Restricted forms of TQBF that are **PSPACE**-complete

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As discussed in lecture, a quantified Boolean formula (QBF) is a propositional formula quantified over its variables. In what follows we will assume that the QBFs under consideration are in prenex normal form (PNF) i.e., they are of the form $Q_1x_1Q_2x_2 \dots Q_nx_n\,\psi$, where each Q_i is a quantifier ($\exists$ or $\forall$) and ψ is a propositional formula (without quantifiers). We can make this assumption without loss of generality since, as you know from CSCB36, for any first-order formula ϕ there is a logically equivalent formula ϕ' in PNF. In fact, if you think back on the proof of this fact, it is easy to see that there is a polytime algorithm that, given a first-order formula, outputs a logically equivalent PNF formula.

In general the truth of a QBF depends on the truth assignment to its free variables, i.e., the variables that are not quantified. For example, consider $\phi = \exists x\,(x \wedge \overline{y} \wedge z)$ whose free variables are y and z. Whether ϕ is true or false depends on the truth values of y and z: It is true under the truth assignment τ to y and z where $\tau(y) = 0$ and $\tau(z) = 1$; in this case, there does exist a truth value for x, namely $x = 1$, that satisfies $x \wedge \overline{y} \wedge z$. On the other hand, ϕ is false under a truth assignment τ' to y and z where $\tau'(y) = 1$, regardless of what $\tau'(z)$ is; in that case, there is no truth value for x that satisfies $x \wedge \overline{y} \wedge z$. If the QBF is fully quantified, i.e., it has no free variables, then it is either true or false, without reference to any truth assignment.

In lecture we proved that the decision problem

$$\text{TQBF} = \{\langle\phi\rangle\colon \phi \text{ is a true QBF}\}$$

is **PSPACE**-complete. In this document we strengthen this result by proving that TQBF remains **PSPACE**-complete even for QBFs of syntactically restricted forms. We say that a QBF is in conjunctive normal form (CNF) if its quantifier-free part is in that form; it is in 3-CNF if it is in CNF and every clause has at most three literals. Similar terminology applies to a QBF being in disjunctive normal form (DNF).

Theorem 12.5 *The decision problems*

$$\text{3CNF-TQBF} = \{\langle\phi\rangle\colon \phi \textit{ is a true QBF whose quantifier-free part is in 3-CNF}\}, \textit{ and}$$
$$\text{3DNF-TQBF} = \{\langle\phi\rangle\colon \phi \textit{ is a true QBF whose quantifier-free part is in 3-DNF}\}$$

are ***PSPACE****-complete.*

Proof. 3CNF-TQBF and 3DNF-TQBF are in **PSPACE** since they are special cases of TQBF, which, as we have proved, is in **PSPACE**. It remains to prove that TQBF polytime mapping-reduces to each of them.

We first prove that 3CNF-TQBF $\leq_m^p$ TQBF. In the document "3-CNF satisfiability" we proved the following result:

> *For any propositional formula ψ with variables $x_1, \dots, x_n$ there is a 3-CNF propositional formula $\widehat{\psi}$ with variables $x_1, \dots, x_n, z_1, \dots, z_m$ such that a truth assignment τ satisfies ψ if and only if there is a truth assignment τ' that extends τ and satisfies $\widehat{\psi}$.*[1] *Furthermore, there is a polytime algorithm that, given ψ, outputs $\widehat{\psi}$.*

[1] Recall that τ' extends τ if the two assignments agree on the common variables $x_1, \dots, x_n$ of ψ and $\widehat{\psi}$

Therefore, the propositional formula ψ is logically equivalent to the 3-CNF QBF $\exists z_1 \ldots \exists z_m \, \widehat{\psi}$ — i.e., both formulas are satisfied by the same truth assignments. Note that the free variables of $\exists z_1 \ldots \exists z_m \widehat{\psi}$ are $x_1, \ldots, x_n$ — i.e., exactly the same as the variables of the (non-quantified) propositional formula ψ.

Given any QBF $\phi = Q_1 x_1 \ldots Q_n x_n \, \psi$, where ψ is quantifier-free, we can replace the quantifier-free part ψ by the logically equivalent 3-CNF QBF $\exists z_1 \ldots \exists z_m \, \widehat{\psi}$ to obtain the QBF $\widehat{\phi} = Q_1 x_1 \ldots Q_n x_n \exists z_1 \ldots \exists z_m \, \widehat{\psi}$, which is a 3-CNF QBF that is logically equivalent to ϕ. Therefore, $\phi \in$ TQBF if and only if $\widehat{\phi} \in$ 3CNF-TQBF. Since $\widehat{\phi}$ can be constructed from ϕ in polytime, we have that TQBF $\leq_m^p$ 3CNF-TQBF.

The fact that 3DNF-TQBF is **PSPACE**-complete follows easily from the fact that 3CNF-TQBF is **PSPACE**-complete, keeping in mind that **PSPACE** is closed under complementation. (Work out the details!) Contrast this with the analogous situation regarding the satisfiability problem: As we have seen 3-CNF satisfiability is **NP**-complete, and therefore likely not in **P**, while 3-DNF satisfiability is in **P**. □