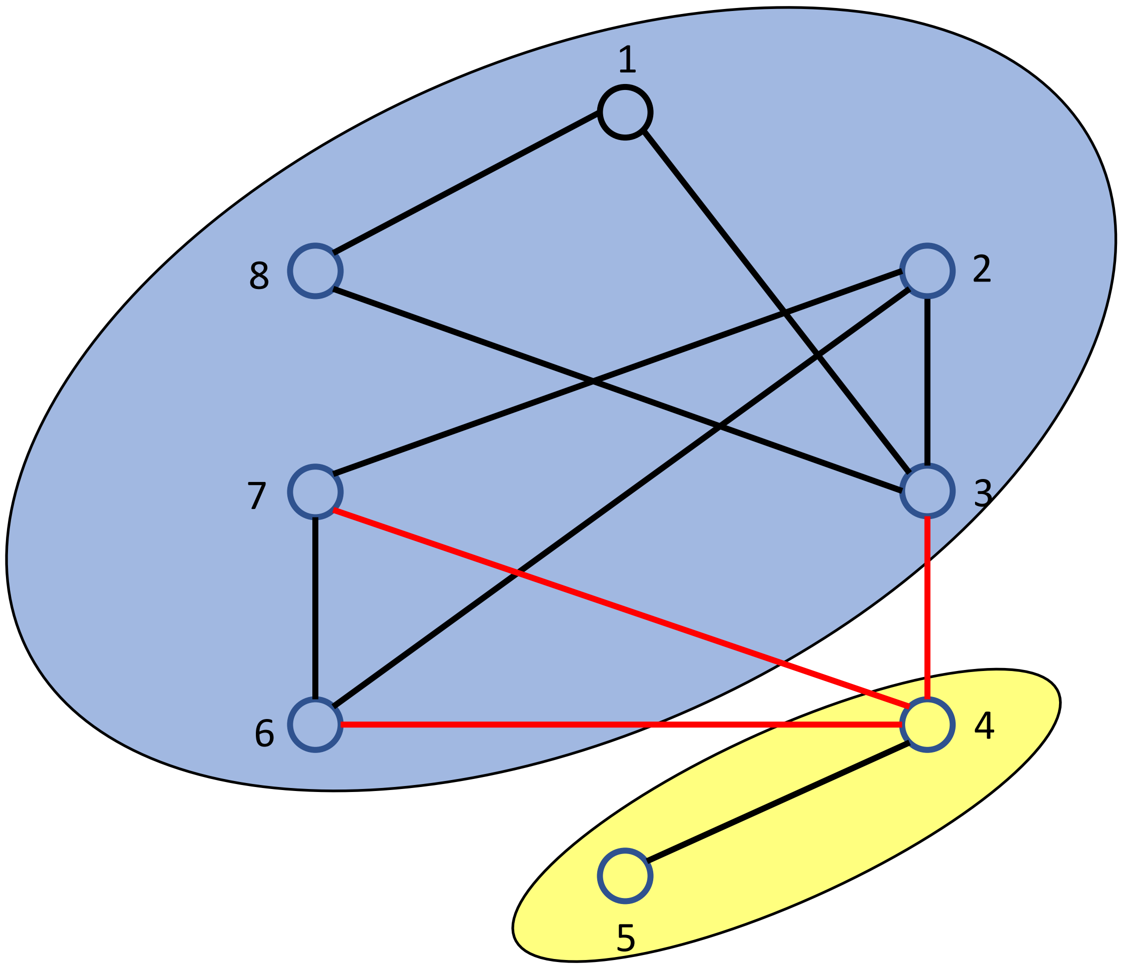
Graph with 8 nodes and 11 edges



A cut of the graph

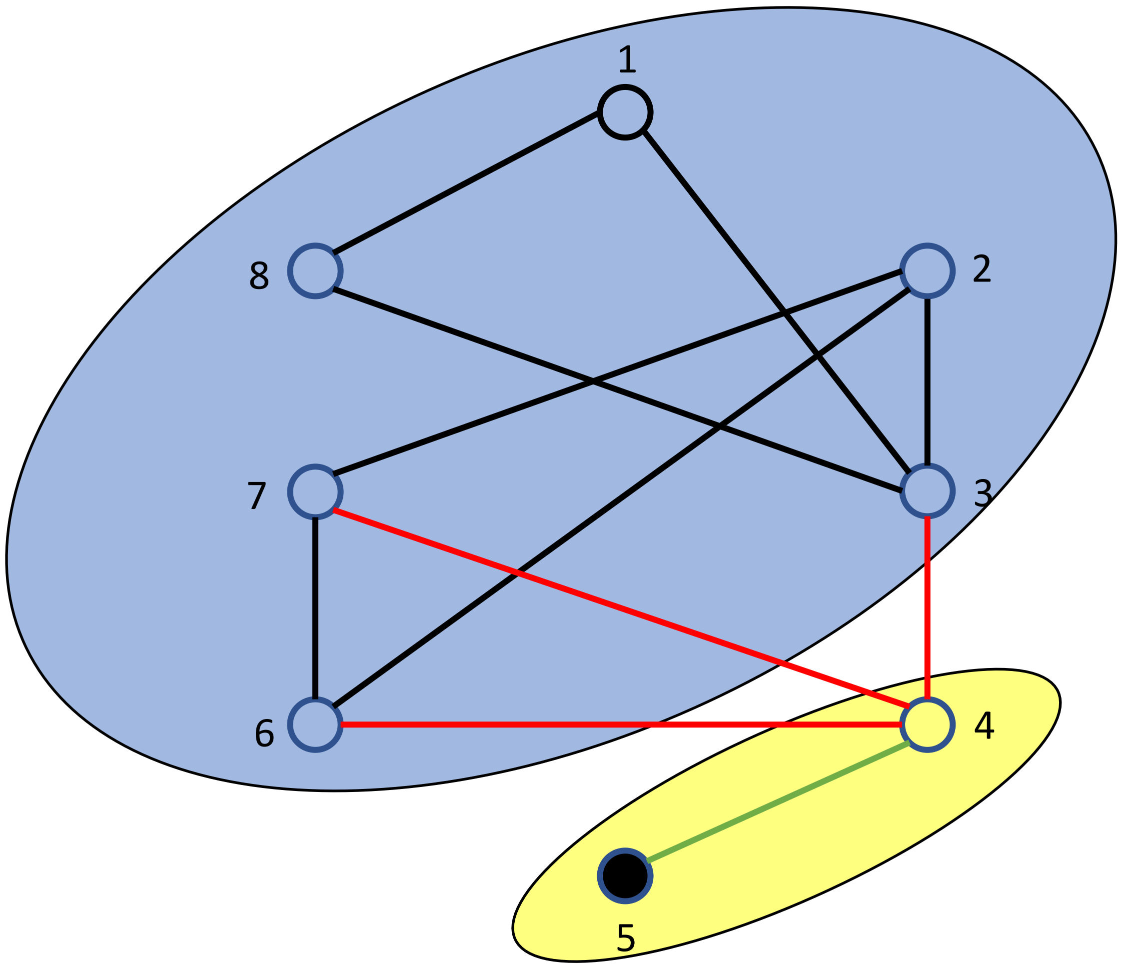


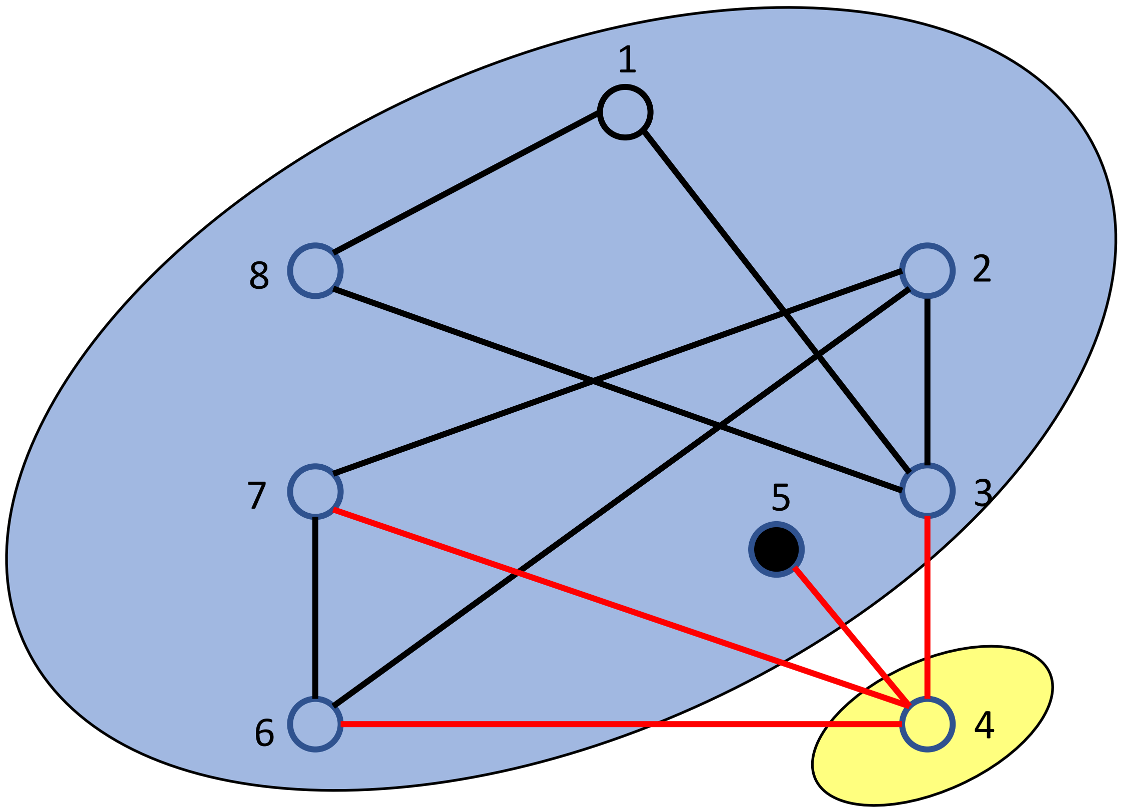
This cut has
3 cross edges



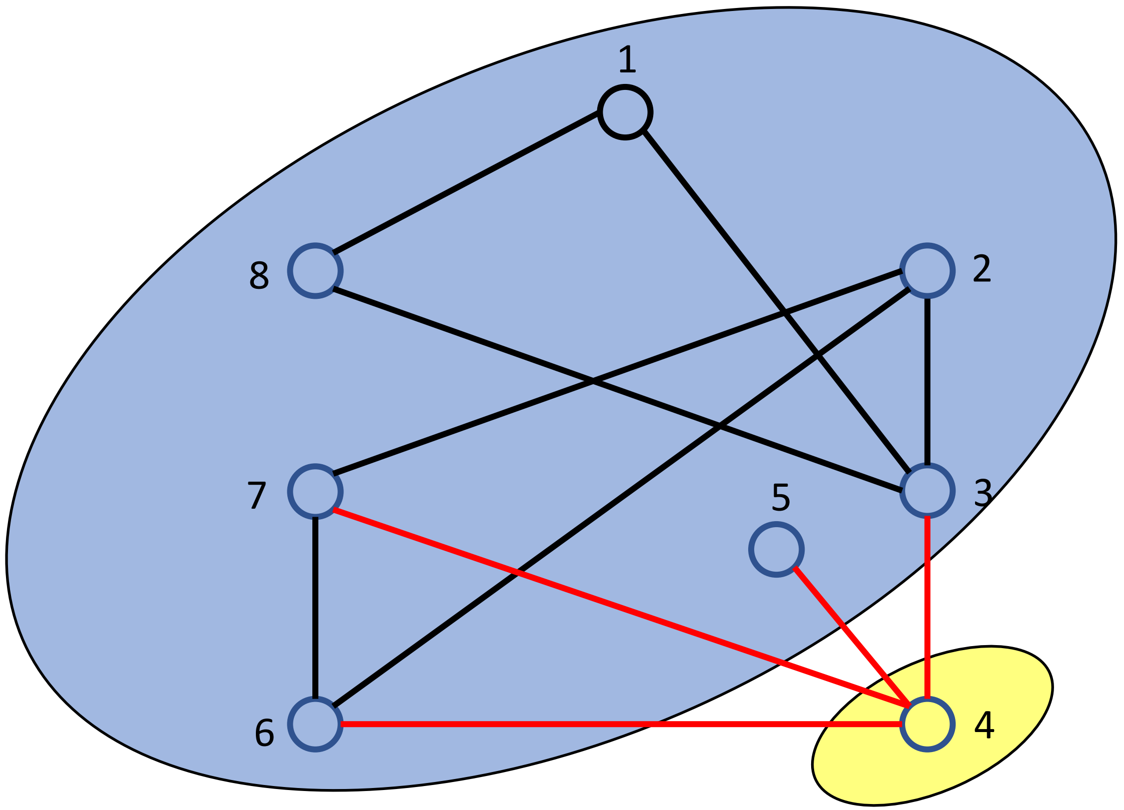
Node 5 has more internal edges (1) than cross edges (0).

Increase the number of cross edges by moving it to the blue side.



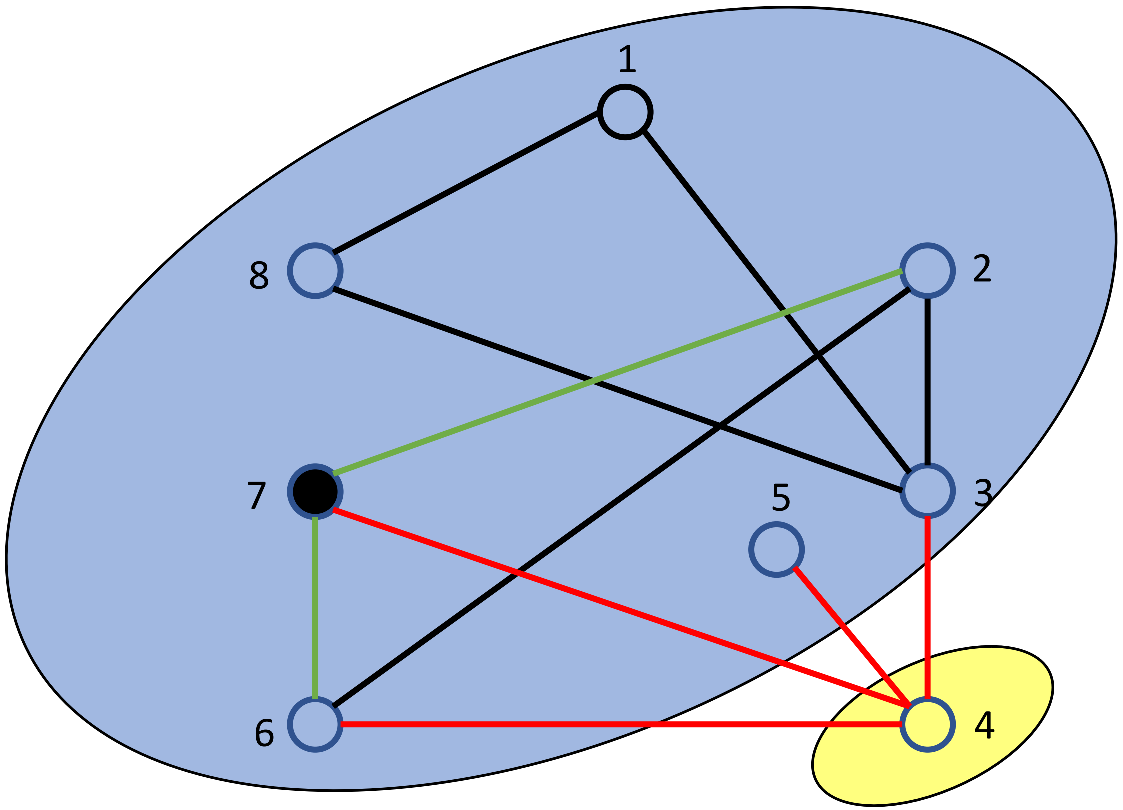

1
8
2
7
5
3
6
4

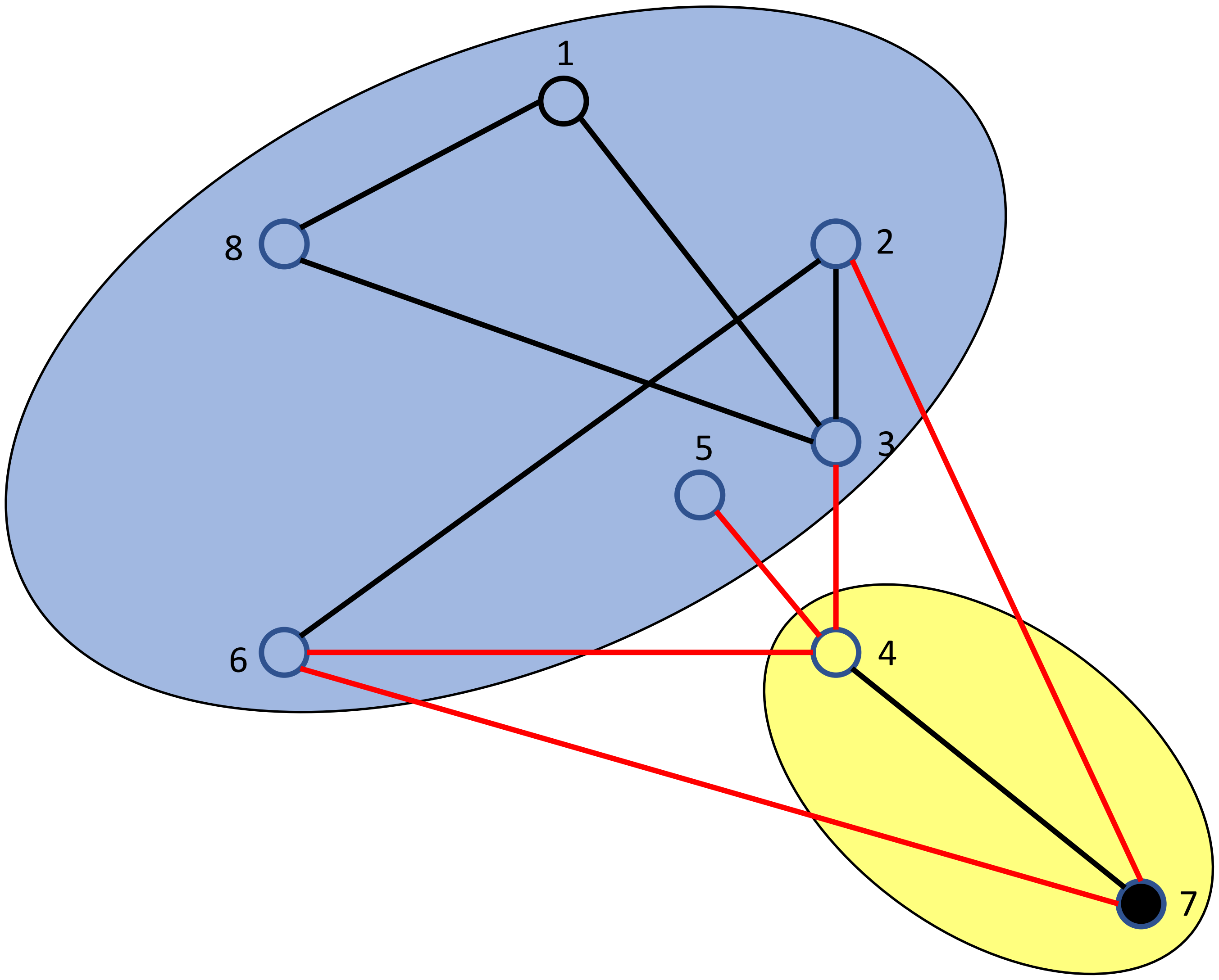
This cut has
4 cross edges



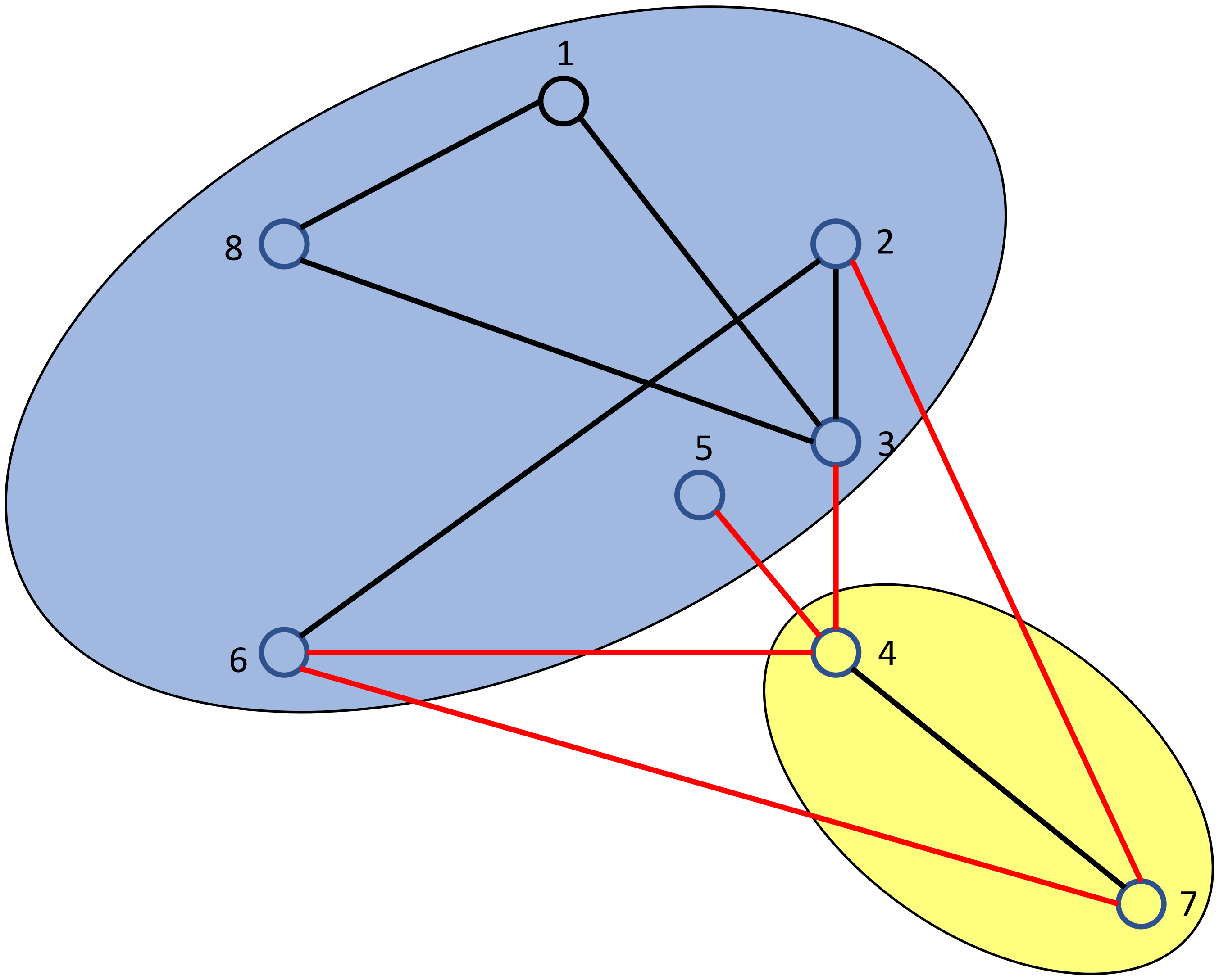
Node 7 has more internal edges (2) than cross edges (1).

Increase the number of cross edges by moving it to the yellow side.



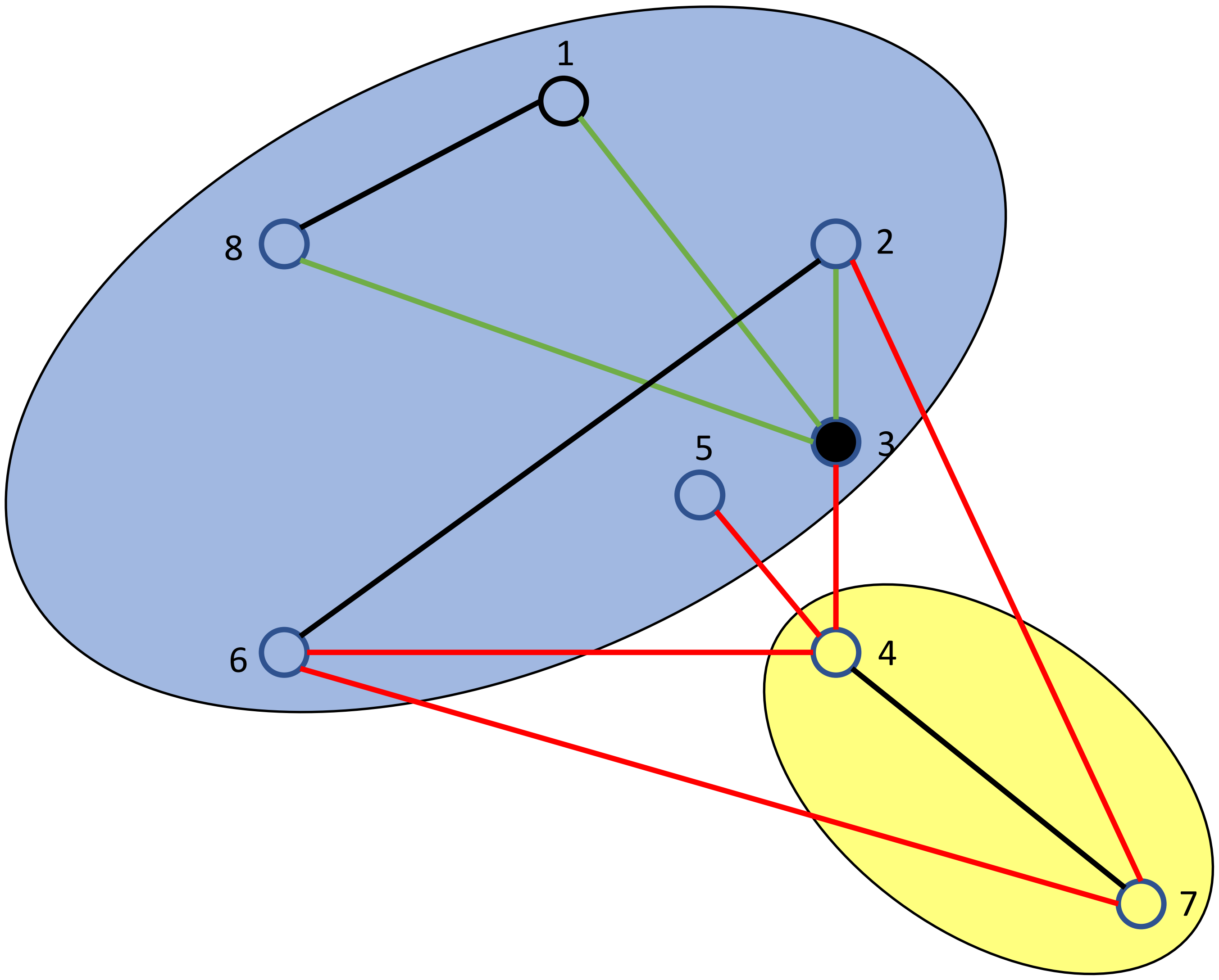

1
8
2
3
5
6
4
7

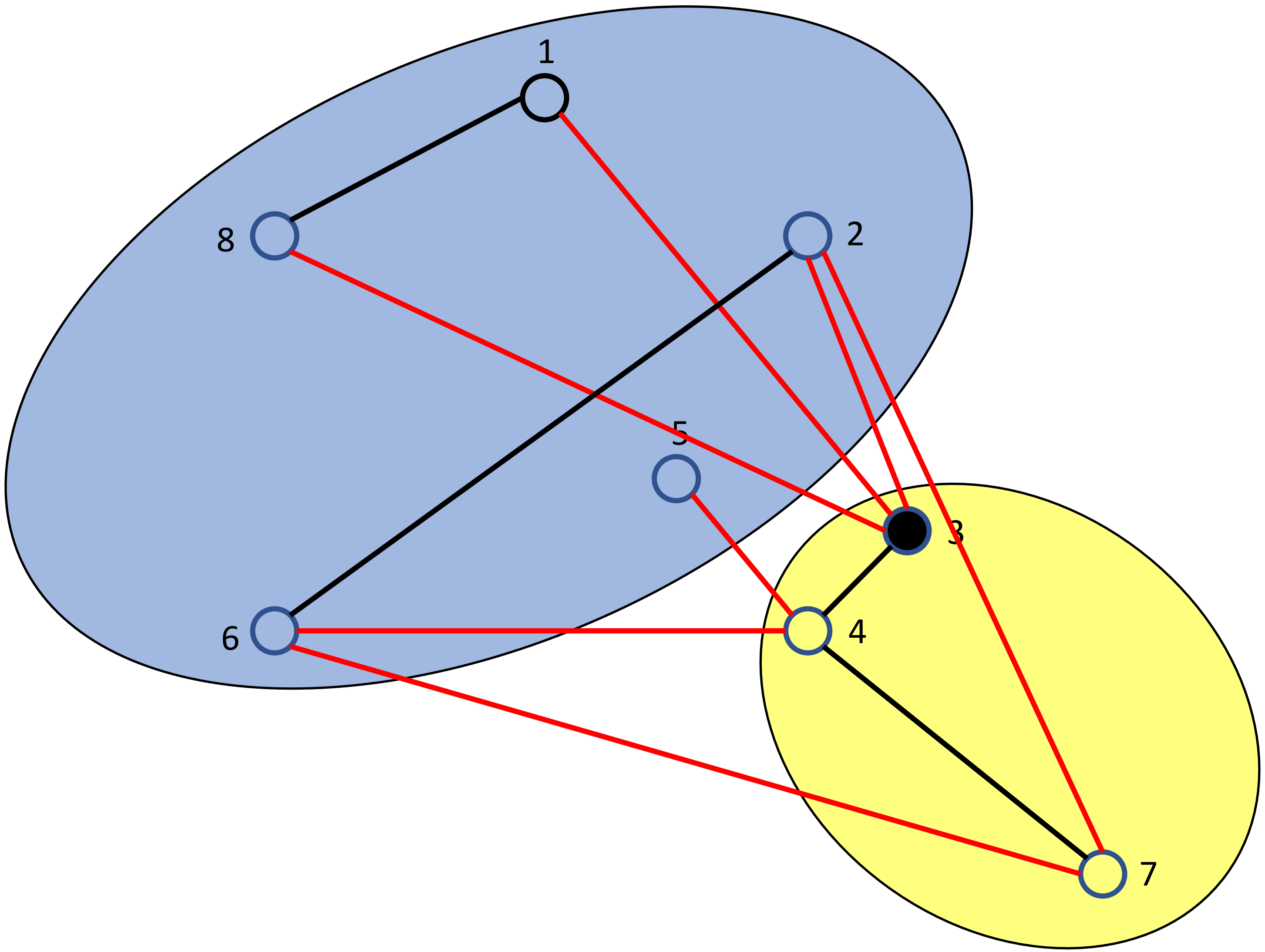
This cut has
5 cross edges



Node 3 has more internal edges (3) than cross edges (1).

Increase the number of cross edges by moving it to the yellow side.

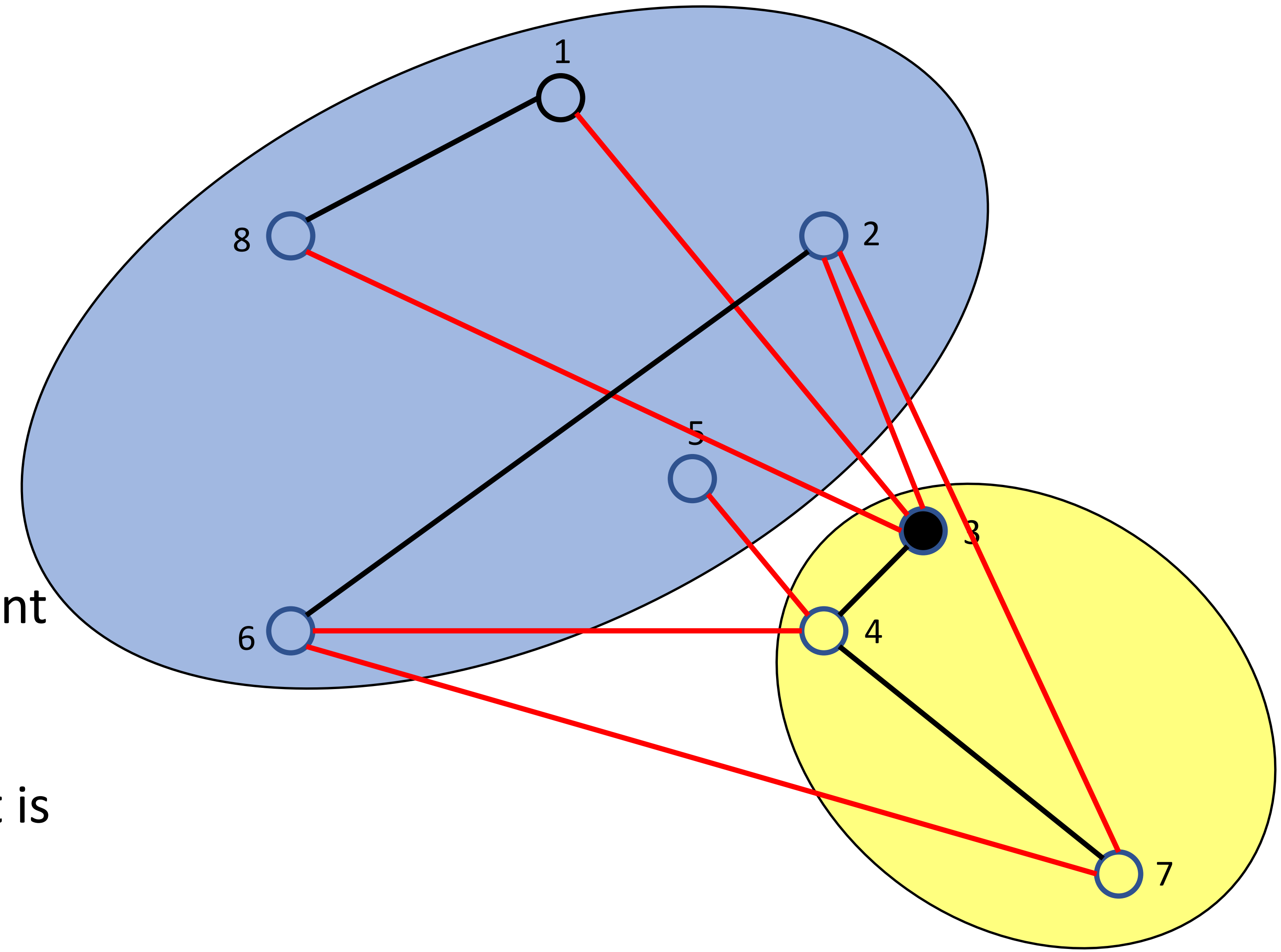



1
8
2
5
3
6
4
7

This cut has
7 cross edges.

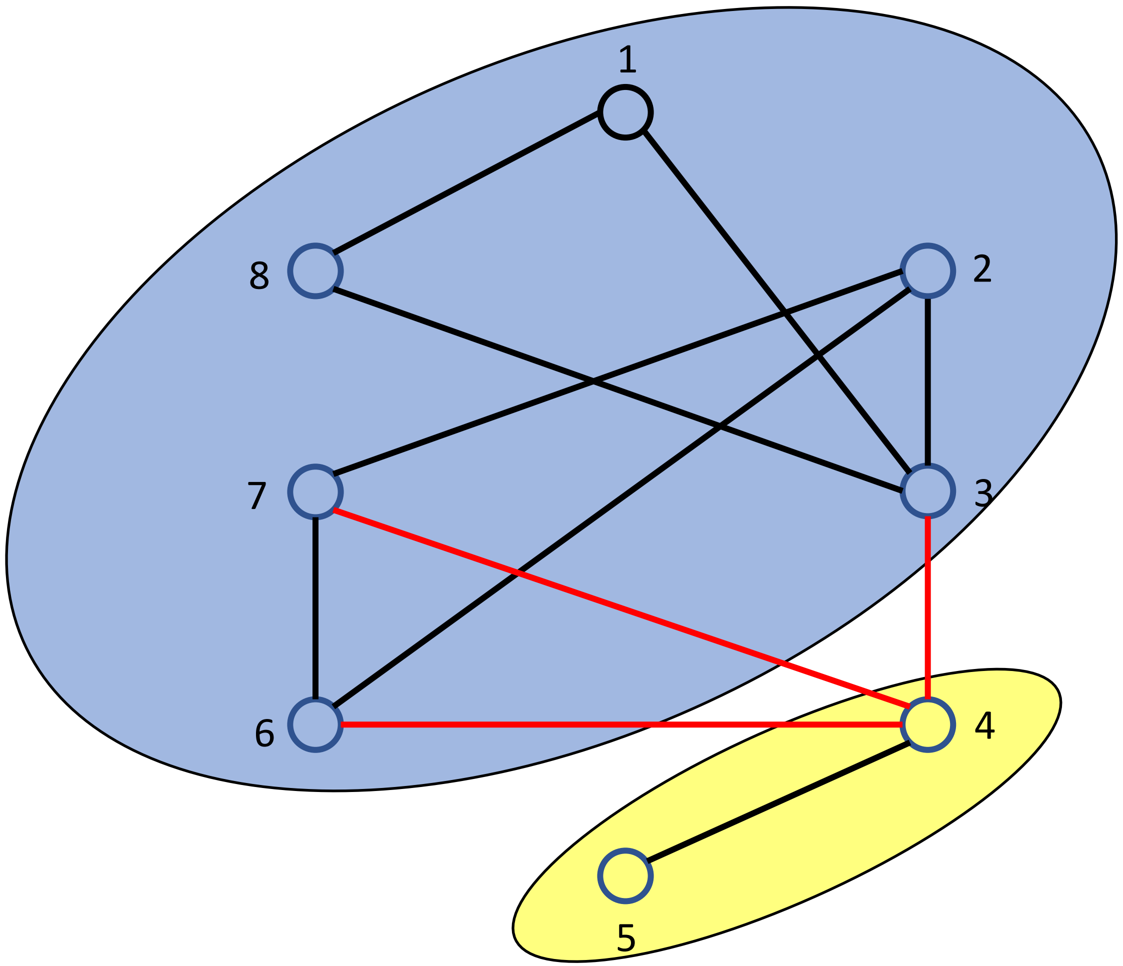
No local improvement
is possible.

But as we will see, it is
<u>not</u> a max cut!



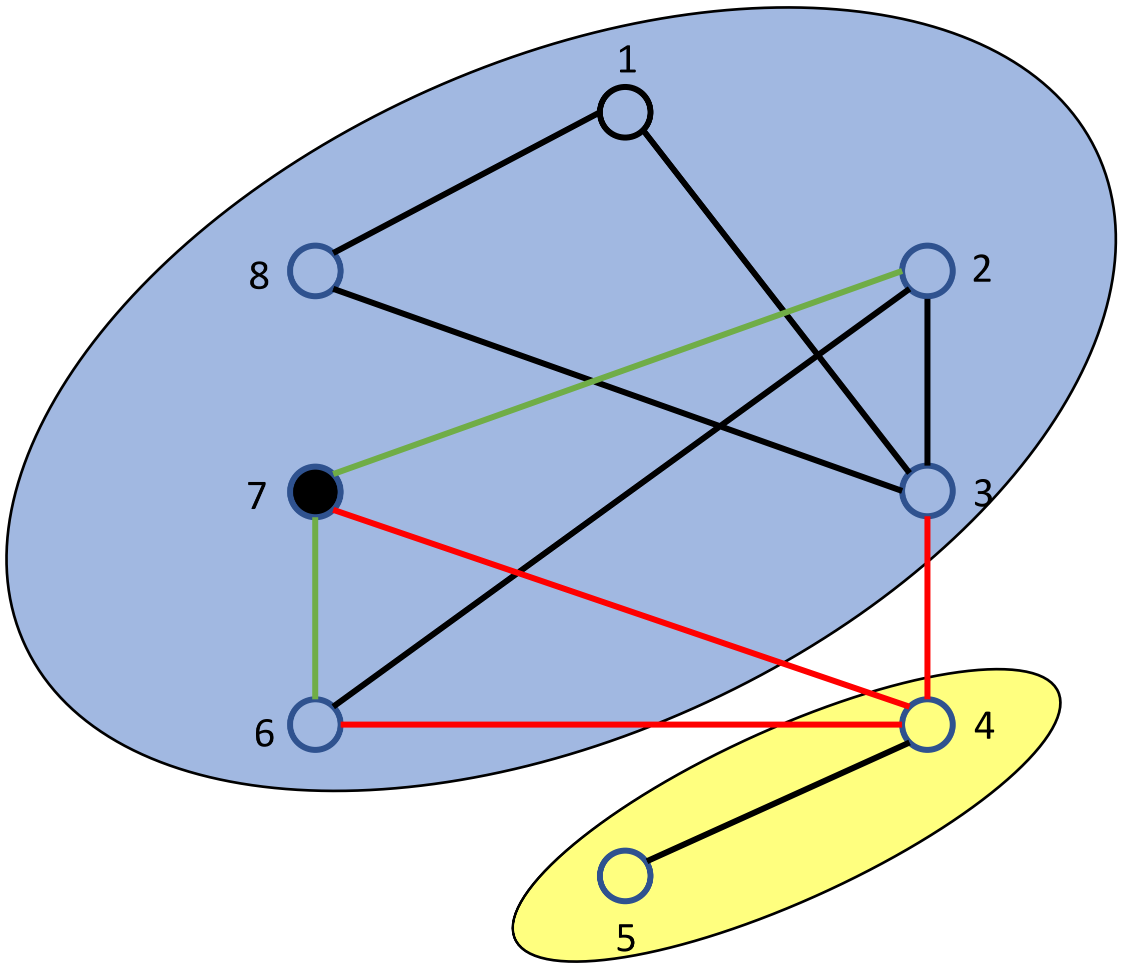
Back to the original cut with 3 cross edges.

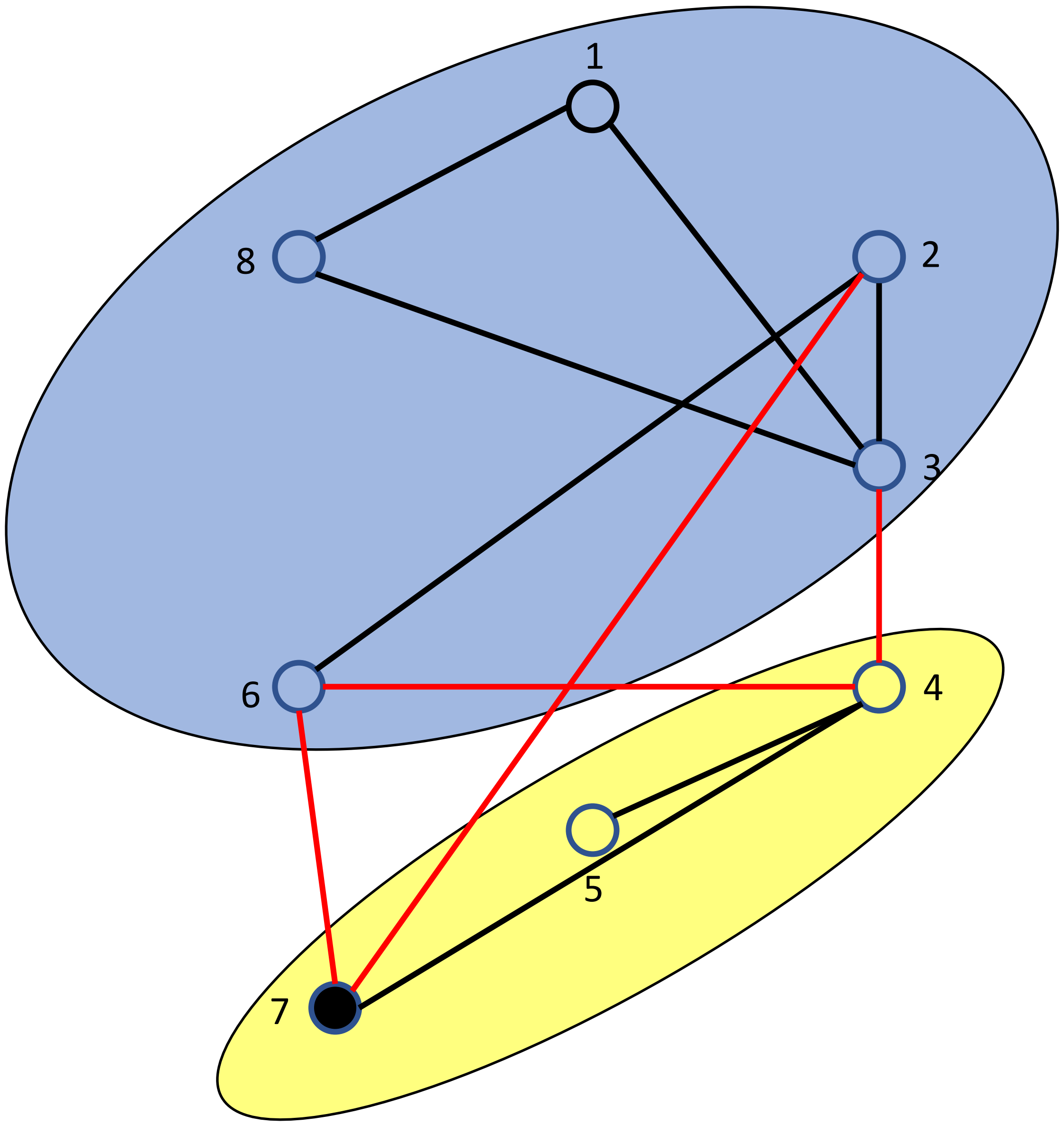
Now move nodes in a different order.



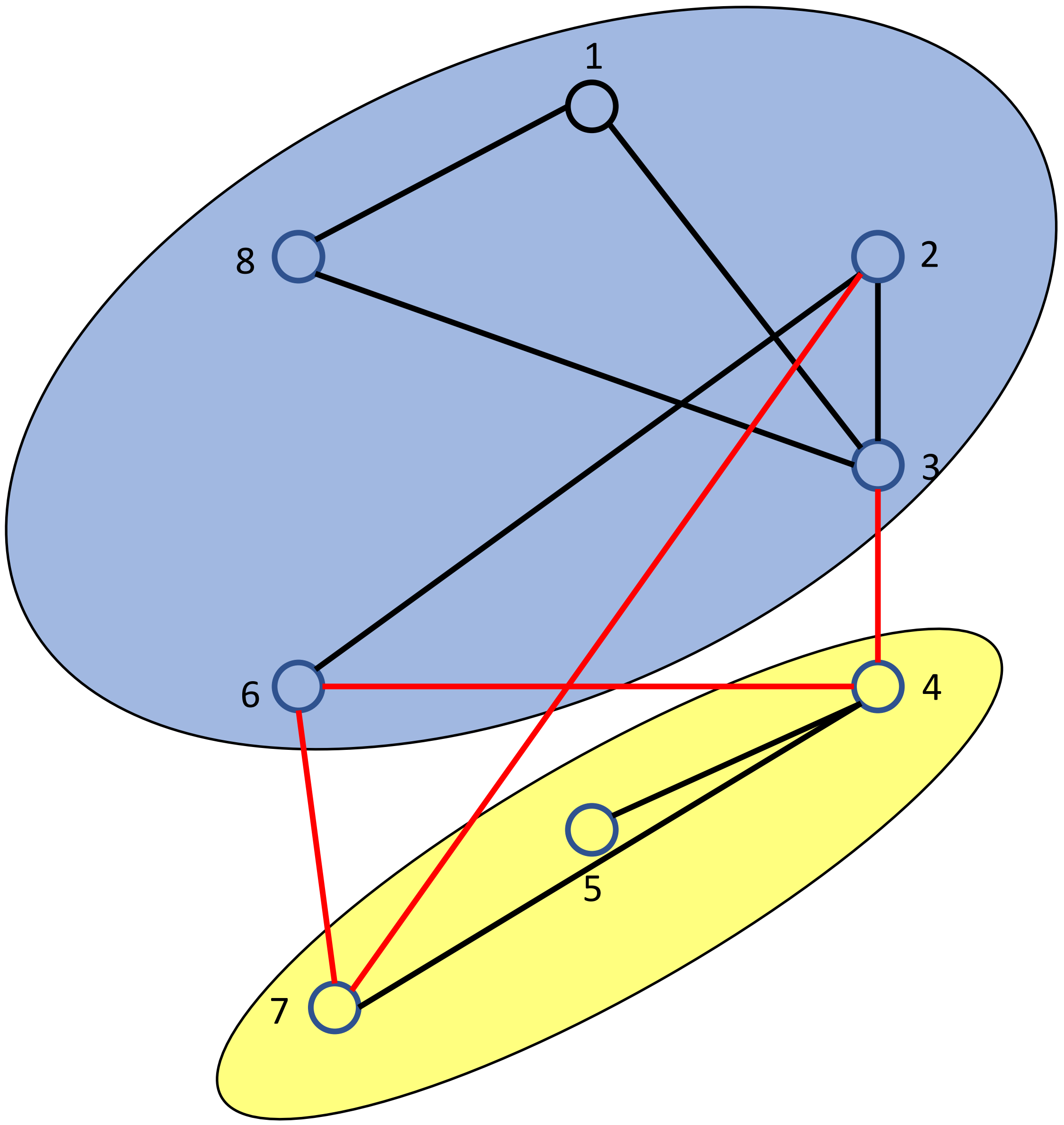
Node 7 has more internal edges (2) than cross edges (1).

Improve the number of cross edges by moving it to the yellow side (instead of moving node 5 to the blue side, as before).



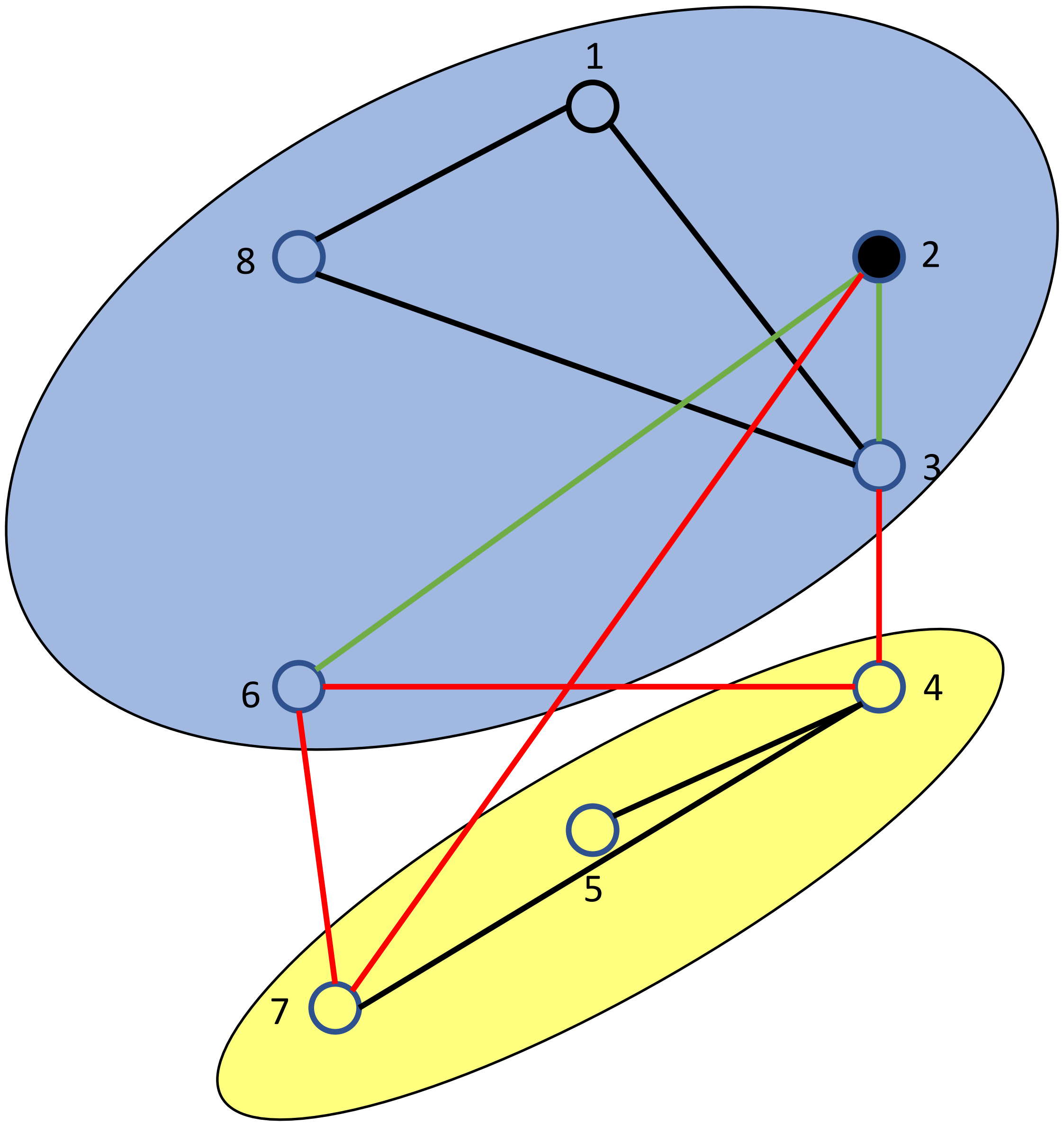

1
8
2
3
6
4
5
7

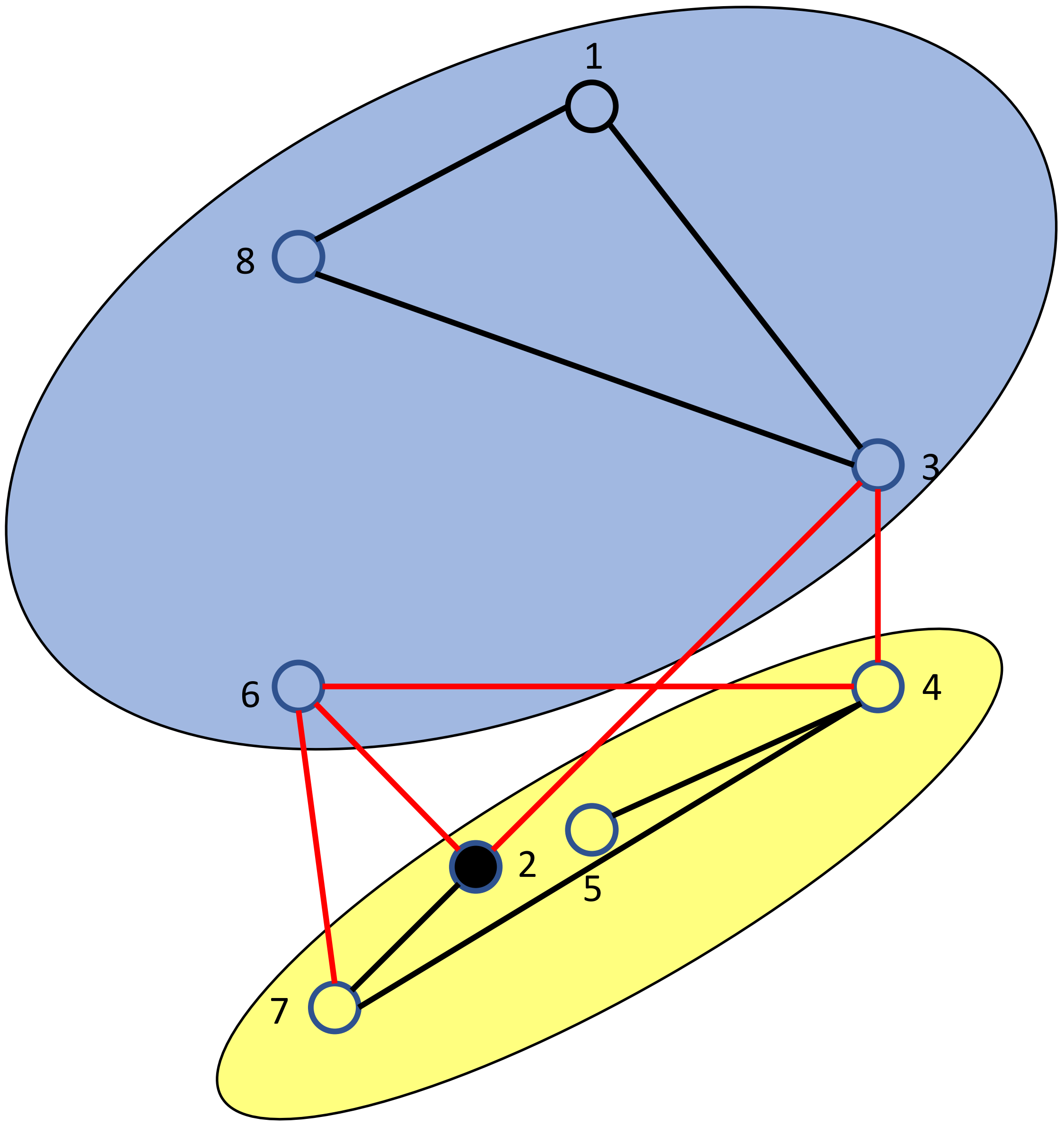
This cut has
4 cross edges



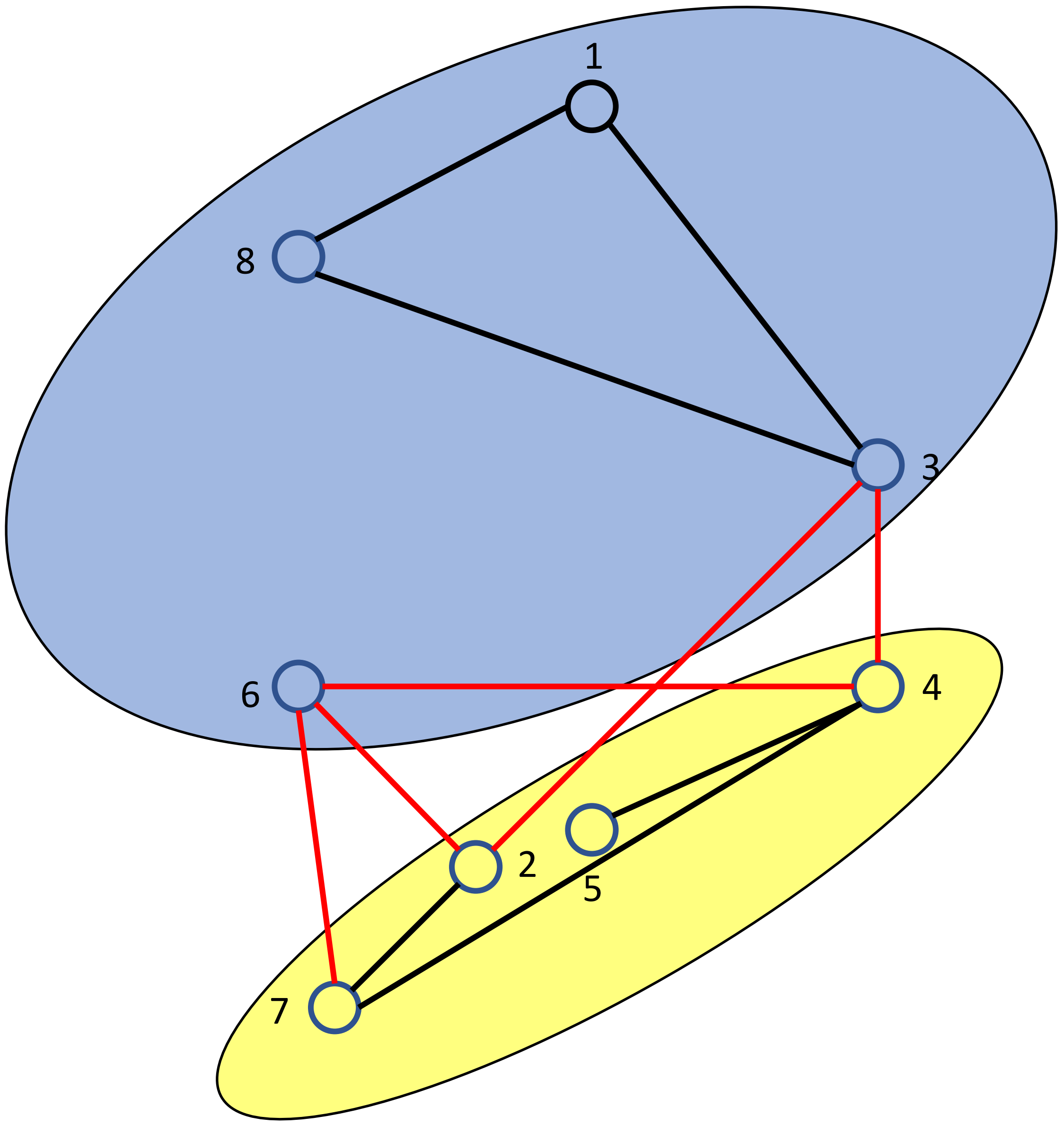
Node 2 has more internal edges (2) than cross edges (1).

Improve the number of cross edges by moving it to the yellow side.




1
8
3
6
4
2
5
7

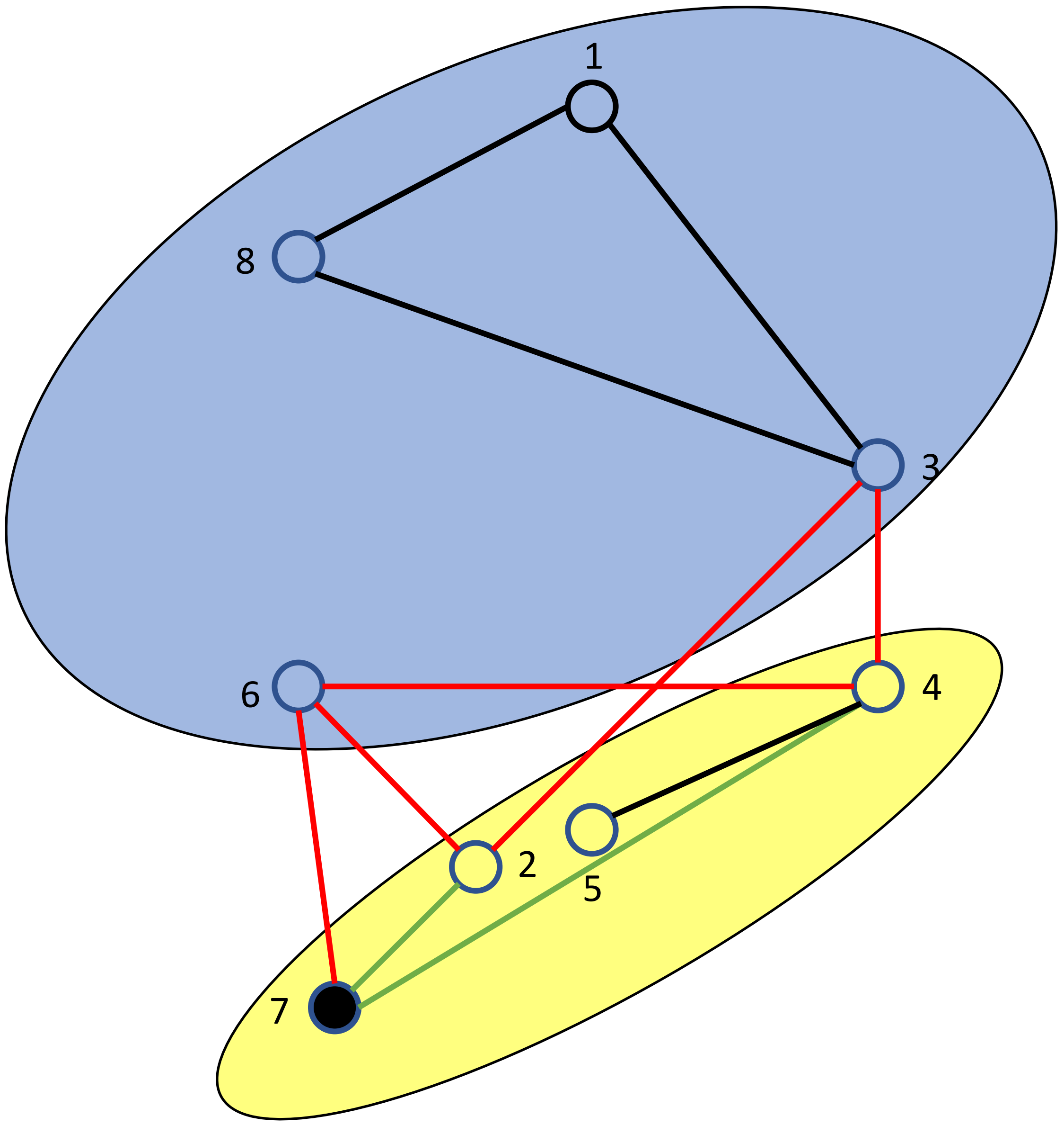
This cut has
5 cross edges

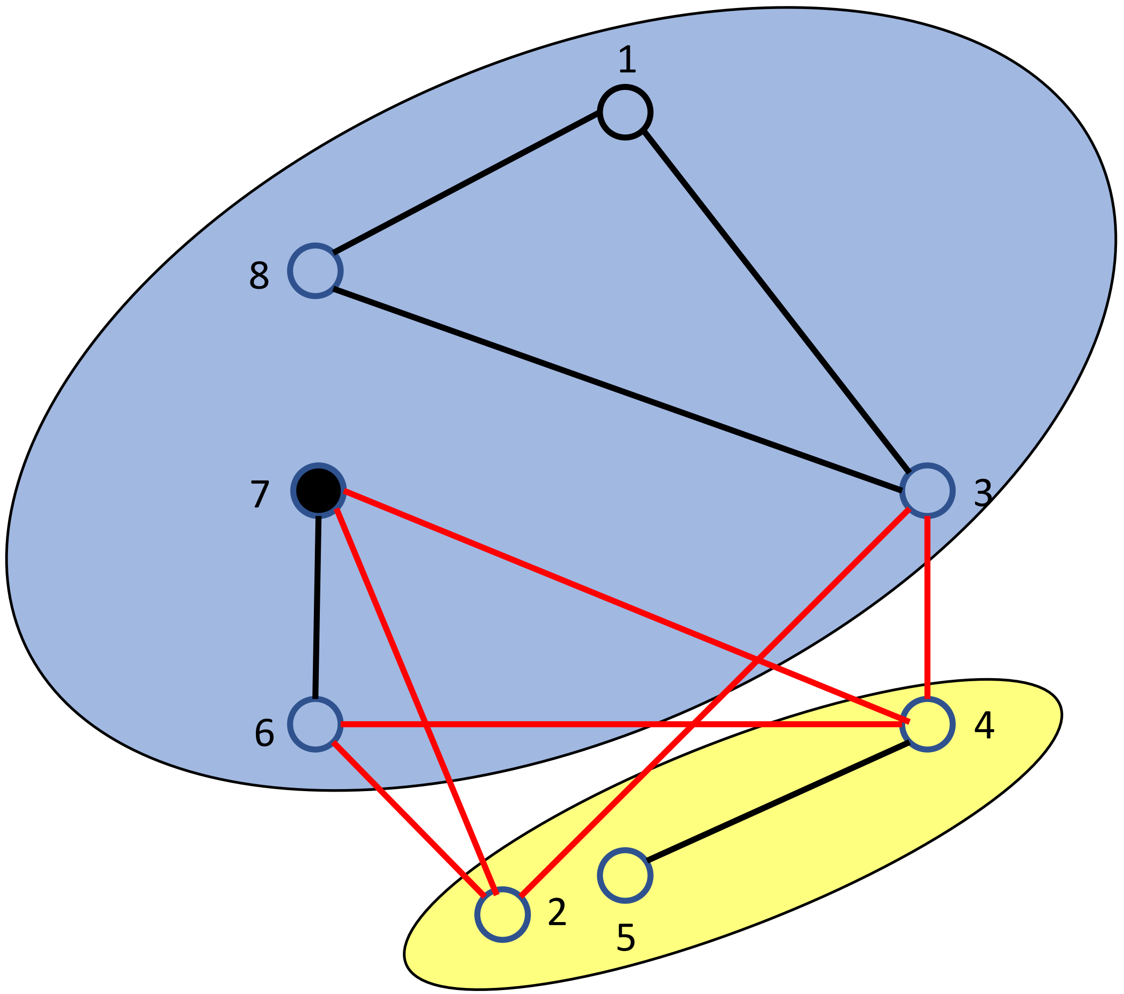


Node 7 has more internal edges (2) than cross edges (1).

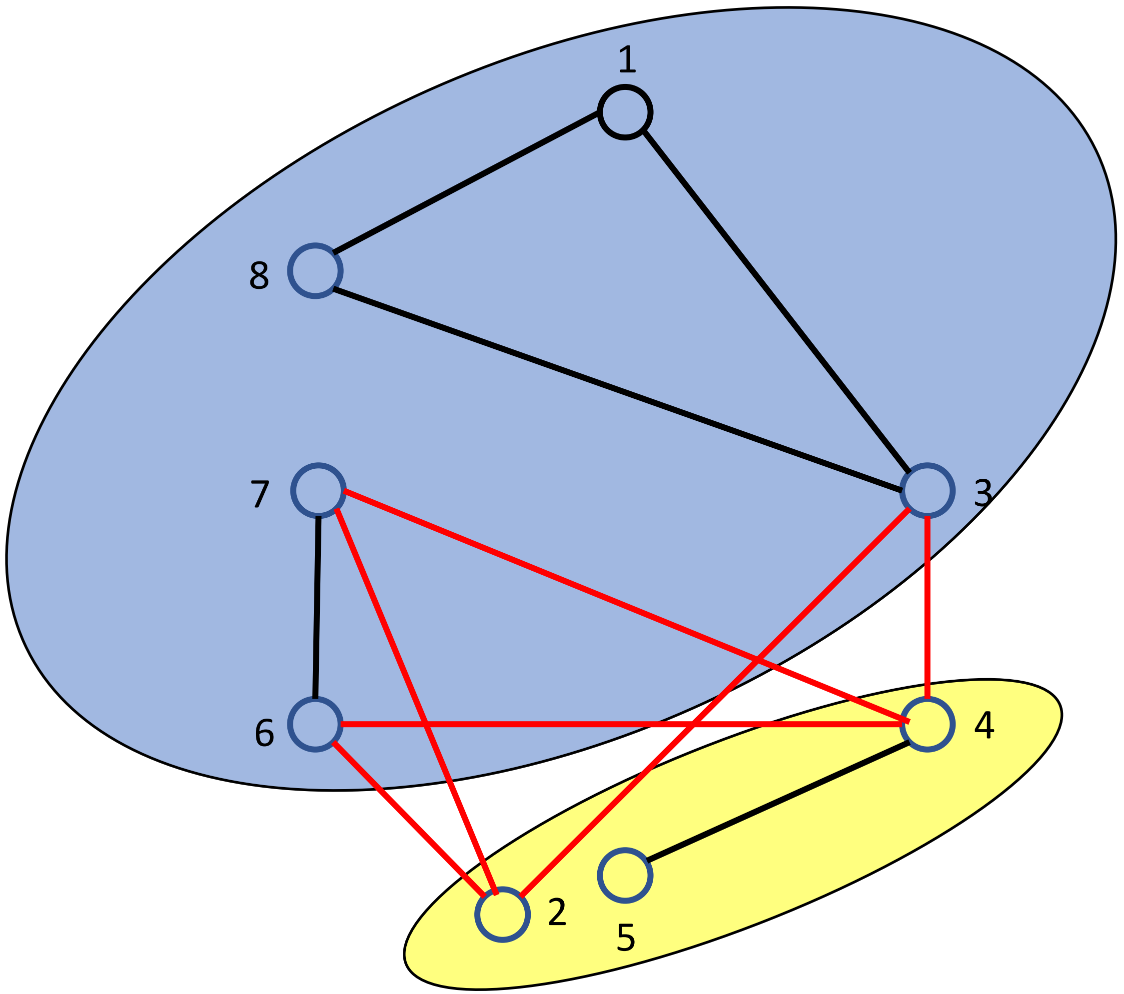
Improve the number of cross edges by moving it to the blue side.

NB: Moving back!



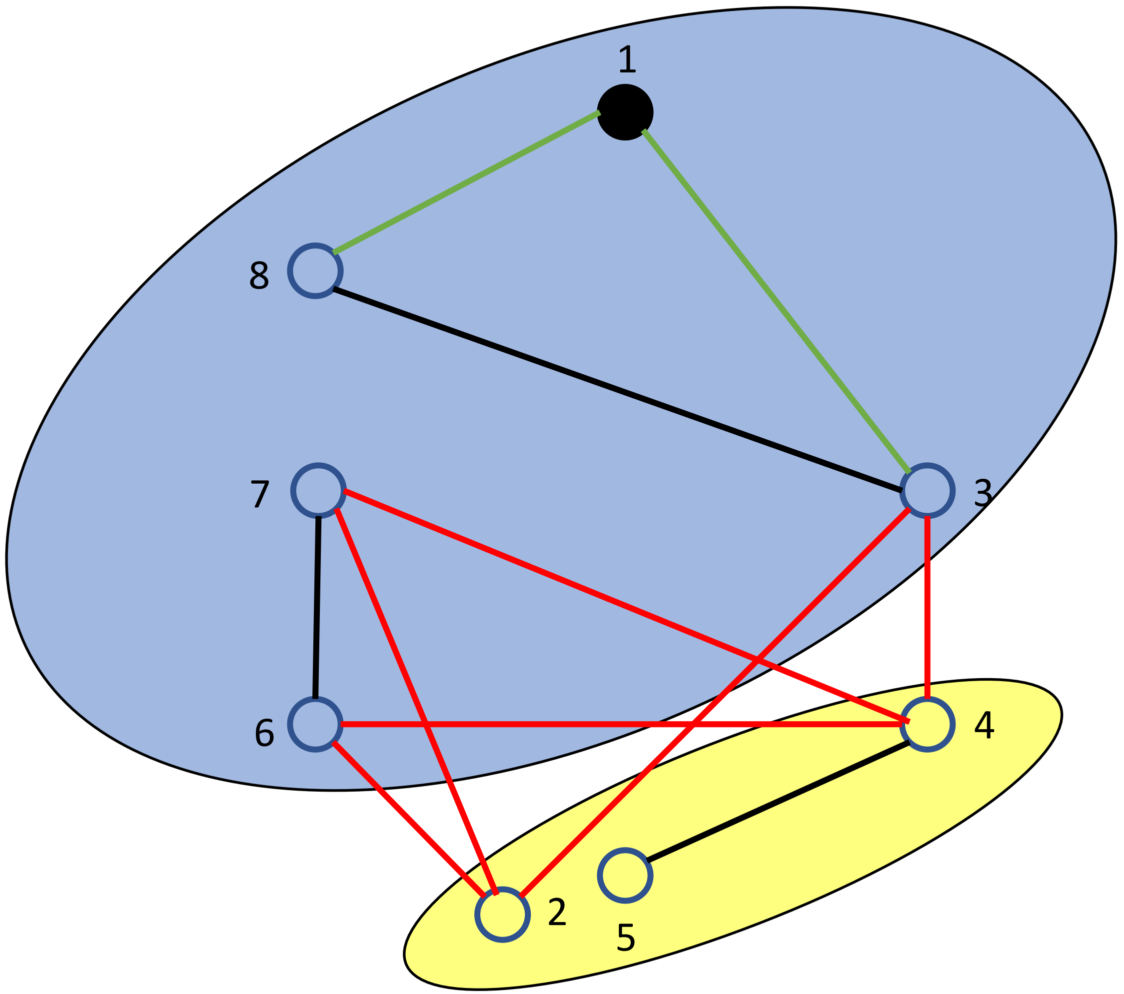

1
8
7
3
6
4
2
5

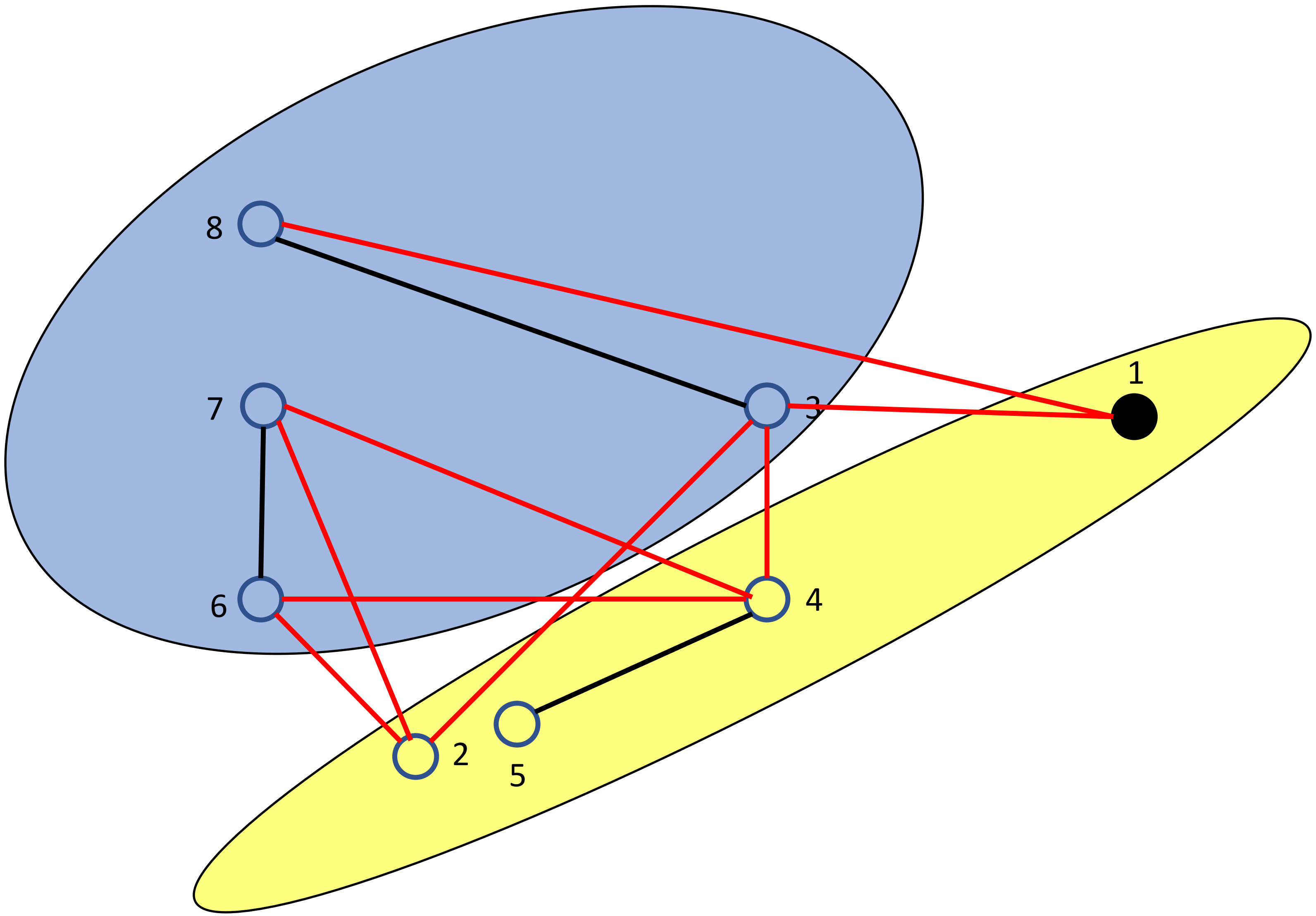
This cut has
6 cross edges



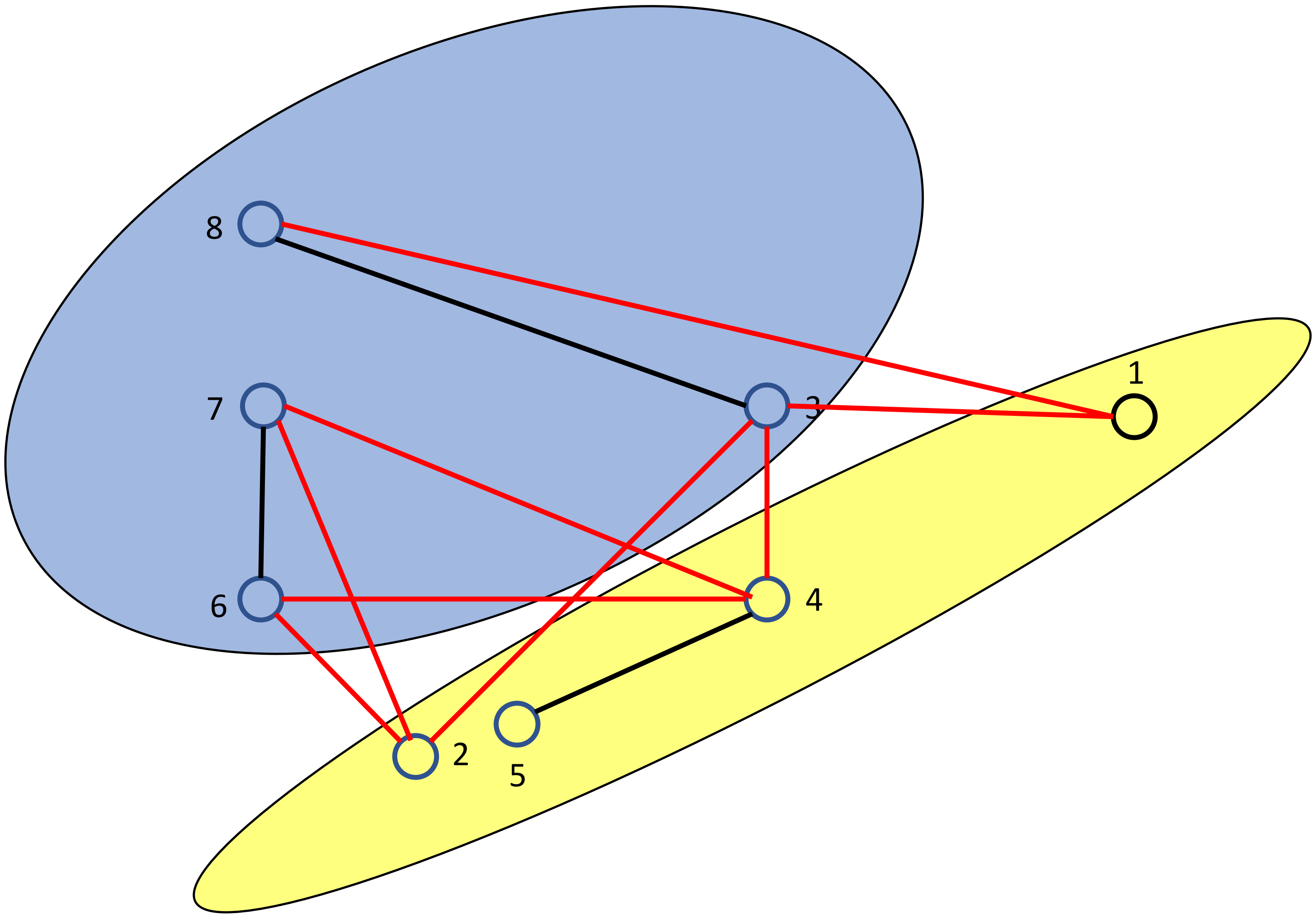
Node 1 has more internal edges (2) than cross edges (0).

Improve the number of cross edges by moving it to the yellow side.



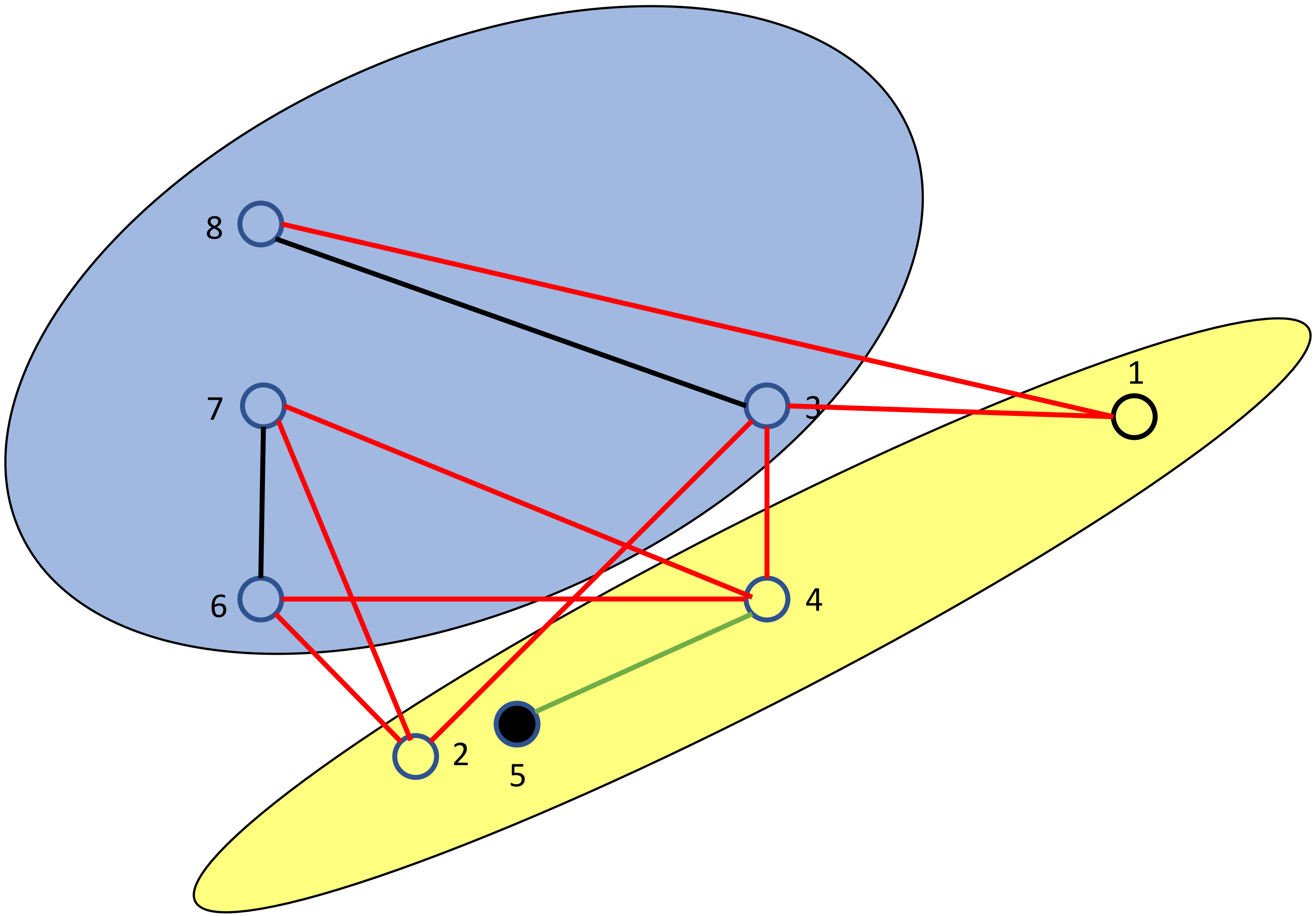

8
7
6
3
1
4
2
5

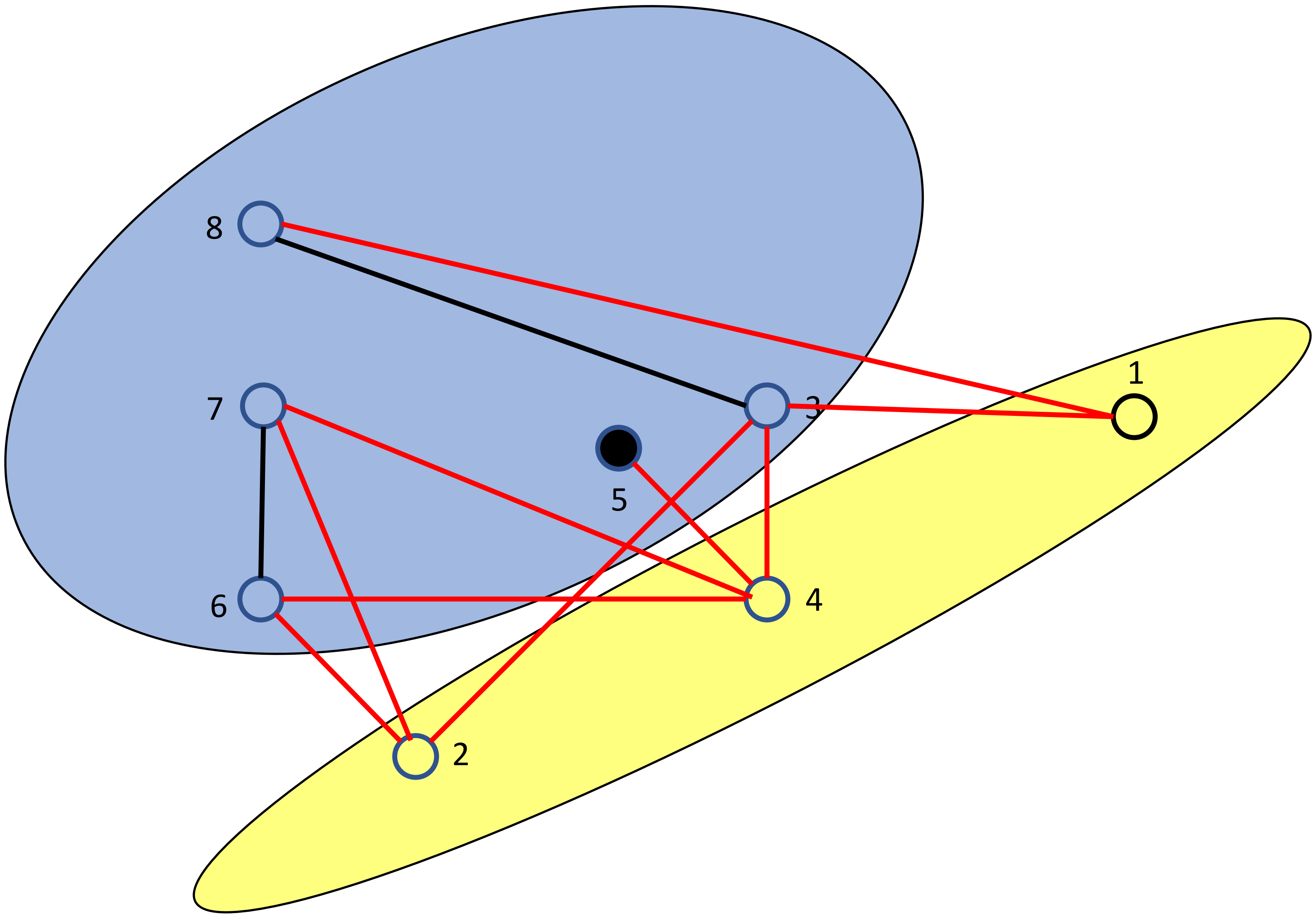
This cut has
8 cross edges



Node 5 has more internal edges (1) than cross edges (0).

Improve the number of cross edges by moving it to the blue side.





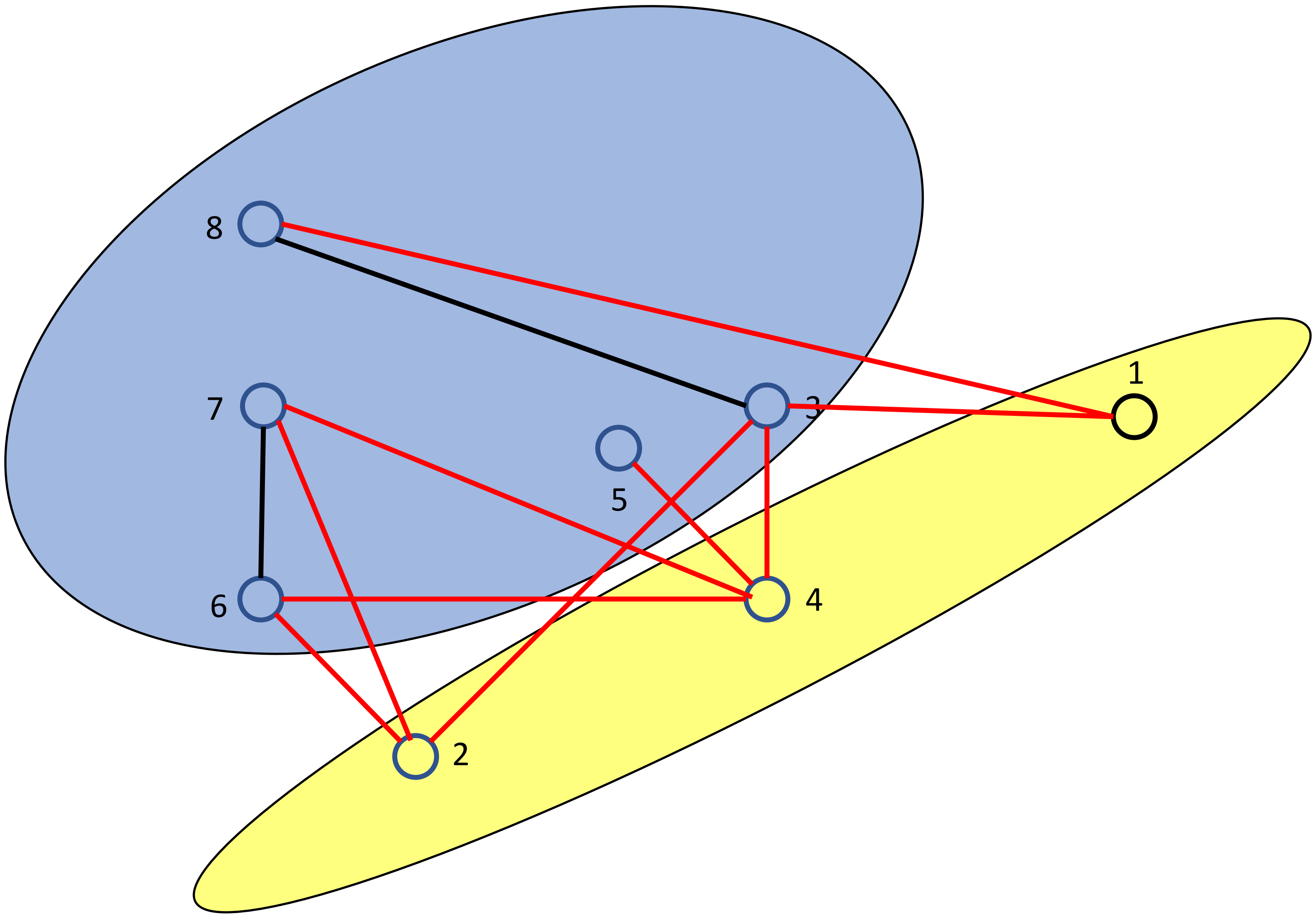
8
7
5
3
1
6
4
2

This cut has
9 cross edges.

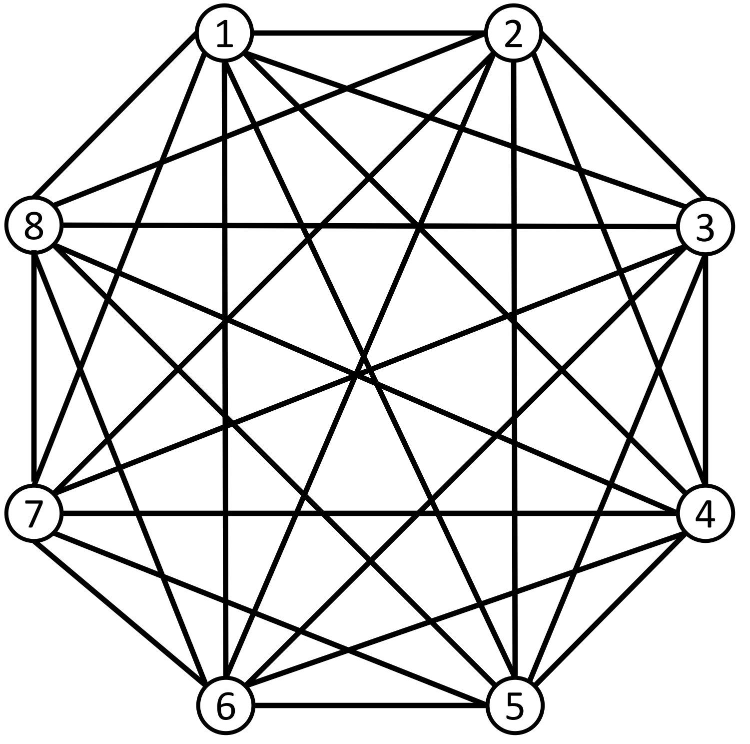
This is a max cut.

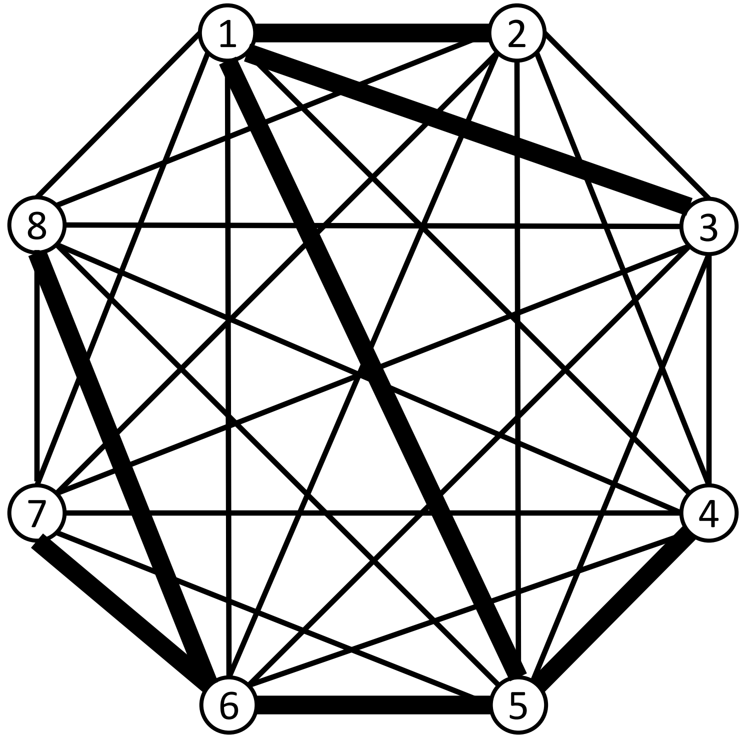
Why?

Hint: Disjoint triangles 1,3,8 and 4,6,7.

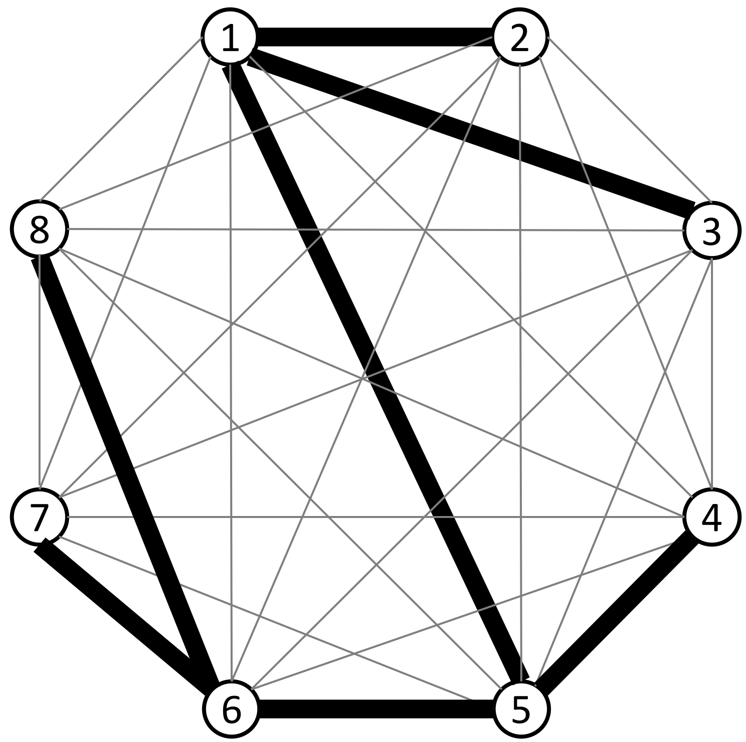


Approximation algorithm for metric TSP

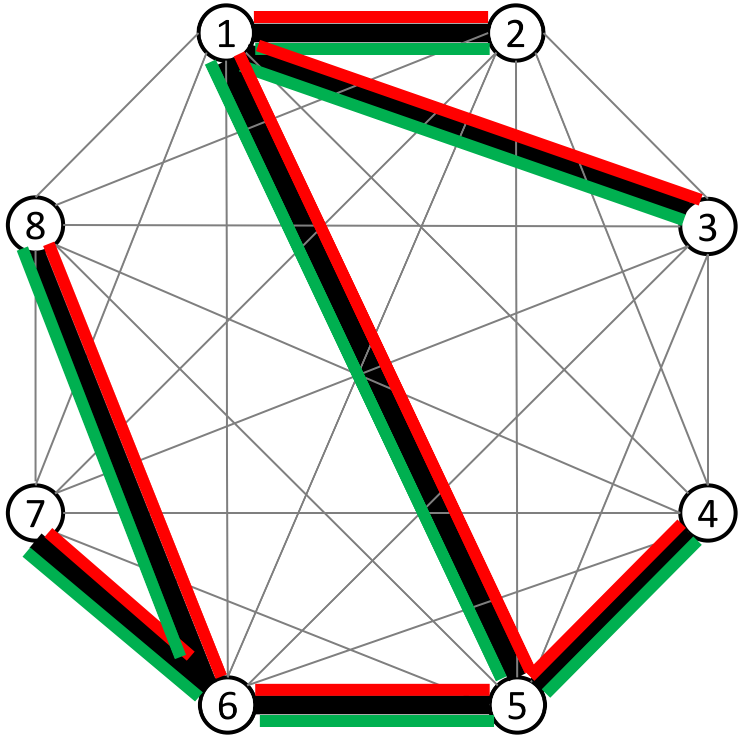

1
2
8
3
7
4
6
5



Step 1: Find a MST of the graph

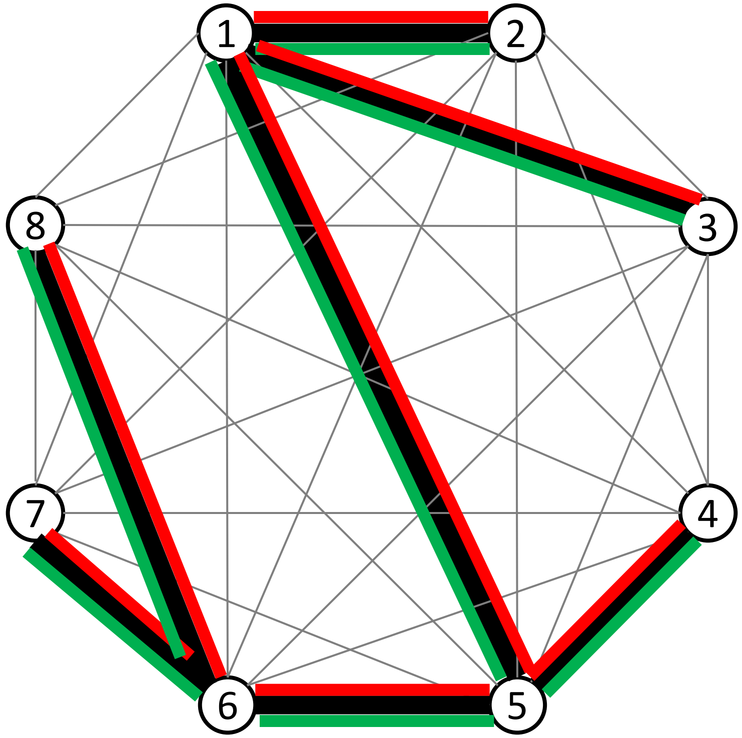


Step 1: Find a MST of the graph



Step 2: Do a DFS of the MST

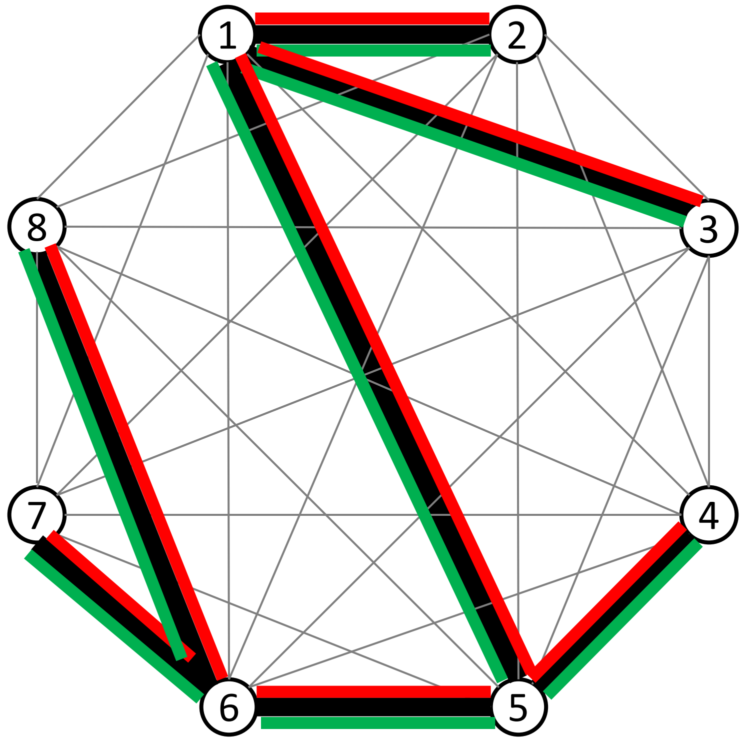
(each edge of the MST is visited twice: once when discovered and once again when backtracking)



1, 2, 1, 3, 1, 5, 4, 5, 6, 8, 6, 7, 6, 5, 1

Step 2: Do a DFS of the MST

Record the sequence of nodes in the order visited



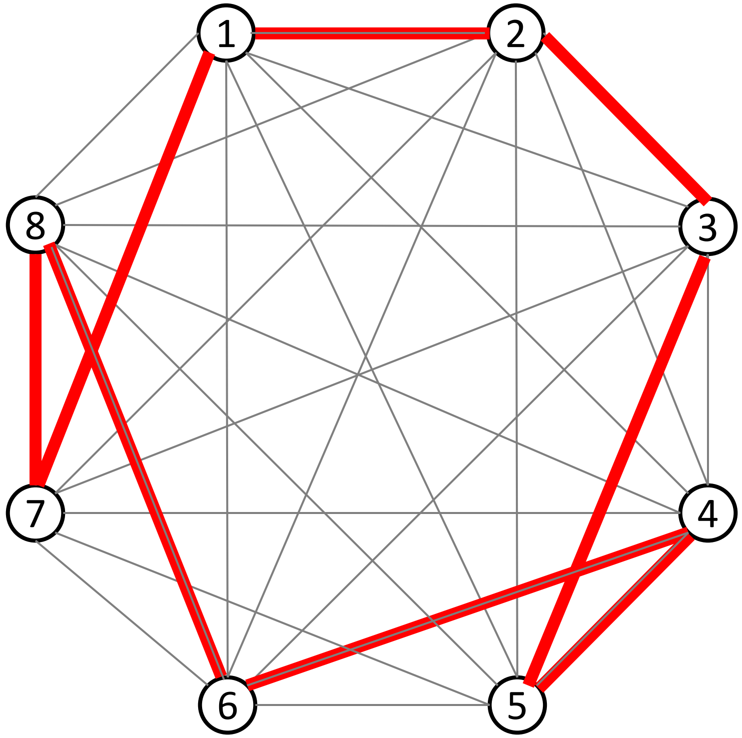
1, 2, 1, 3, 1, 5, 4, 5, 6, 8, 6, 7, 6, 5, 1

"Trail" produced by the DFS of the MST

1, 2, 3, 5, 4, 6, 8, 7, 1

Tour produced by the algorithm

Step 3: Keep only the <u>first</u> occurrence of each node, then back to the first node



1, 2, 3, 5, 4, 6, 8, 7, 1

Tour produced by the algorithm

Step 3: This is the algorithm's tour

(its cost is at most twice the cost of the optimal tour)

Metric TSP approximation algorithm

1. Find a MST T* of the graph
2. Do a DFS of T*
3. S' := sequence of nodes in the order visited by the DFS
   # S' is not a tour
4. S := subsequence of S' containing only the first
   occurrence of each node, followed by the first node
   # S is a tour
5. return S