Independent Set, Clique, and Vertex Cover problems

$G = (V, E)$ undirected graph.

- Independent set of G: $V' \subseteq V$ s.t. there is no edge between any two nodes in V'.
- Clique of G: $V' \subseteq V$ s.t. there is an edge between every two nodes in V'.
- Vertex cover of G: $V' \subseteq V$ s.t. every edge of G has (at least) one endpoint in V'.

Independent Set/Clique/Vertex Cover problem:

Instance: $\langle G, k \rangle$, G is an undirected graph, $k \in \mathbb{Z}^+$.
Question: Does G have an independent set/clique/vertex cover of size k?

Independent Set, Clique, and Vertex Cover problems

$G = (V, E)$ undirected graph.

Complement of undirected graph $G = (V, E)$: Undirected graph $\bar{G} = (V, \bar{E})$, where $\bar{E}$ is the set of edges connecting nodes in V that are not in E.

V' is an independent set of $G \quad \Leftrightarrow \quad V - V'$ is a vertex cover of G
V' is an independent set of $G \quad \Leftrightarrow \quad V'$ is a clique of $\bar{G}$

G has an independent set of size k
$\Leftrightarrow$ G has vertex cover of size $|V| - k$
$\Leftrightarrow$ $\bar{G}$ has an clique of size k

Independent Set, Clique, and Vertex Cover problems

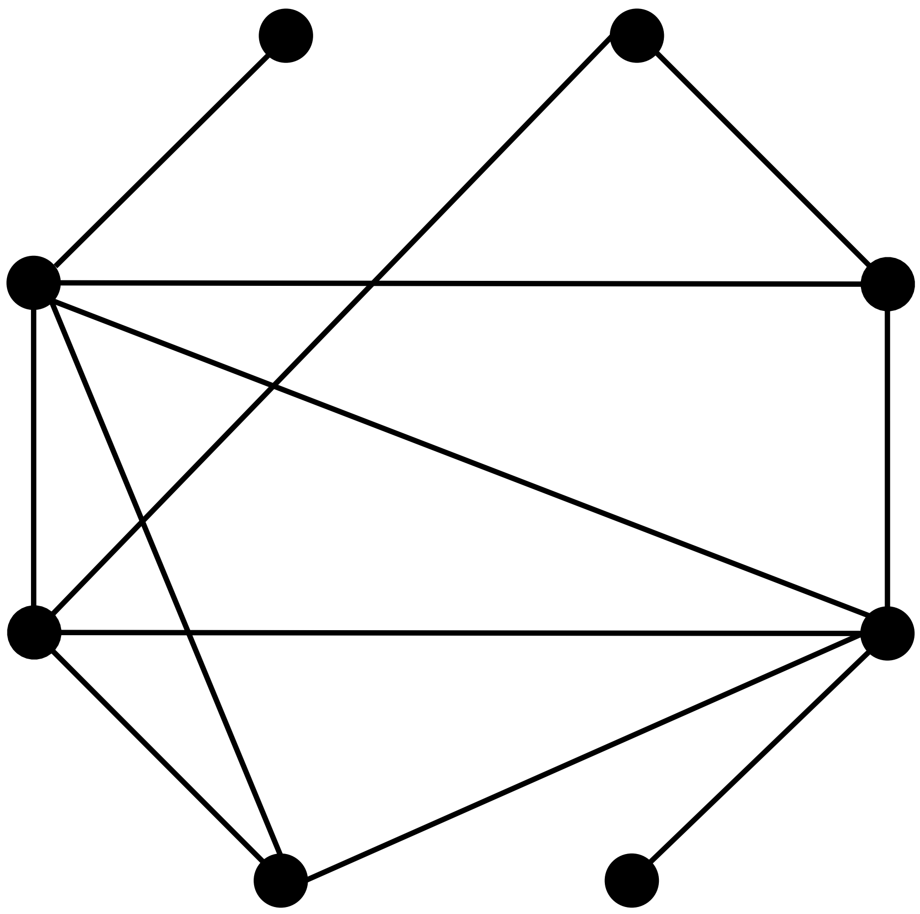
So:

IS $\leq_m^p$ CLIQUE and IS $\leq_m^p$ VC

Since CLIQUE and VC are in NP (easy to show), they are both NP-complete.

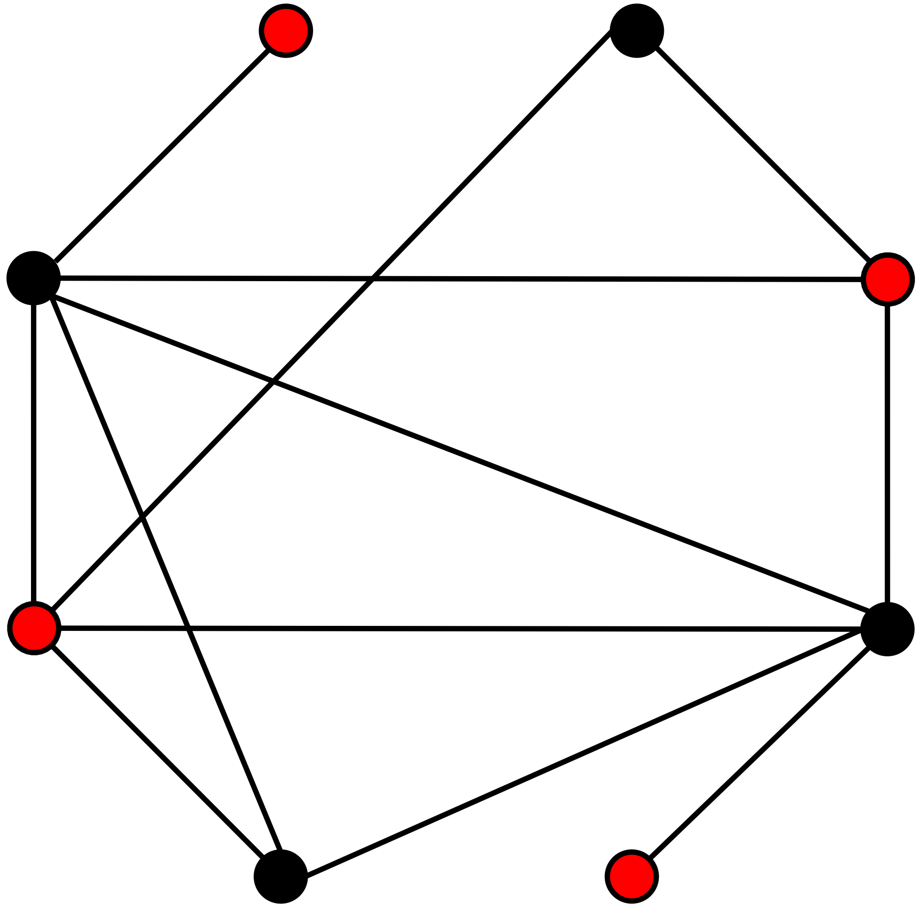
Independent Set, Clique, and Vertex Cover problems

A graph G



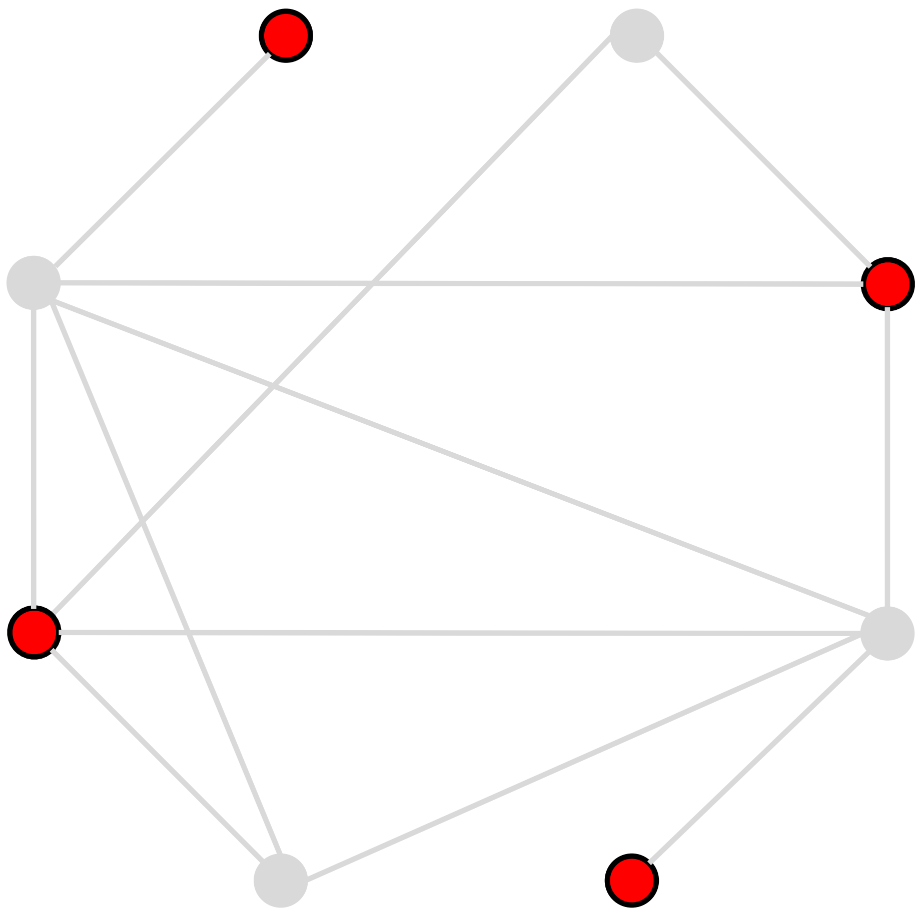
Independent Set, Clique, and Vertex Cover problems

An independent set of graph G



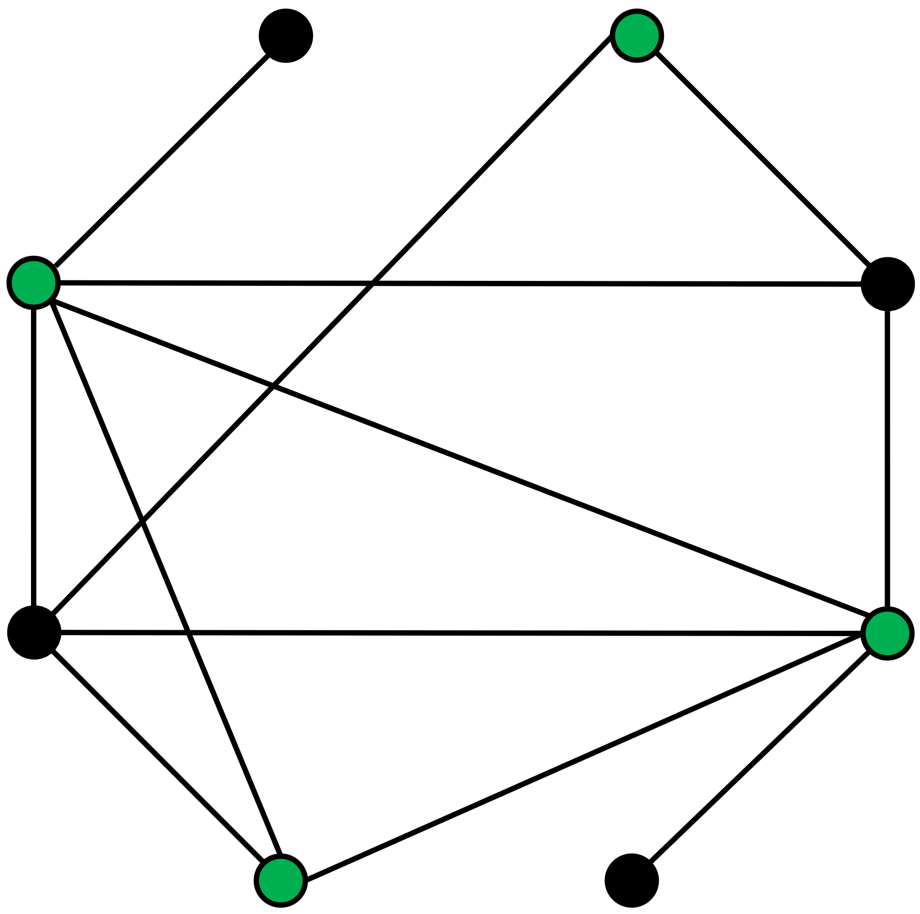
Independent Set, Clique, and Vertex Cover problems

An independent set of graph G



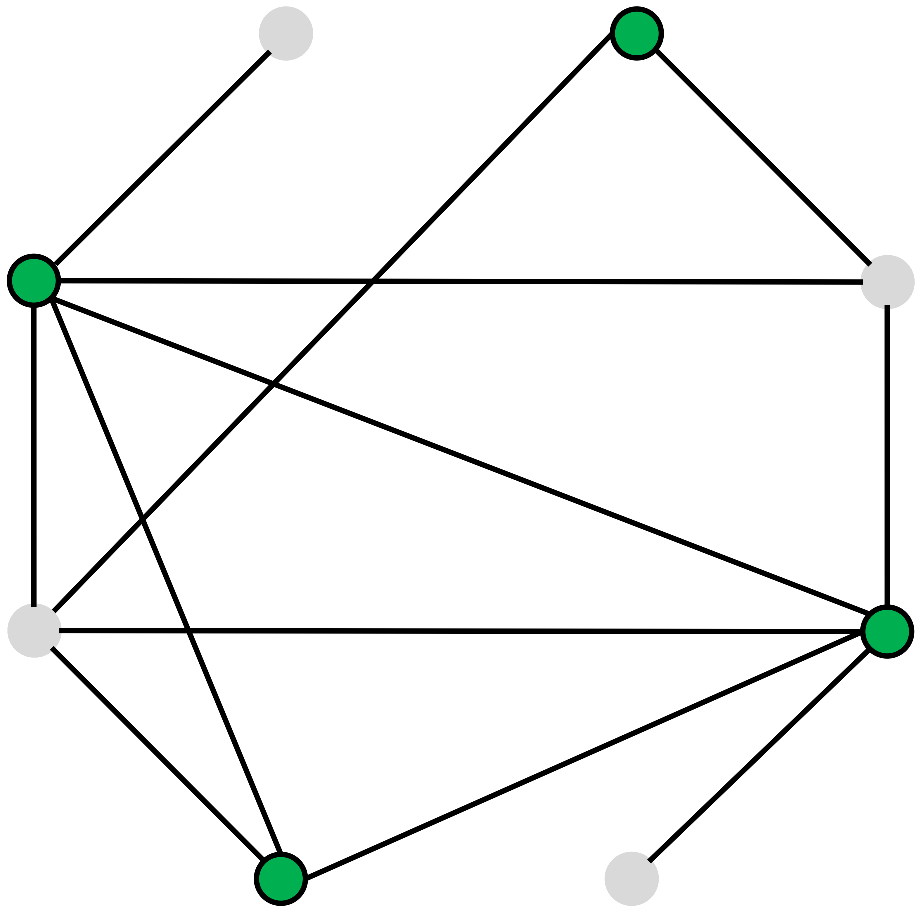
Independent Set, Clique, and Vertex Cover problems

A vertex cover of graph G



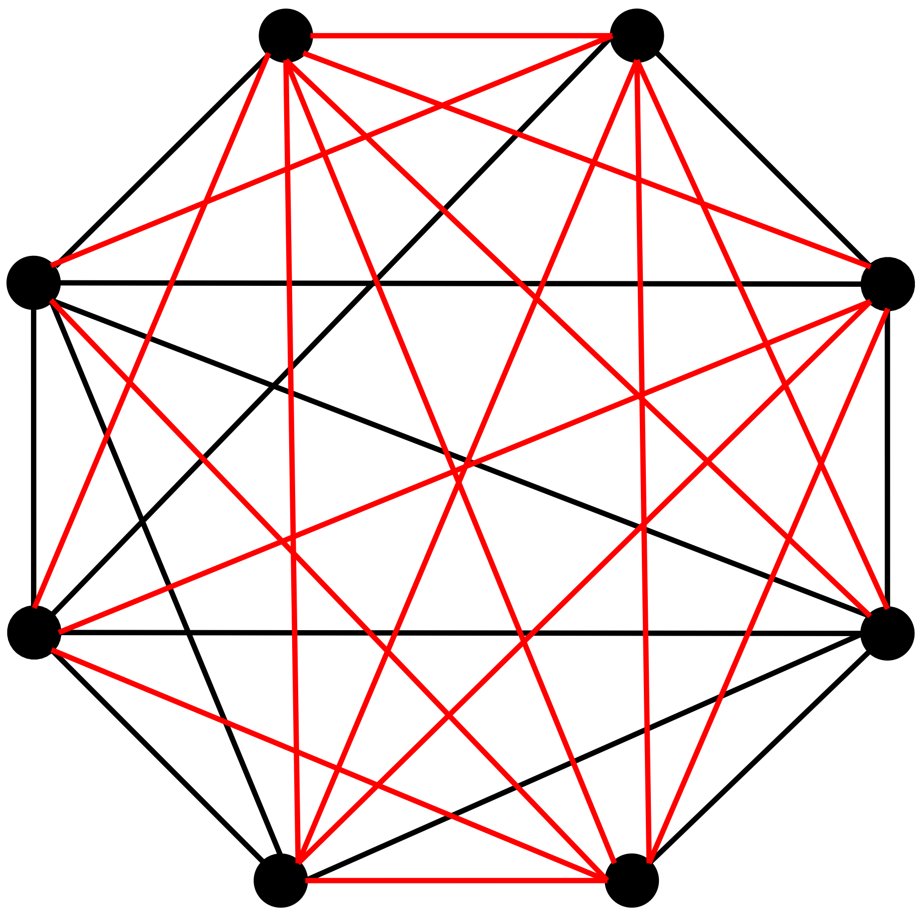
Independent Set, Clique, and Vertex Cover problems

A vertex cover of graph G



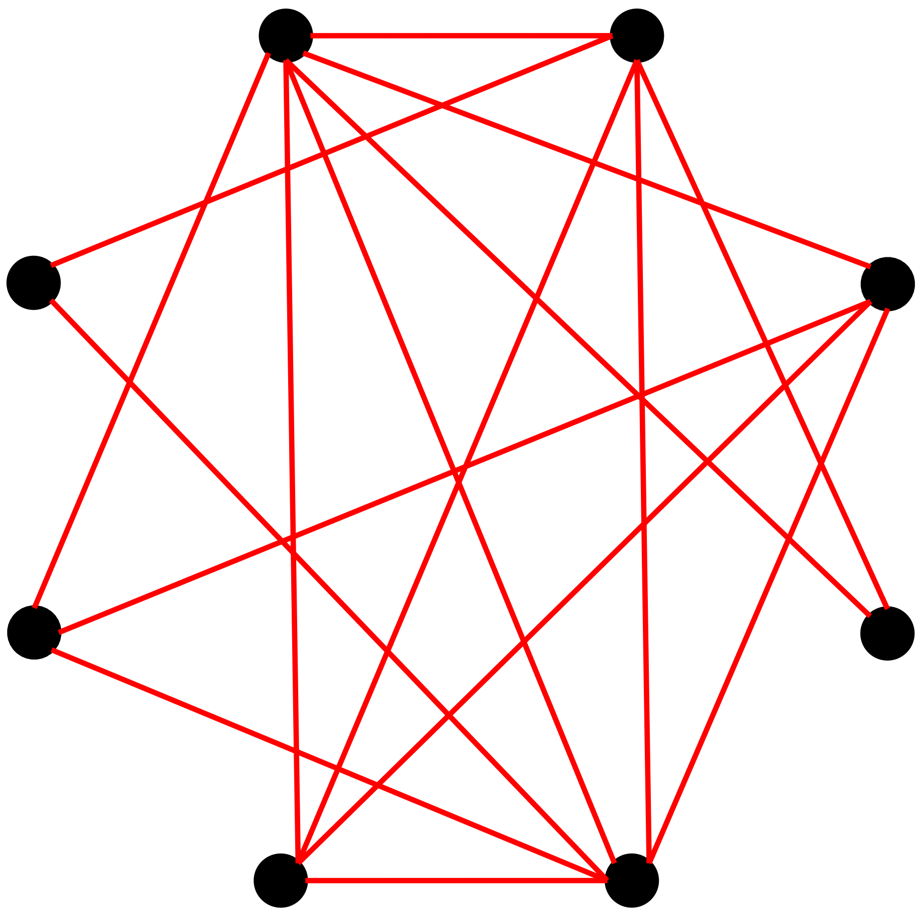
Independent Set, Clique, and Vertex Cover problems

G and its complement $\bar{G}$



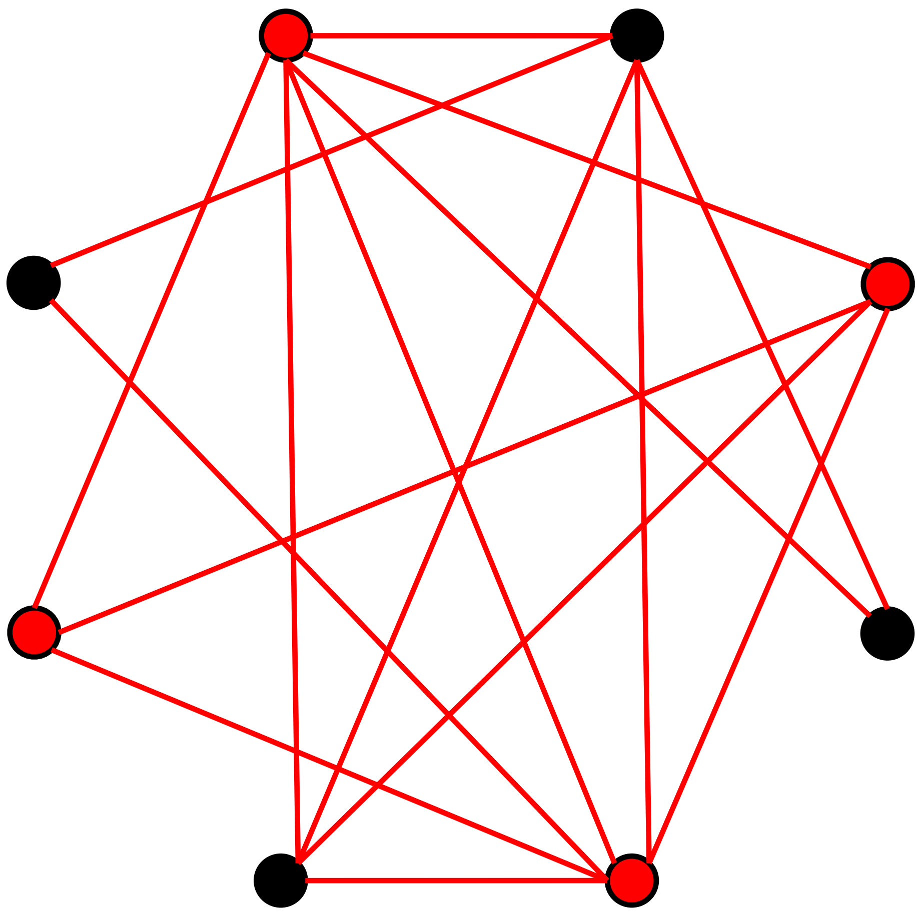
Independent Set, Clique, and Vertex Cover problems

The complement $\bar{G}$



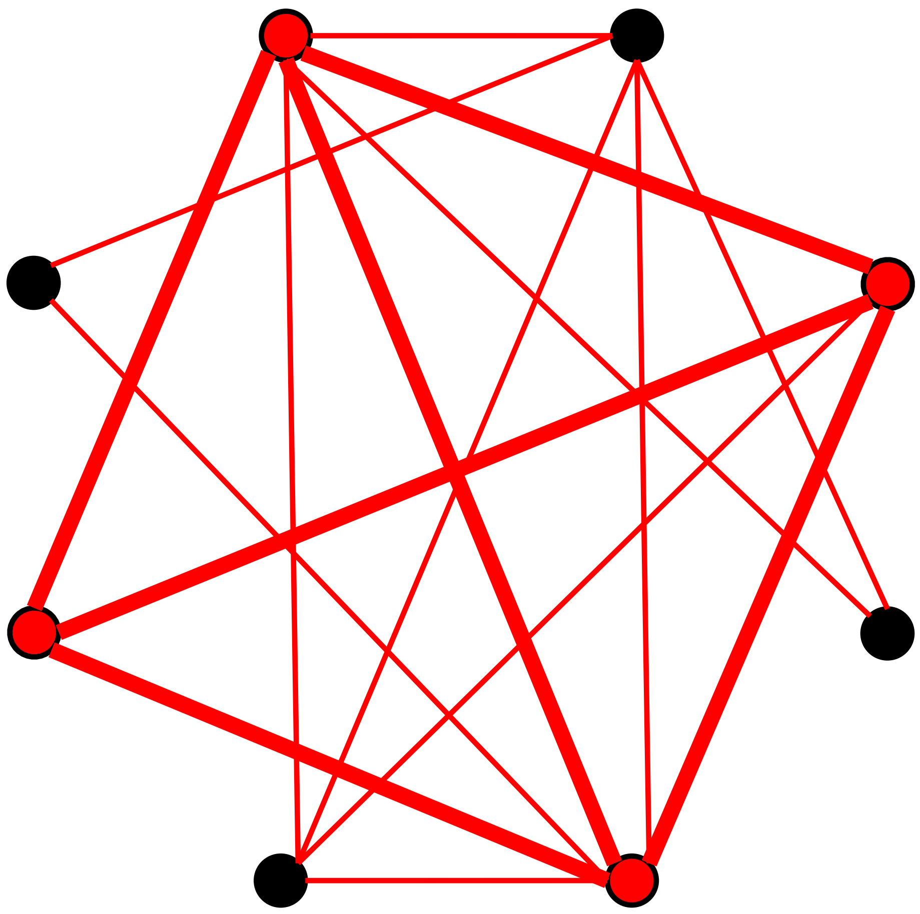
Independent Set, Clique, and Vertex Cover problems

A clique in the complement $\bar{G}$



Independent Set, Clique, and Vertex Cover problems

A clique in the complement $\bar{G}$




I bet she's thinking about another guy
How the heck do you prove that 3D matching is NP-complete?

3-dimensional matching

$A = \{a, b, c, d\}$
$B = \{w, x, y, z\}$
$C = \{1, 2, 3, 4\}$

$$
\begin{aligned}
T = \{&(a, x, 1), (a, z, 4), \\
&(b, x, 2), (b, x, 3), \\
&(c, w, 3), (c, x, 3), \\
&(d, y, 1), (d, x, 2)\}
\end{aligned}
$$

Tripartite matching

$A = \{a, b, c, d\}$
$B = \{w, x, y, z\}$
$C = \{1, 2, 3, 4\}$

$$T = \{(a, x, 1), (a, z, 4), \\ (b, x, 2), (b, x, 3), \\ (c, w, 3), (c, x, 3), \\ (d, y, 1), (d, x, 2)\}$$

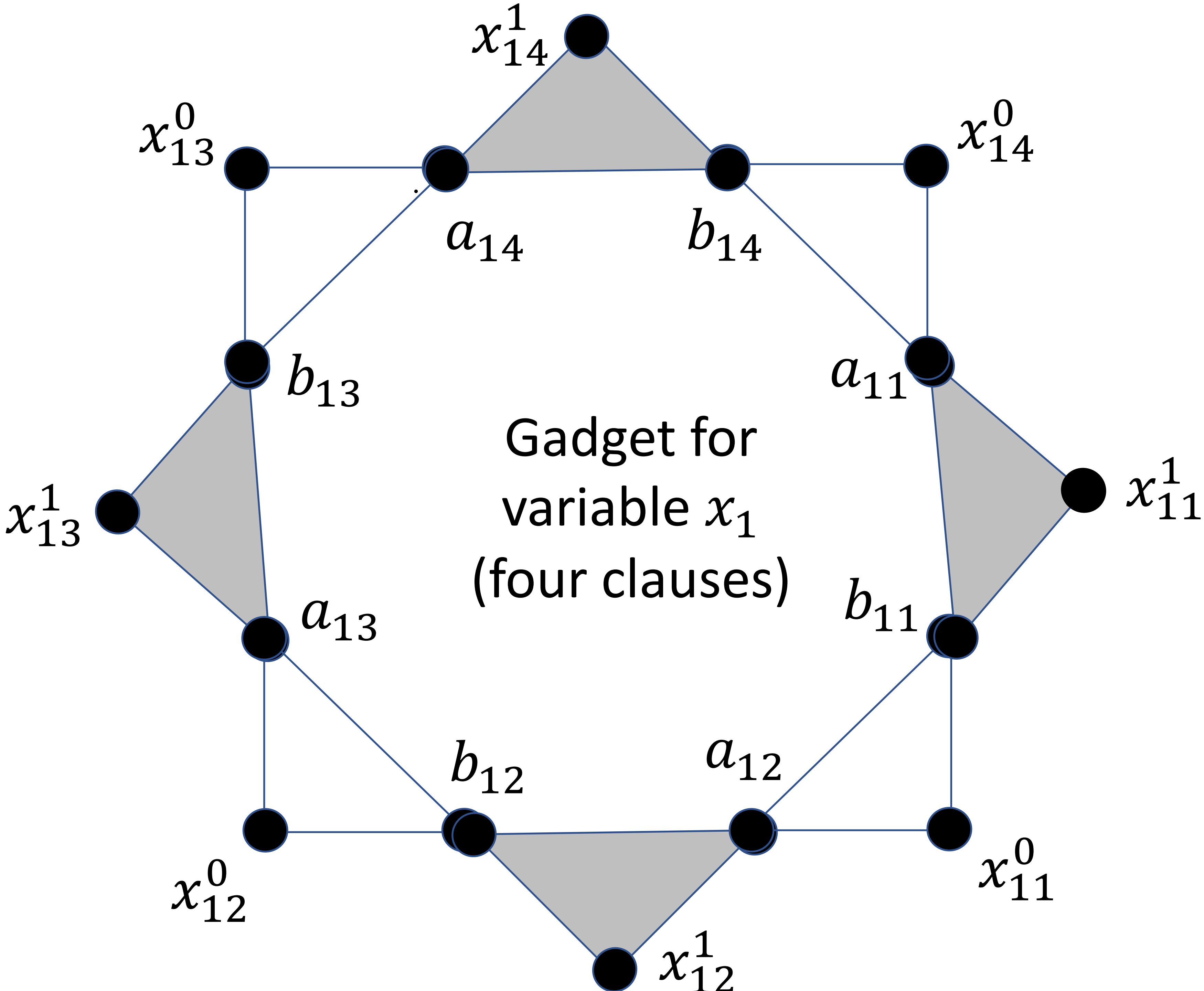
Matching M

Tripartite matching

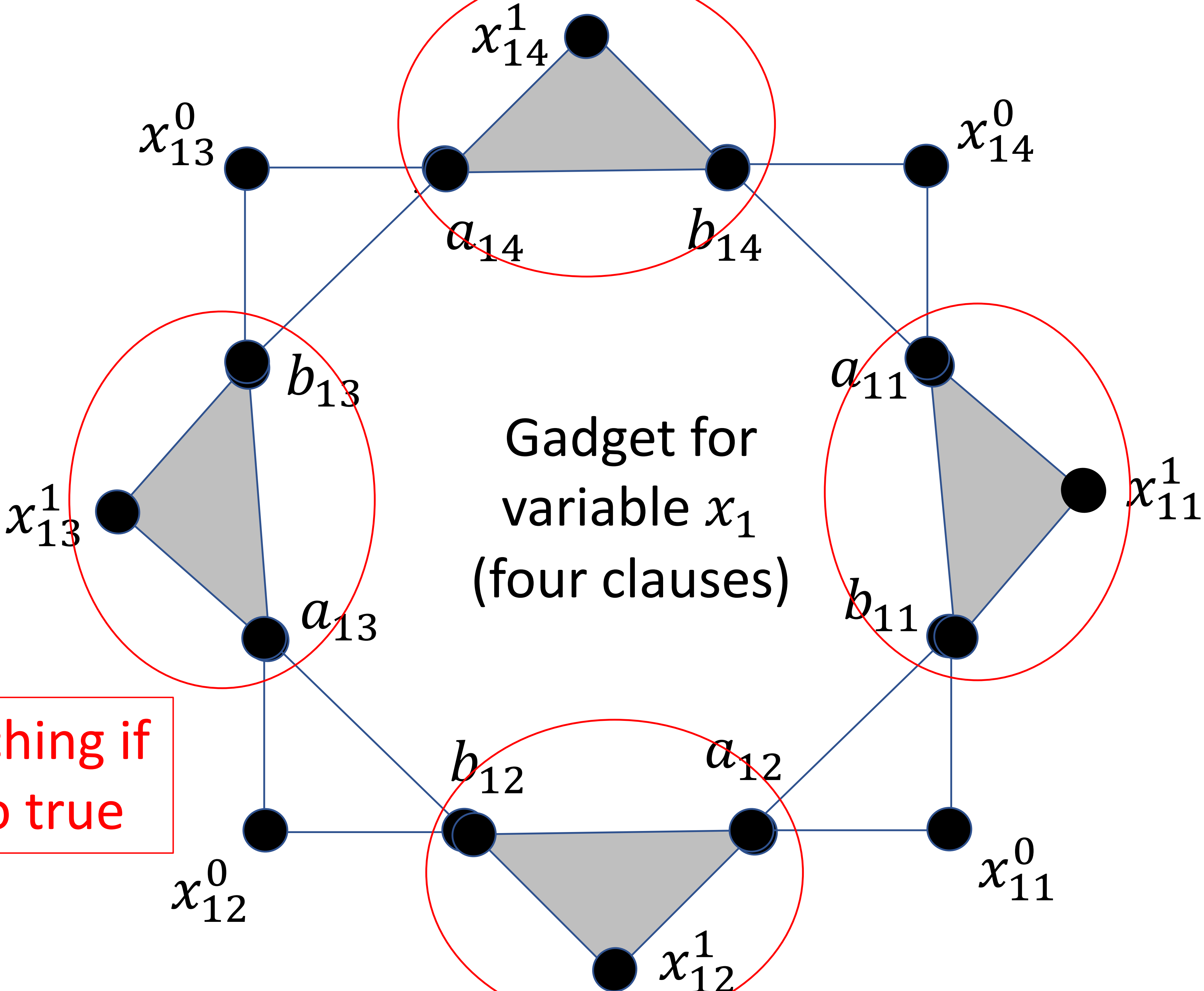
$A = \{a, b, c, d\}$
$B = \{w, x, y, z\}$
$C = \{1, 2, 3, 4\}$

$$T' = \{(a, x, 1), (a, z, 4), (b, x, 2), (b, x, 3), (c, w, 1), (c, x, 3), (d, y, 1), (d, x, 2)\}$$

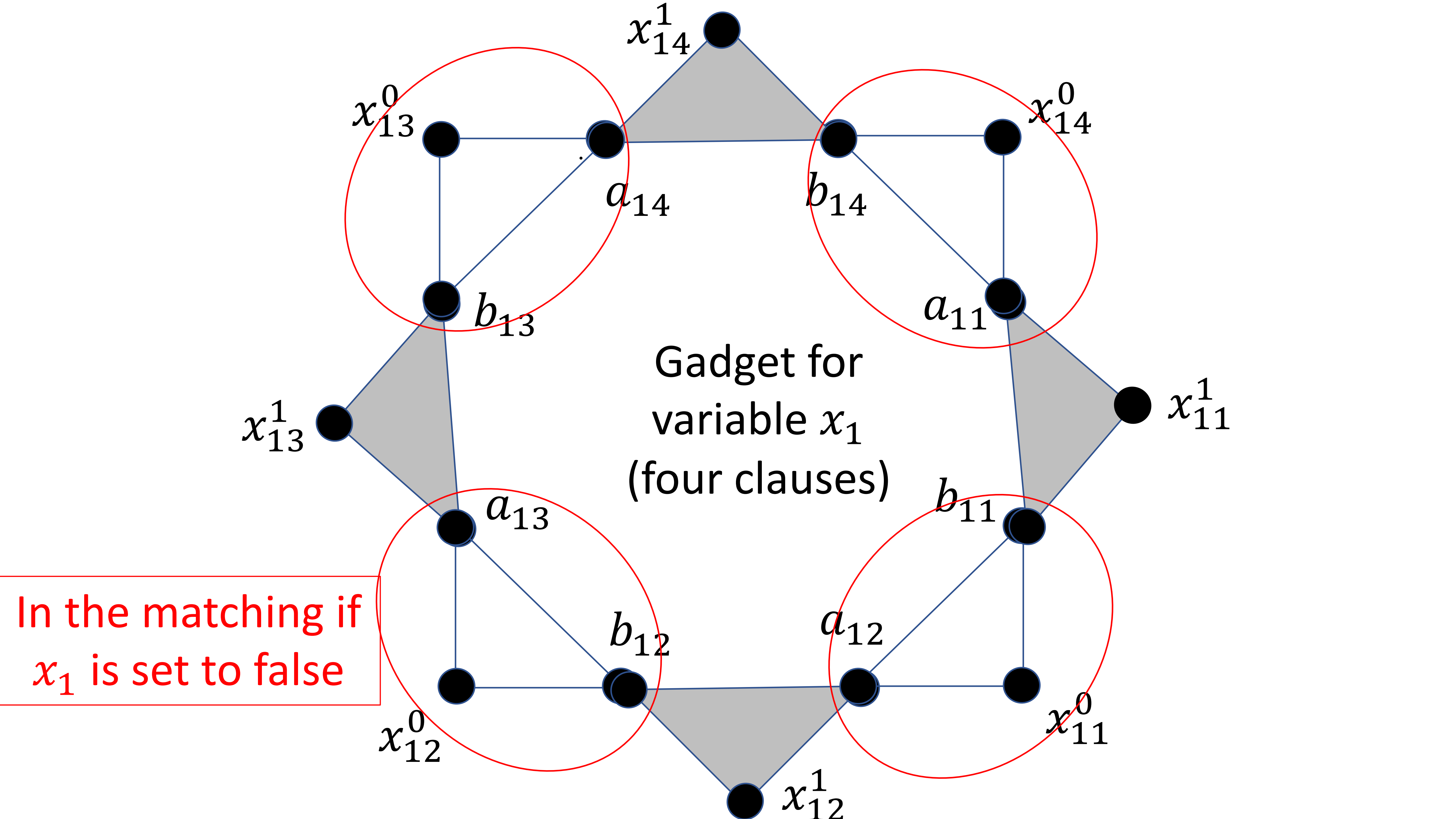
No matching!



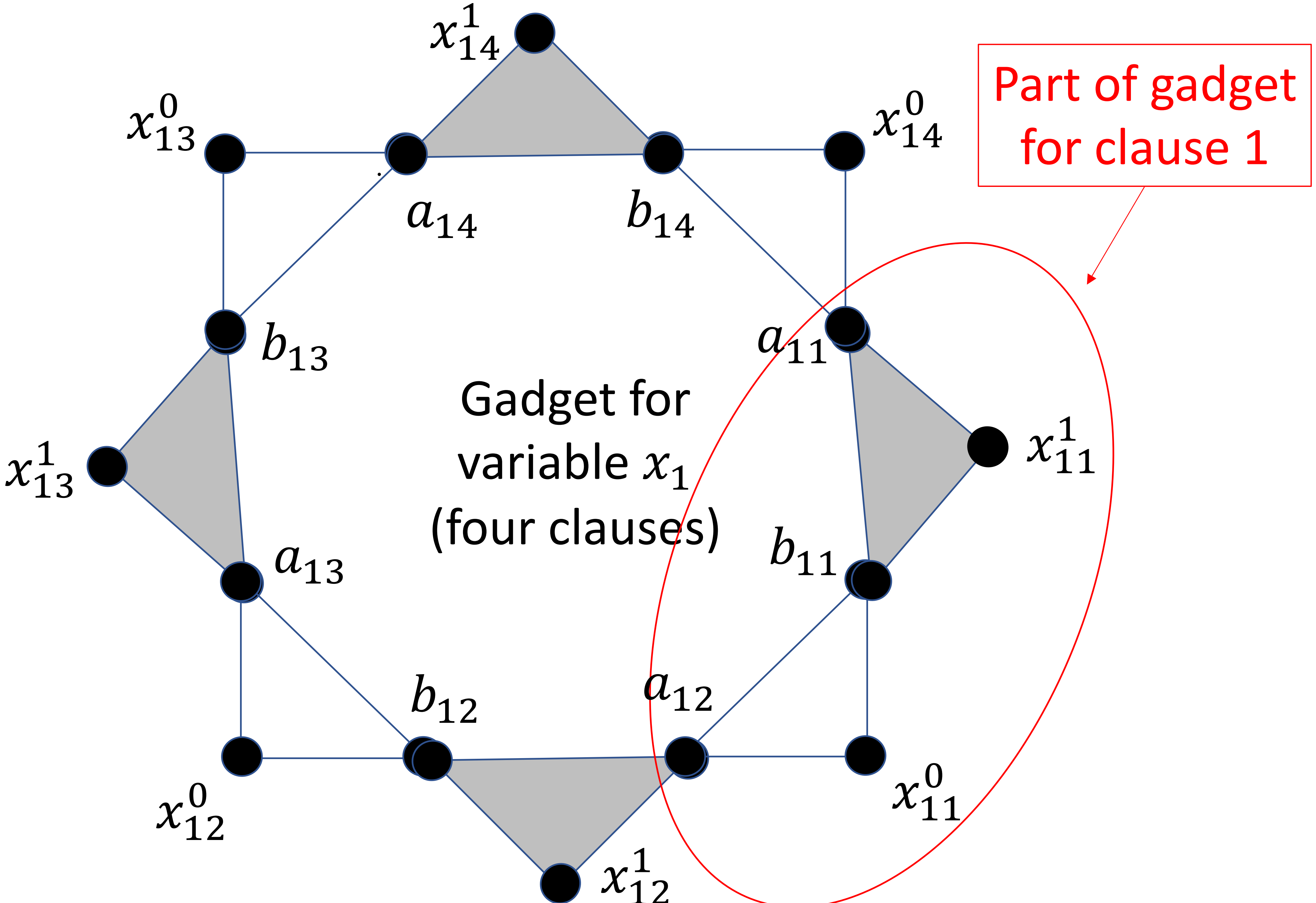
x_{14}^1
x_{13}^0
x_{14}^0
a_{14}
b_{14}
b_{13}
a_{11}
Gadget for variable x_1 (four clauses)
x_{13}^1
x_{11}^1
a_{13}
b_{11}
b_{12}
a_{12}
x_{12}^0
x_{11}^0
x_{12}^1



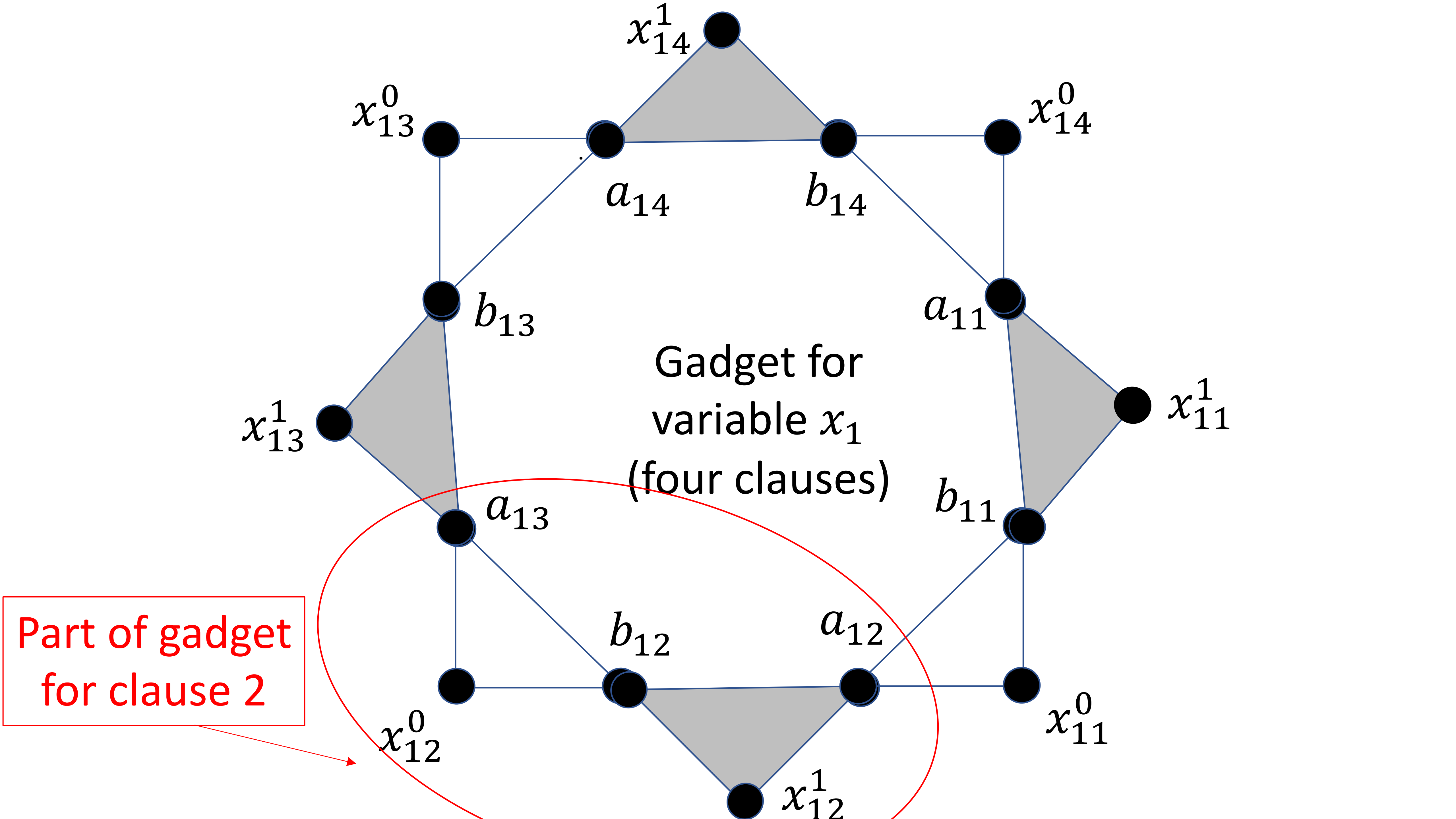
In the matching if x_1 is set to true



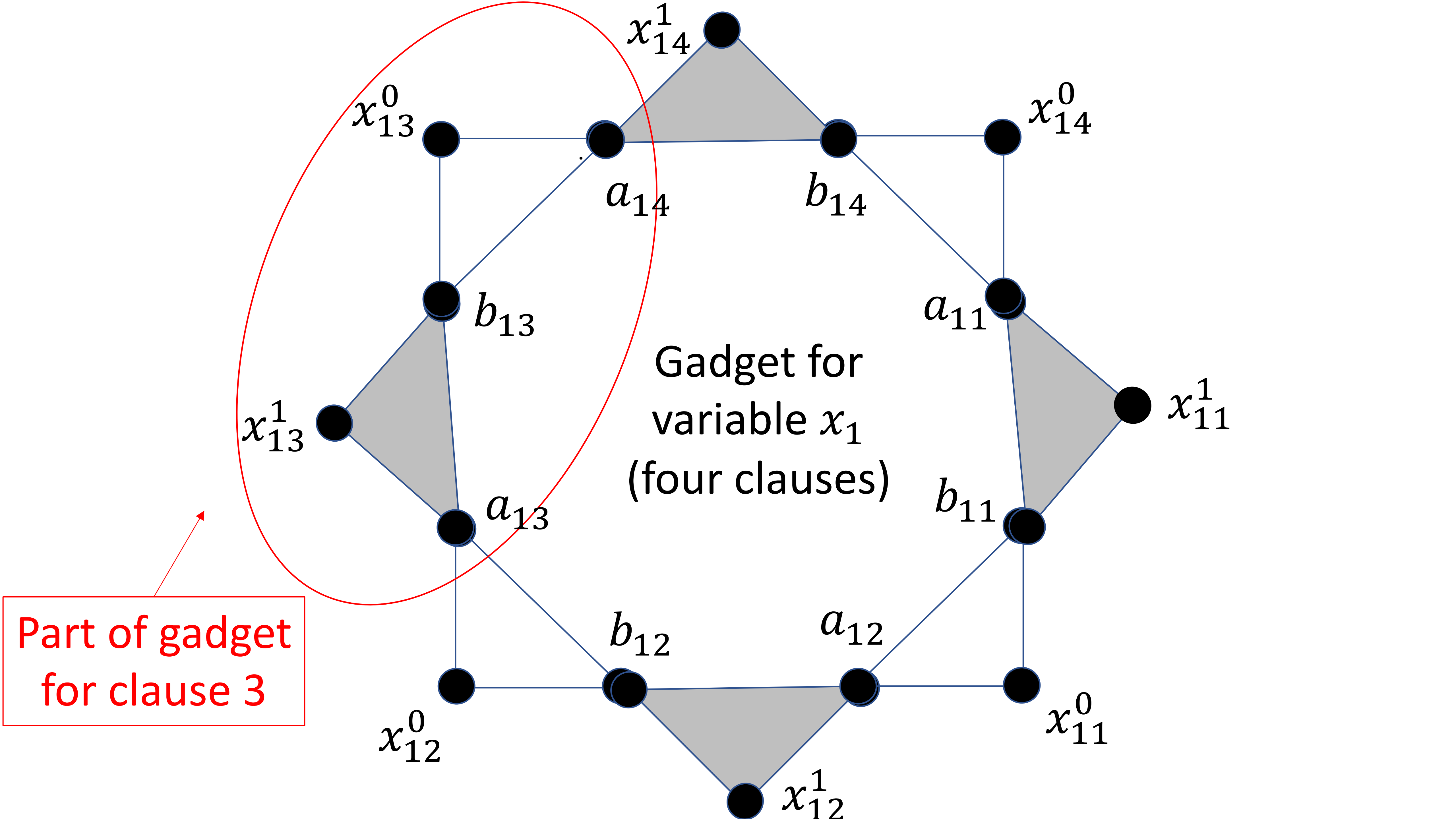
x_{14}^1
x_{13}^0
x_{14}^0
a_{14}
b_{14}
b_{13}
a_{11}
Gadget for
variable x_1
(four clauses)
x_{13}^1
x_{11}^1
a_{13}
b_{11}
In the matching if
x_1 is set to false
b_{12}
a_{12}
x_{12}^0
x_{11}^0
x_{12}^1

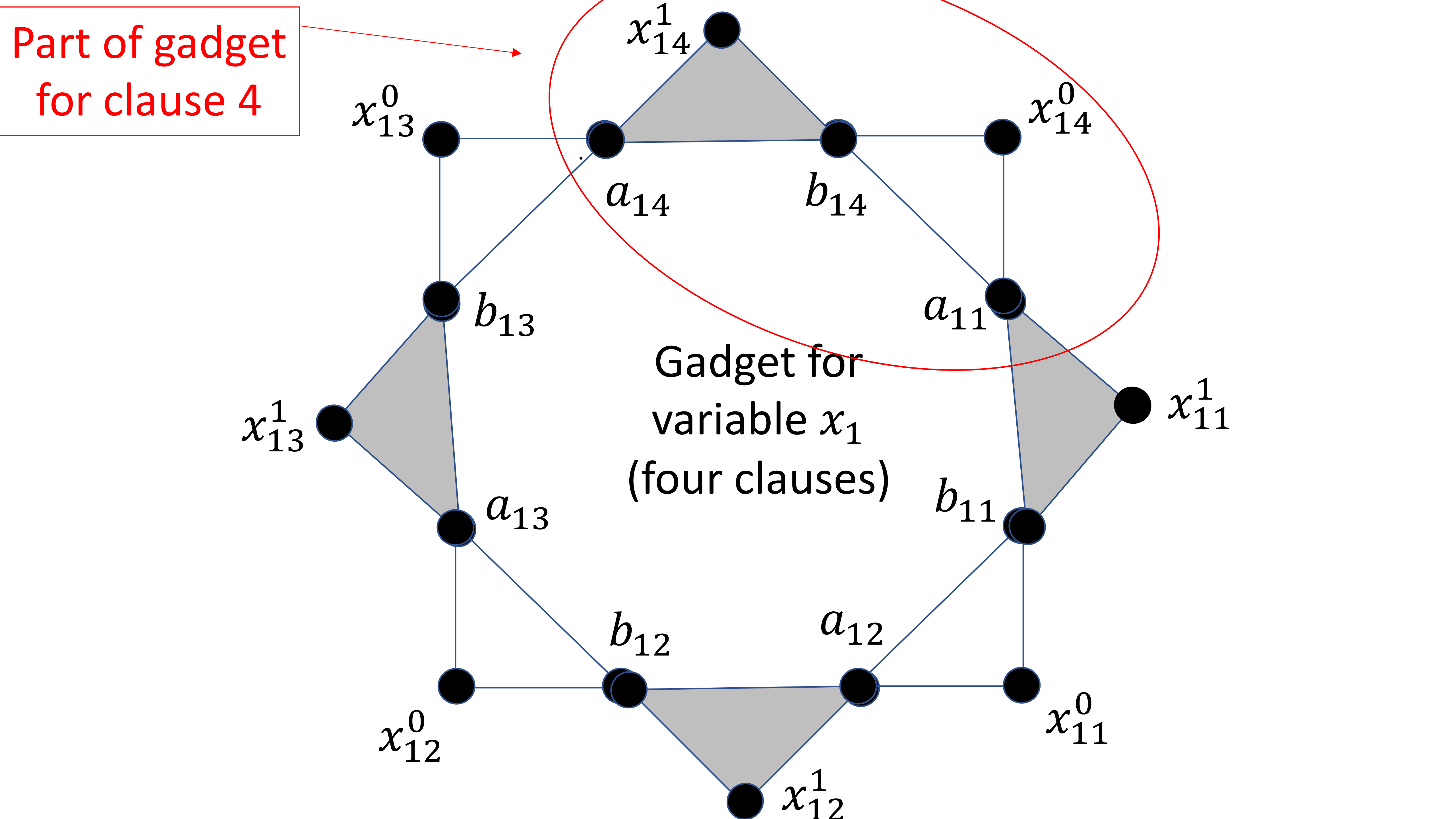


Part of gadget for clause 1
Gadget for variable x_1 (four clauses)
x_{14}^1
x_{13}^0
x_{14}^0
a_{14}
b_{14}
b_{13}
a_{11}
x_{13}^1
x_{11}^1
a_{13}
b_{11}
b_{12}
a_{12}
x_{12}^0
x_{11}^0
x_{12}^1

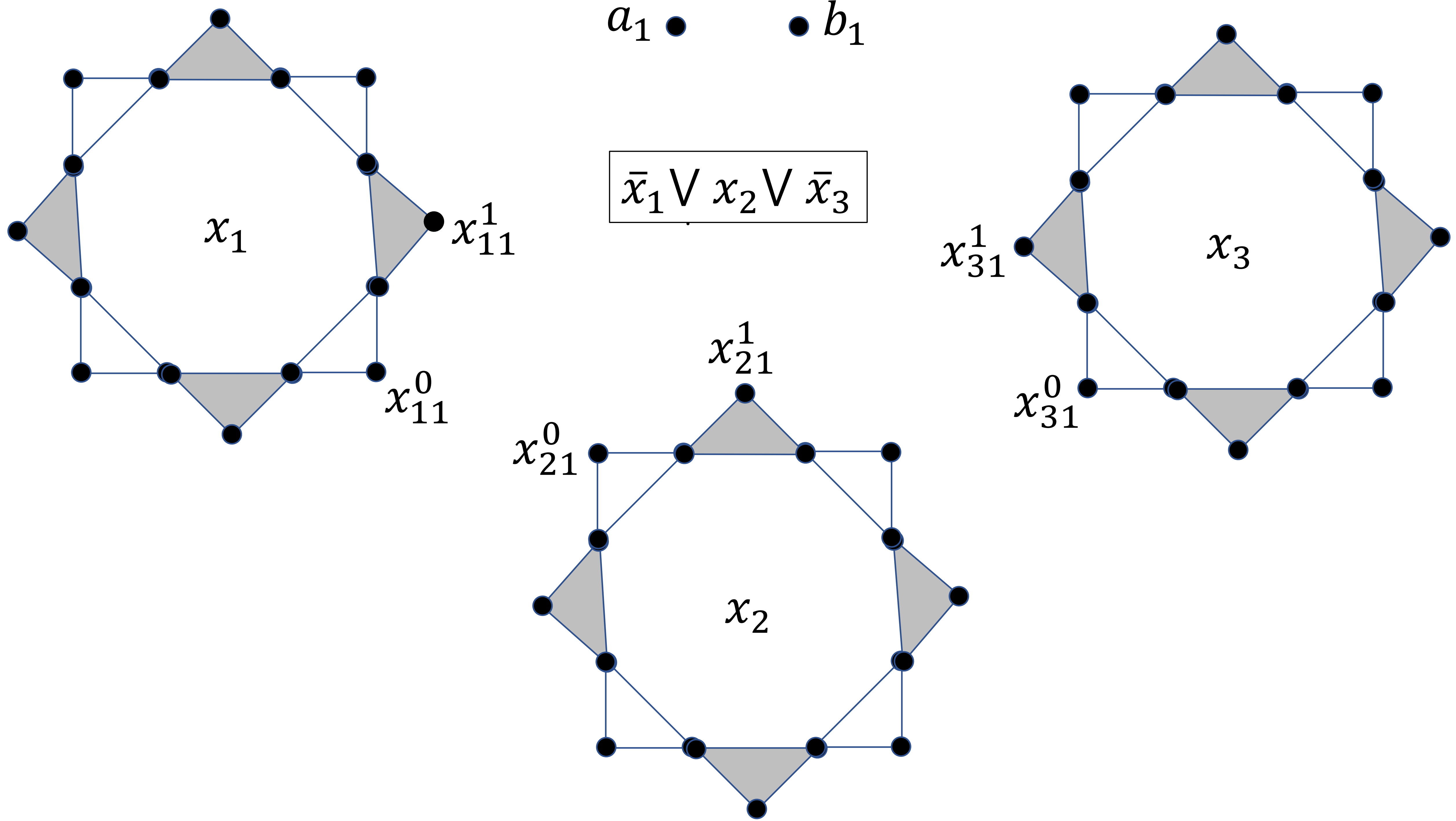


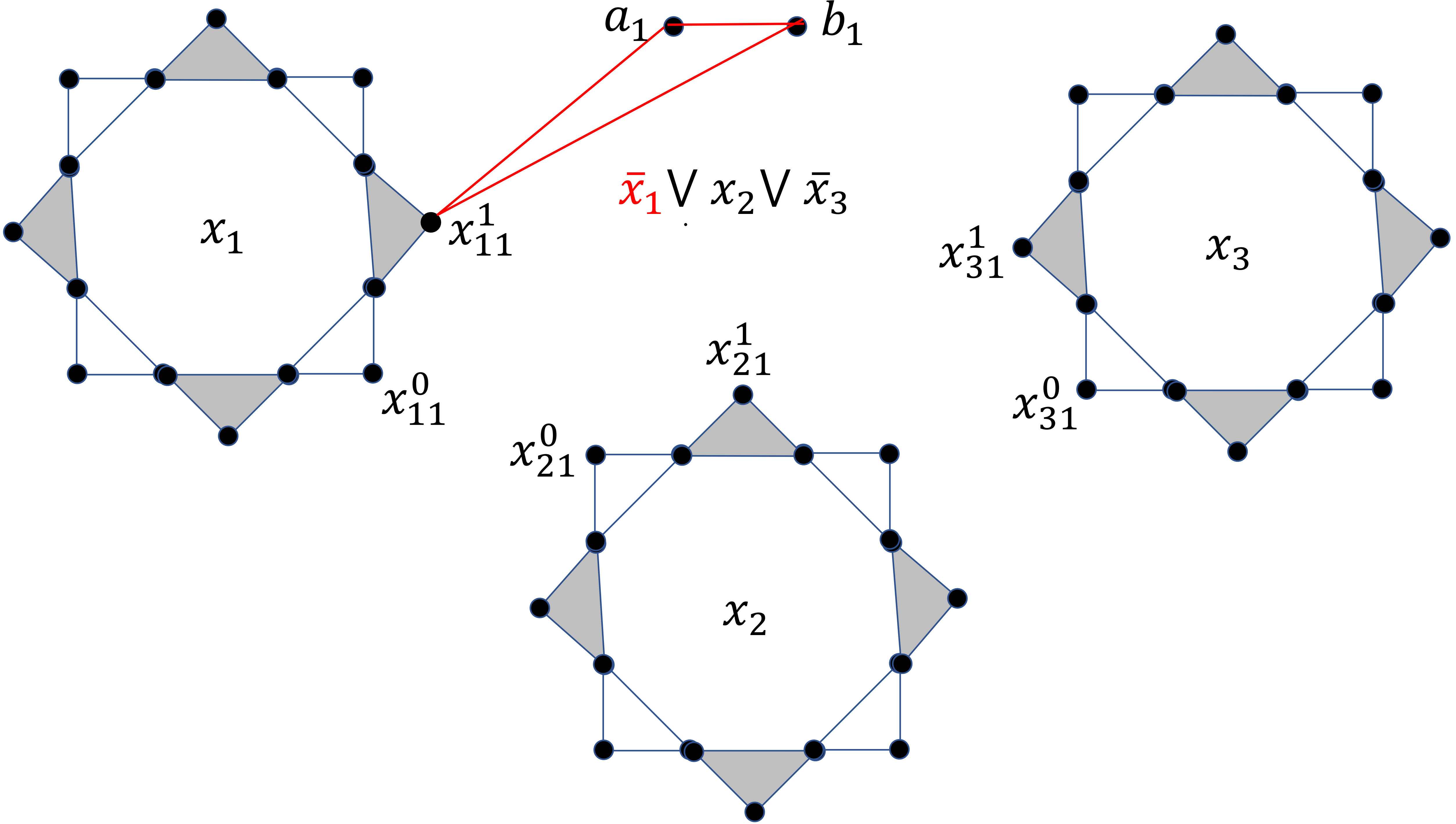
x_{14}^1
x_{13}^0
x_{14}^0
a_{14}
b_{14}
b_{13}
a_{11}
Gadget for
variable x_1
(four clauses)
x_{13}^1
x_{11}^1
a_{13}
b_{11}
Part of gadget
for clause 2
b_{12}
a_{12}
x_{12}^0
x_{11}^0
x_{12}^1


x^1_{14}
x^0_{13}
a_{14}
b_{14}
x^0_{14}
b_{13}
a_{11}
Gadget for
variable x_1
(four clauses)
x^1_{13}
x^1_{11}
a_{13}
b_{11}
Part of gadget
for clause 3
b_{12}
a_{12}
x^0_{12}
x^0_{11}
x^1_{12}

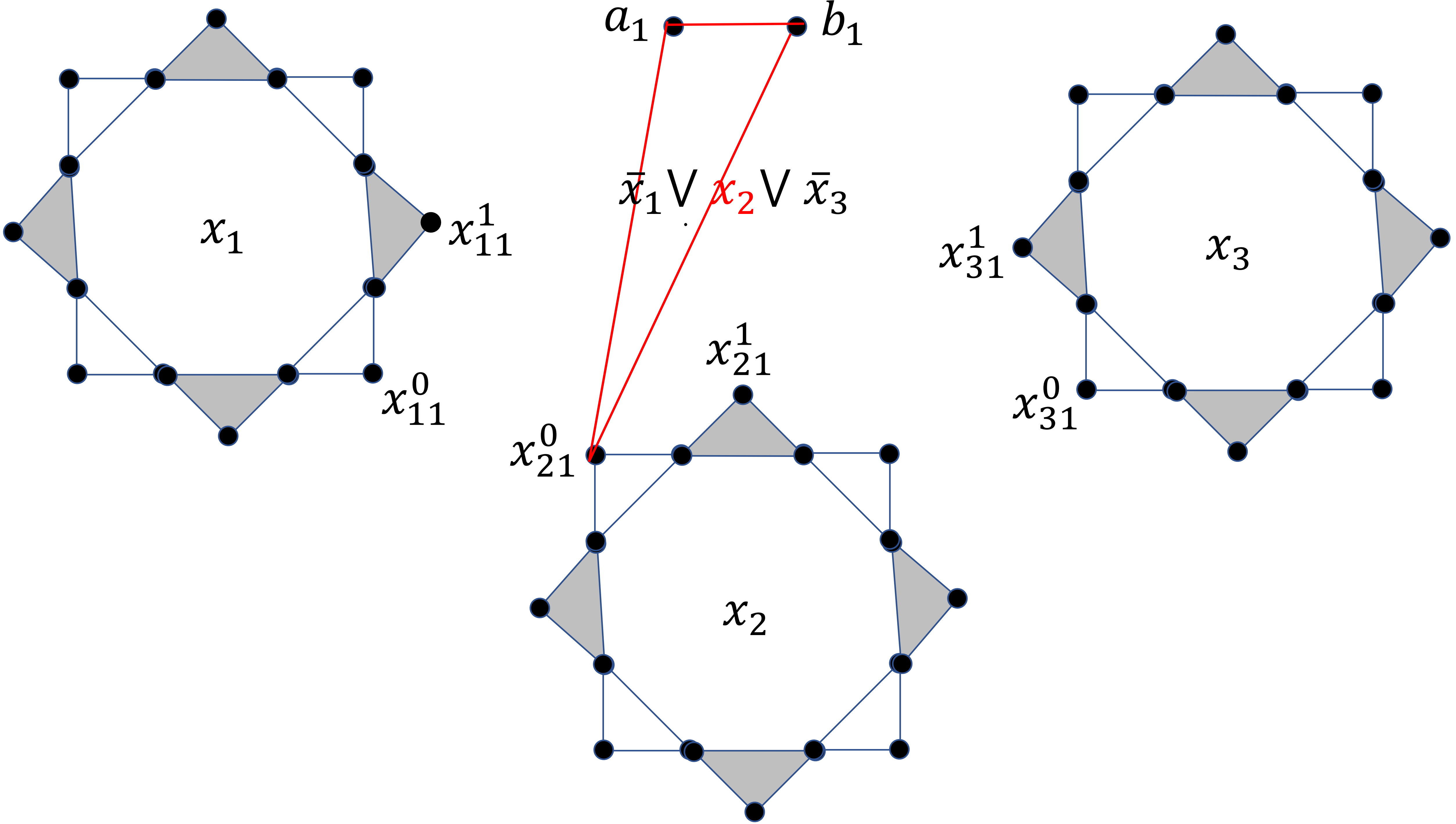


Part of gadget for clause 4
x_{14}^{1}
x_{13}^{0}
x_{14}^{0}
a_{14}
b_{14}
b_{13}
a_{11}
Gadget for variable x_1 (four clauses)
x_{13}^{1}
x_{11}^{1}
a_{13}
b_{11}
b_{12}
a_{12}
x_{12}^{0}
x_{11}^{0}
x_{12}^{1}

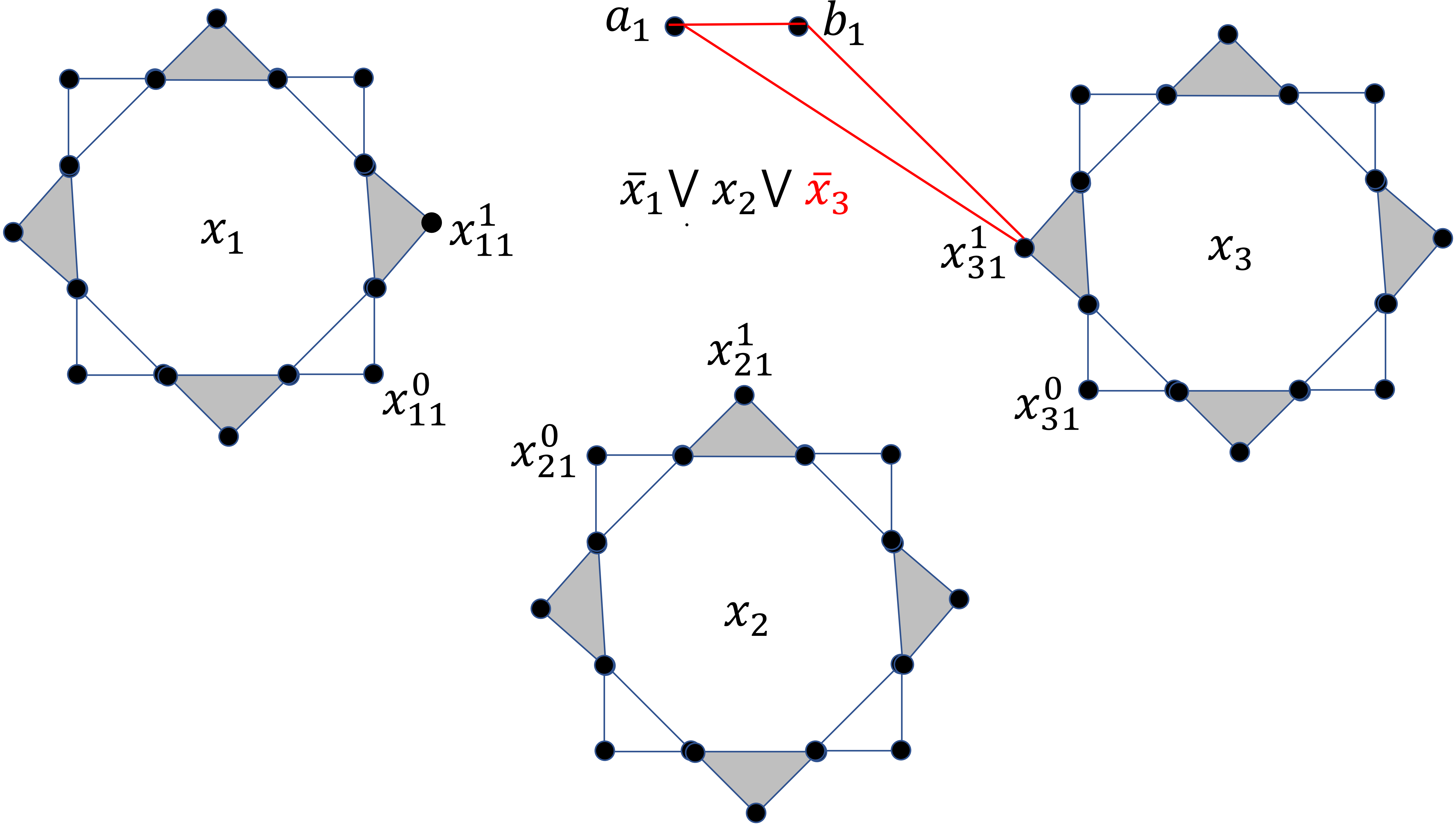

a_1
b_1
$\bar{x}_1 \vee x_2 \vee \bar{x}_3$
x_1
x_{11}^1
x_{11}^0
x_3
x_{31}^1
x_{31}^0
x_{21}^1
x_{21}^0
x_2



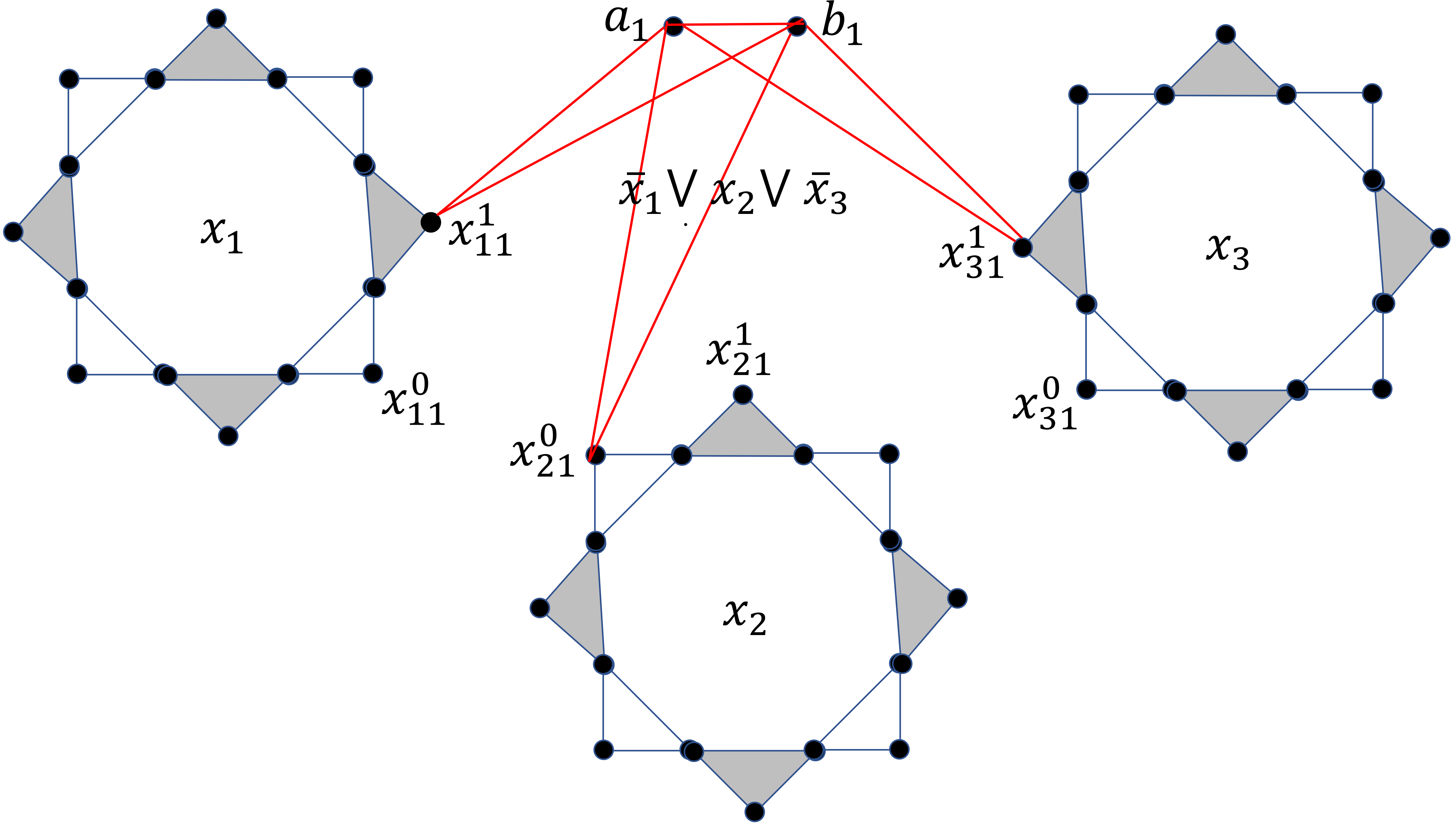
a_1
b_1
$\bar{x}_1 \vee x_2 \vee \bar{x}_3$
x_1
x_{11}^1
x_{11}^0
x_2
x_{21}^1
x_{21}^0
x_3
x_{31}^1
x_{31}^0



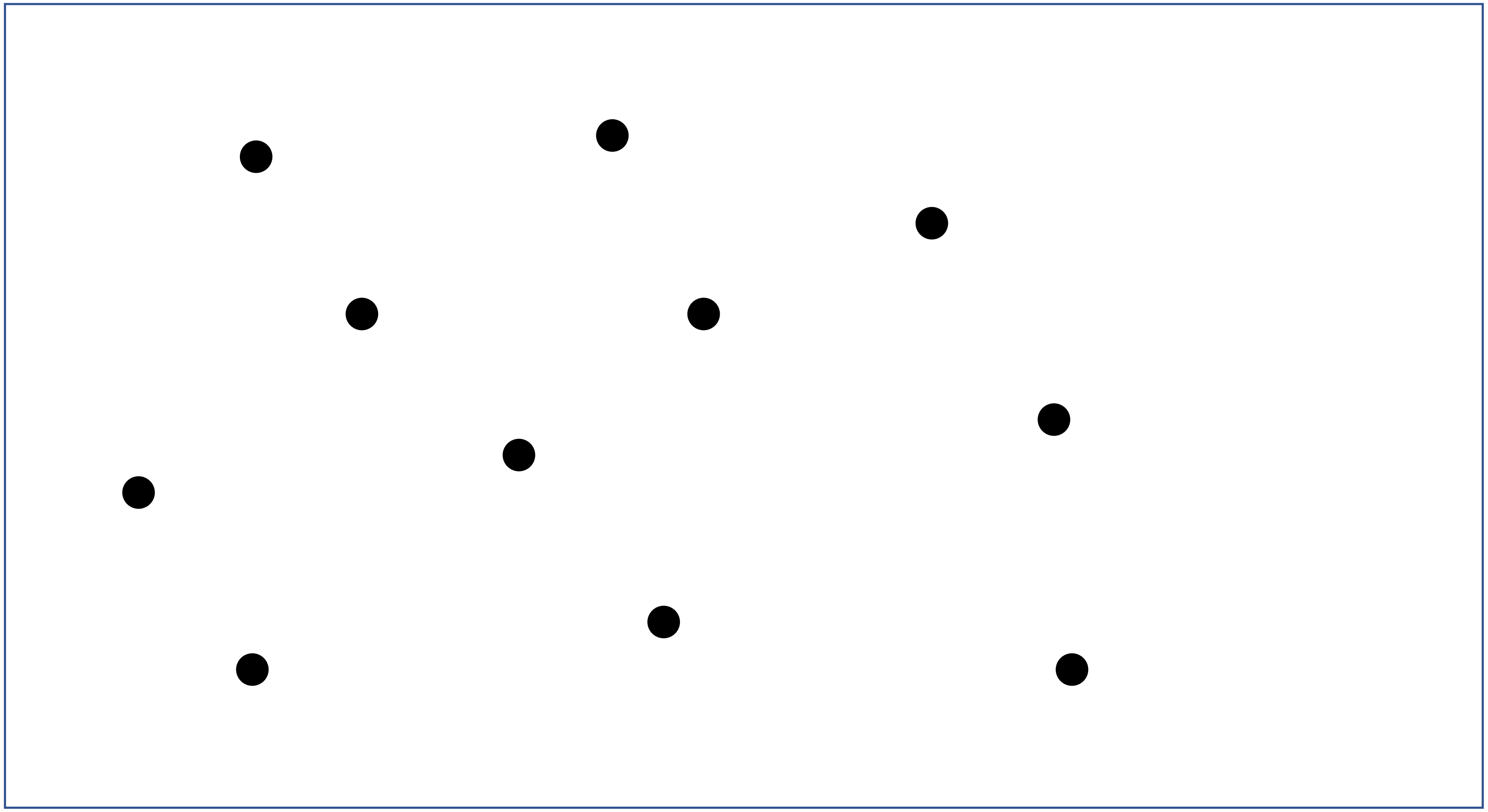
a_1
b_1
$\bar{x}_1 \vee x_2 \vee \bar{x}_3$
x_1
x_{11}^1
x_{11}^0
x_2
x_{21}^1
x_{21}^0
x_3
x_{31}^1
x_{31}^0



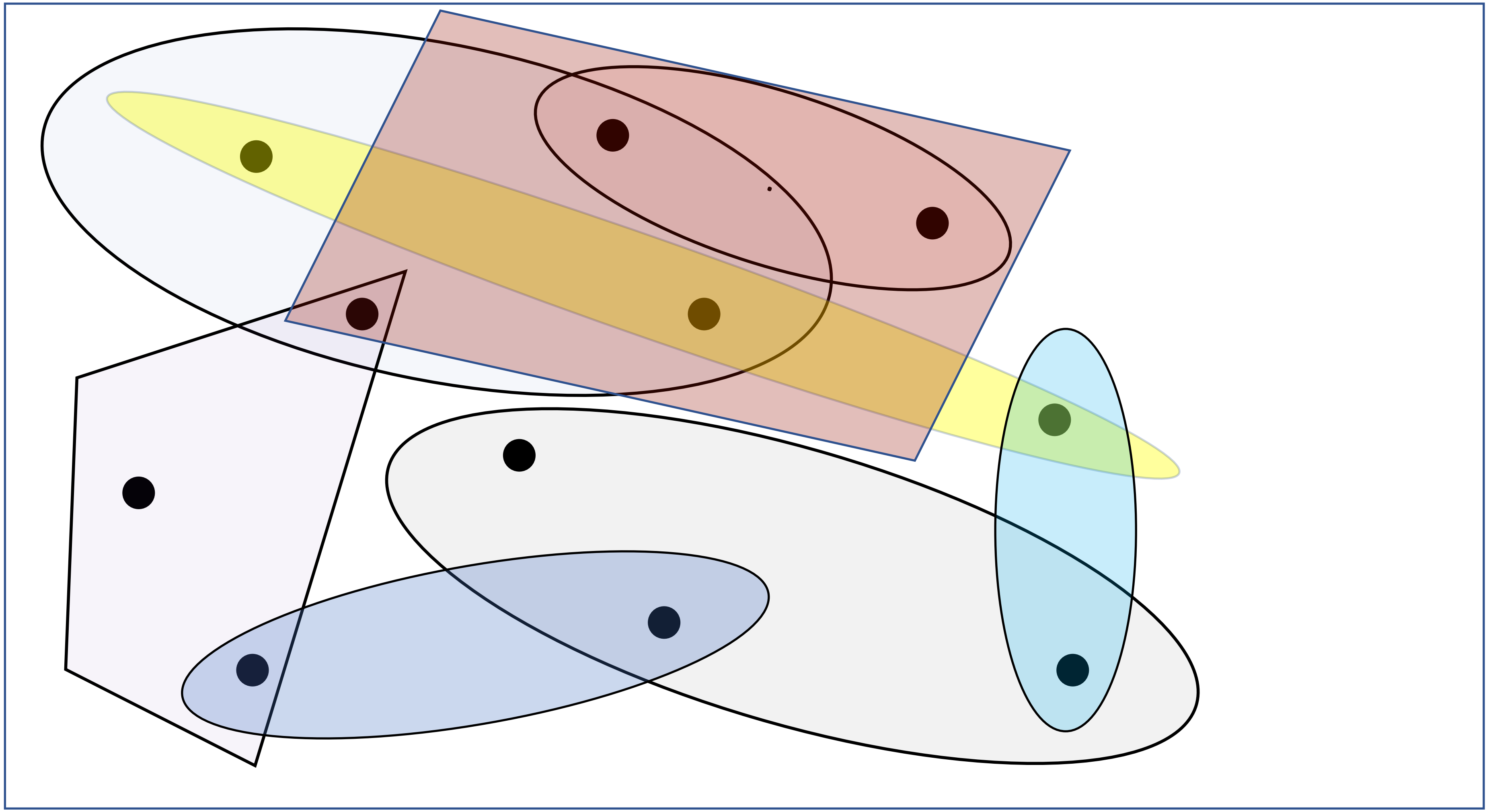
a_1
b_1
$\bar{x}_1 \vee x_2 \vee \bar{x}_3$
x_1
x_{11}^1
x_{11}^0
x_{31}^1
x_3
x_{31}^0
x_{21}^1
x_{21}^0
x_2


a_1
b_1
$\bar{x}_1 \vee x_2 \vee \bar{x}_3$
x_1
x_{11}^1
x_{11}^0
x_{21}^1
x_{21}^0
x_2
x_{31}^1
x_{31}^0
x_3

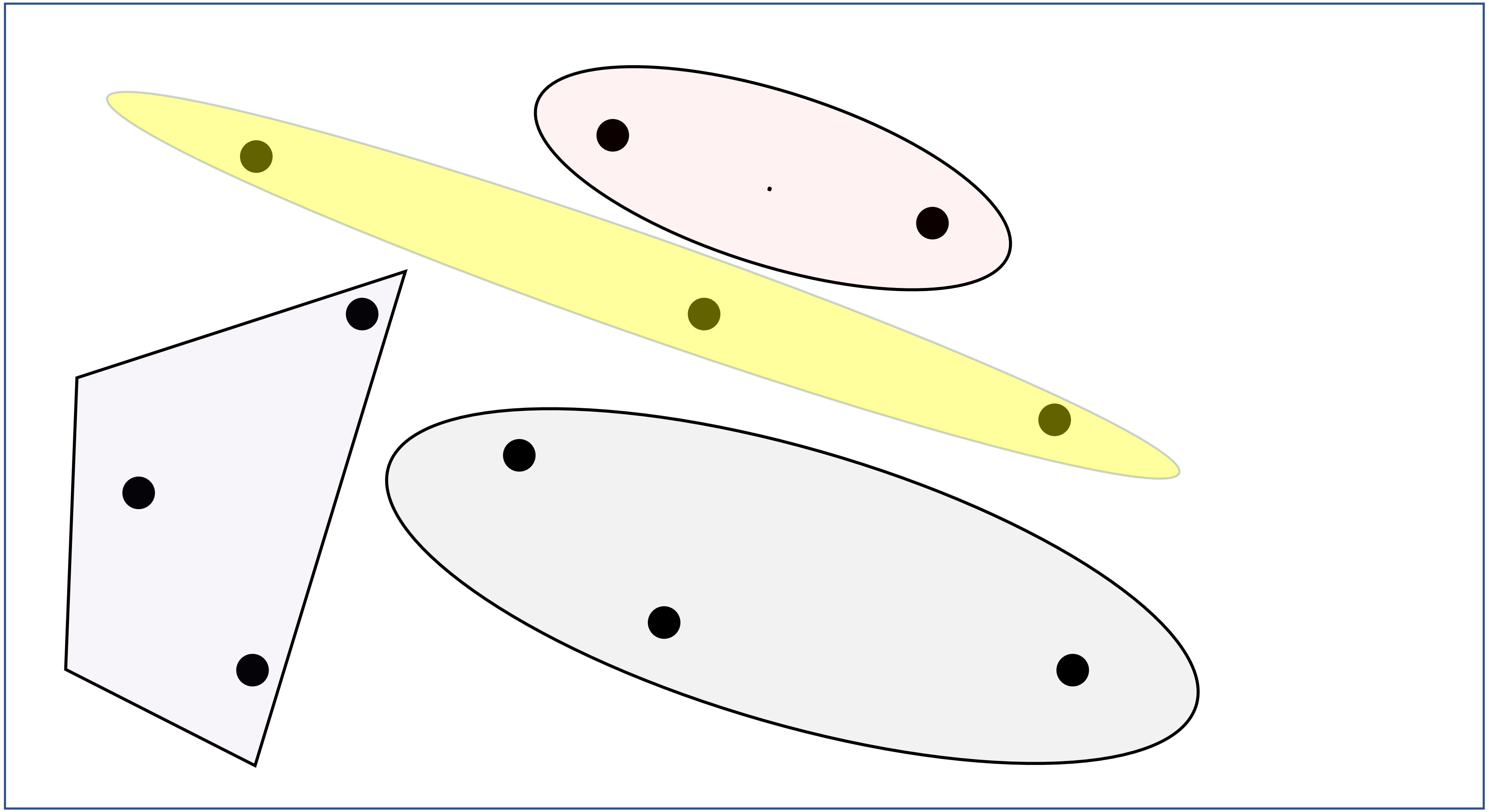
The Universe of elements U



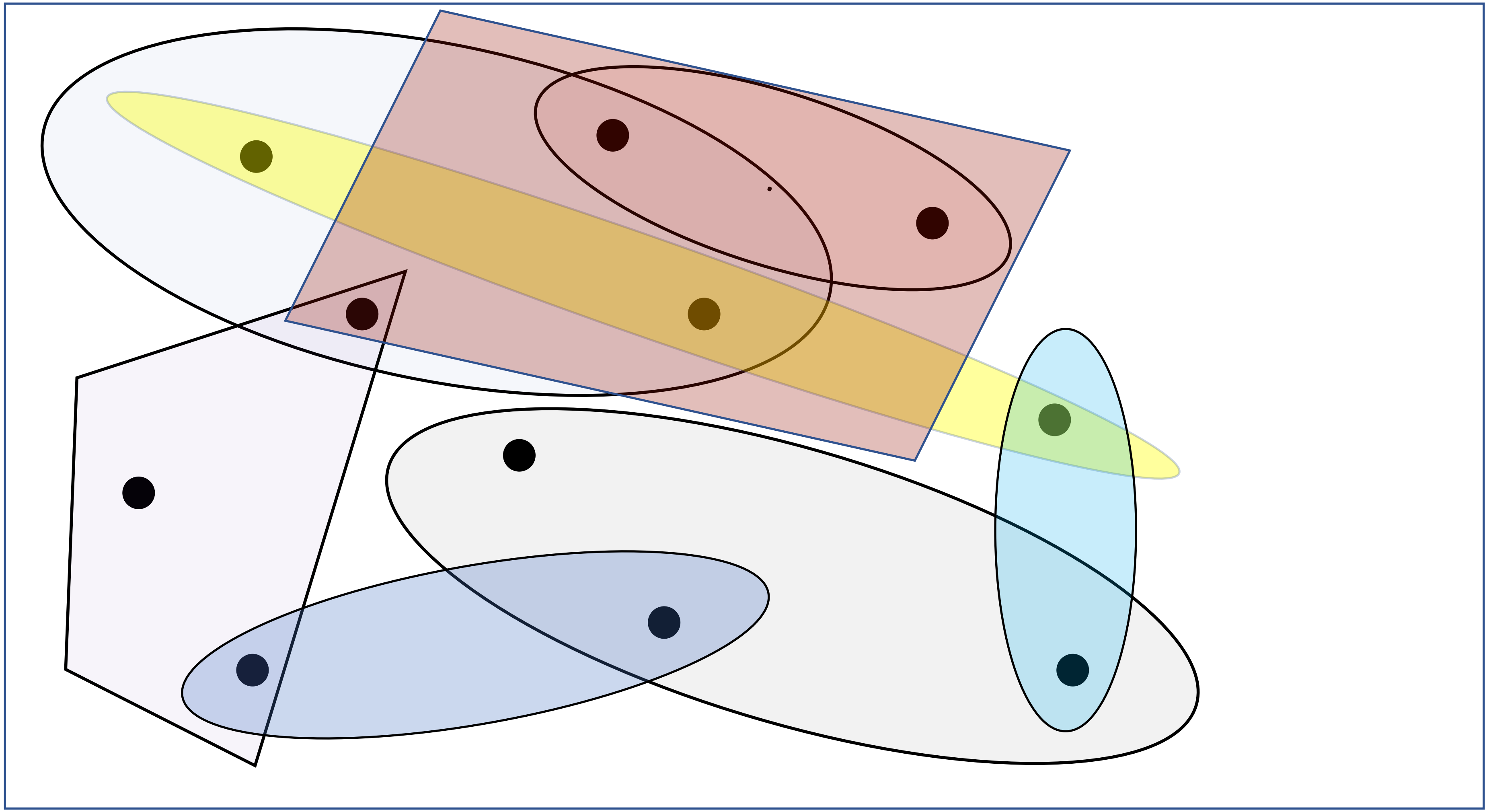
A set $\mathbb{S}$ of subsets of U



An exact cover of $\mathbb{S}$



A set $\mathbb{S}$ of subsets of U



A set $\mathbb{S}$‘ of subsets of U with no exact cover