Computer Science C63 — January 22, 2025
Scarborough Campus — University of Toronto

Homework Assignment #3
Due: January 29, 2025, by 11:59 pm

- ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
- *To minimize bias in grading, Crowdmark does not reveal your name to the grader. Please do not subvert this feature by including your name(s) on the files you submit.*
- *It is your responsibility to ensure that the files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
- *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
- *For any question, you may use facts previously proved in this course, its prerequisites, or in the assigned sections of the textbook.*
- *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness a of your answers, and the clarity, precision, and conciseness of your presentation.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC63. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.** *Nota Bene:* 'Other source' includes, but is not limited to, the use of AI-based tools, even for (allegedly) 'just improving' your work. What you submit must be entirely your own creation."

Question 1. (20 marks) Let $f : \{0,1\}^* \to \{0,1\}^*$ be a (possibly partial) function. The ***graph*** of f is the set $\mathcal{G}(f) = \{\langle x, f(x)\rangle : \ x \in \{0,1\}^*\}$, i.e., the set of encodings of pairs of strings $(x, f(x))$, where $x \in \{0,1\}^*$. (For example we can take $\langle x, f(x)\rangle$ to be $x\#f(x)$ — the two strings separated by #). Prove the following facts:

(a) for every total function f, f is computable if and only if $\mathcal{G}(f)$ is decidable.

(b) for every partial function f, f is computable if and only if $\mathcal{G}(f)$ is recognizable.

(For the definitions of computable function and computable partial function see the document "Turing machines as function computers" on the course web page.) For part (b) it suffices to explain clearly, but without getting into details, how your proof differs from that for part (a) — or why it does not need to change, for the parts where it does not.

Question 2. (10 marks) Informally but clearly describe a ***nondeterministic*** Turing machine (multitape, if this helps) that recognizes the language

$$A = \big\{x \in \{0,1\}^* \colon \exists y, w_1, w_2, w_3 \in \{0,1\}^* \text{ such that } |y| = 15 \text{ and } x = w_1 y w_2 y w_3\big\}.$$

That is, A is the set of strings x in which the same substring y of length 15 appears in (at least) two disjoint parts.[1]

Your description should be in point-form English, but should not be too high-level. For example, you shouldn't simply say "guess a substring of the input"; instead you should explain how nondeterminism is used to this end. You don't need to explain in detail how a TM does things that have been discussed in class, tutorial, or the textbook, such as returning a tape head to the leftmost cell, or scanning in some direction until a particular symbol is found.

Question 3. (20 marks) For each of the following statements, indicate whether it is true or false, and justify your answer.

a. For any languages $A, B \subseteq \Sigma^*$, if $A \cup B$ and $A \cap B$ are both decidable, then A and B are both decidable.

b. For any languages $A, B \subseteq \Sigma^*$, if A is decidable and $A \cap B$ is recognizable, then B is recognizable.

c. For any languages $A, B \subseteq \Sigma^*$, if A and B are both recognizable, and $A \cup B$ and $A \cap B$ are both decidable, then A and B are both decidable.

[1] A multitape nondeterministic Turing machine can be simulated by a single-tape NTM by a construction similar to the one we saw for multitape deterministic TMs. (You should not include this construction in your answer, but you should think about it.)