Definition of the "yields" relation $\vdash$

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Let $M = (Q, \Sigma, \Gamma, \delta, q_0, h_A, h_R)$ be a Turing machine. Without loss of generality, we assume that $Q \cap \Gamma = \varnothing$, so that state symbols cannot be confused with tape symbols.

Notational conventions:

- Lower case characters near the beginning of the alphabet $(a, b, c, \ldots)$ denote tape symbols (elements of Γ).
- Lower case characters near the end of the alphabet $(w, x, y, z, \ldots)$ denote strings of tape symbols (elements of Γ^*).
- p, q (decorated with accents, subscripts, superscripts etc.) denote states (elements of Q).
- $\sqcup$ is the blank symbol (element of Γ)

A ***configuration*** of M is a string of the form xqy, where $x, y \in \Gamma^*$ and $q \in Q$, where y does not end with the blank symbol $\sqcup$. This describes the complete state of the Turing machine at some point in its computation: The machine is in state q, its tape contains the string xy starting in cell 1 (the leftmost cell) followed by an infinite number of blanks; and the tape head is positioned over cell $|x| + 1$, i.e., the first symbol of y, if $y \neq \epsilon$, or the leftmost of the infinite sequence of trailing blanks, if $y = \epsilon$.

We define the relation $\vdash_M$ between configurations (written simply $\vdash$, if M is clear from the context) to hold if M can move from one configuration to the other in a single step, based on its transition function.

More precisely, let $C = xqy$; then $C \vdash_M C'$ if and only if:

CASE 1. $y = ay'$, for some $a \in \Gamma$. (Thus, $y \neq \epsilon$, and if $a = \sqcup$ then $y' \neq \epsilon$.)

SUBCASE 1(a). $\delta(q, a) = (p, b, R)$: $C' = xbpy'$.

SUBCASE 1(b). $\delta(q, a) = (p, b, L)$ and $x = x'c$, for some $c \in \Gamma$:

$$C' = \begin{cases} x'pcby', & \text{if } b \neq \sqcup \text{ or } y' \neq \epsilon \\ x'pc, & \text{if } b = \sqcup \text{ and } y' = \epsilon \text{ and } c \neq \sqcup \\ x'p, & \text{if } b = \sqcup \text{ and } y' = \epsilon \text{ and } c = \sqcup \end{cases}$$

SUBCASE 1(c). $\delta(q, a) = (p, b, L)$ and $x = \epsilon$ (thus the head is on cell 1):

$$C' = \begin{cases} pby', & \text{if } b \neq \sqcup \text{ or } y' \neq \epsilon \\ p, & \text{if } b = \sqcup \text{ and } y' = \epsilon \end{cases}$$

CASE 2. $y = \epsilon$. (Thus, in C the tape head is on the leftmost of the infinitely many trailing blanks.)

SUBCASE 2(a). $\delta(q, \sqcup) = (p, b, R)$: $= xbp$.

SUBCASE 2(b). $\delta(q, \sqcup) = (p, b, L)$ and $x = x'c$, for some $c \in \Gamma$ (thus $x \neq \epsilon$ and the head is not on cell 1):

$$C' = \begin{cases} x'pcb, & \text{if } b \neq \sqcup \\ x'pc, & \text{if } b = \sqcup \text{ and } c \neq \sqcup \\ x'p, & \text{if } b = \sqcup \text{ and } c = \sqcup \end{cases}$$

Subcase 2(c). $\delta(q, \sqcup) = (p, b, L)$ and $x = \epsilon$ (thus the head is on cell 1):

$$C' = \begin{cases} pb, & \text{if } b \neq \sqcup \\ p, & \text{if } b = \sqcup \end{cases}$$

Note that if $C = yh_A z$ of $C = yh_R z$, there is no C' such that $C \vdash_M C'$: No case applies then, since the transition function is not defined for the two halt states.

The transitive closure of the $\vdash_M$ relation and is denoted $\vdash_M^*$. Intuitively, $C \vdash_M^* C'$ if and only if the TM M transforms C to C' in a finite number of steps (including zero). More precisely, $C \vdash_M^* C'$ if and only if:

- $C' = C$, or
- for some integer $k > 1$, there are configurations $C_1, C_2, \ldots, C_k$ such that $C_1 = C$, $C_k = C'$, and for all i, $1 \leq i < k$, $C_i \vdash_M C_{i+1}$.

Based on the $\vdash_M^*$ relation we can now define what it means for a TM M to accept a string, to recognize a language, and to decide a language:

- M ***accepts*** $x \in \Sigma^*$ if and only if, for some strings $y, z \in \Gamma^*$, $q_0 x \vdash_M^* yh_A z$. That is, started in the initial state q_0 with only the input x on the tape, and the head on the leftmost cell, after a finite number of steps M enters the accept state h_A with some string yz on its tape — we don't care what yz is.
- M ***rejects*** $x \in \Sigma^*$ if and only if, for some strings $y, z \in \Gamma^*$, $q_0 x \vdash_M^* yh_R z$.
- M ***loops on*** $x \in \Sigma^*$ if and only if there is an infinite sequence of confituations $C_0, C_1, C_2, \ldots$ such that $C_0 = q_0 x$ and, for all $n \in \mathbb{N}$, $C_i \vdash_M C_{i+1}$.
- M ***recognizes*** a language L if and only if $L = \{x \in \Sigma^* \colon M \text{ accepts } x\}$. That is, for every $x \in L$, M accepts x, and for every $x \notin L$, M rejects x or loops on x. In this case, we say that M is a ***recognizer*** for L. A language is ***recognizable*** if there is a TM that recognizes it. Common alternative terms for recognizable language are ***recursively enumerable*** language or ***semi-decidable*** language.
- M ***decides*** a language L if and only if M is a recognizer for L and halts on every input. That is, for every $x \in L$, M accepts x, and for every $x \notin L$, M rejects x. In this case, we say that M is a ***decider*** for L. A language is ***decidable*** if there is a TM that decides it. A common alternative term for decidable language is ***recursive*** language.

Recalling that a language is a set (of strings) and that a decision problem can be thought of as a language (the set of strings that represent yes-instances of the problem), we sometimes speak of recognizable (or recursively enumerable or semi-decidable) sets or decision problems; as well as of decidable (or recursive) sets or decision problems.