Tracking and Grouping

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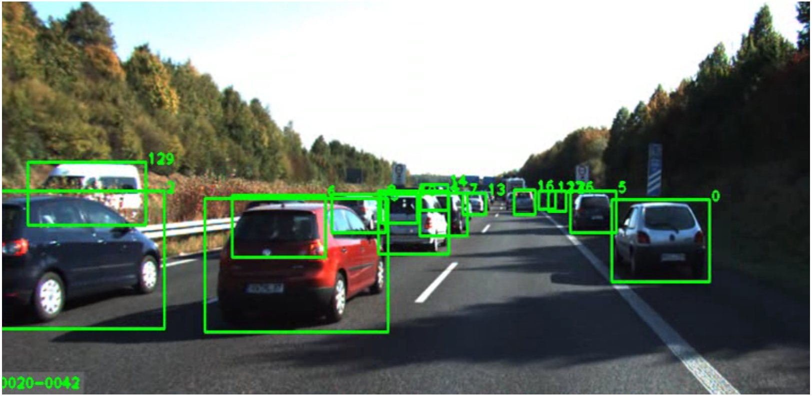
A different view on tracking

Tracking as a graph minimization

- **Goal**: Given a set of detections in video, link the detections into tracks
- Discover which detections are of the same object, and how many objects there are



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Notation and Problem Definition

- Let $\mathcal{X} = \{\mathbf{x}_i\}$ be a set of object observations
- Each $\mathbf{x}_i$ is detection response $\mathbf{x}_i = (x_i, s_i, a_i, t_i)$, where x_i is the position, s_i is the scale, a_i is the appearance and t_i is the time step (frame index)

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- We want to solve the following optimization

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- $P(\mathbf{x}_i|\mathcal{T})$ is the **likelihood** of observation $\mathbf{x}_i$. We can use a Bernoulli distribution for example to represent being an inlier or outlier

$$P(\mathbf{x}_i|\mathcal{T}) = \begin{cases} 1 - \beta_i & \text{if } \exists \mathcal{T}_k \in \mathcal{T}, \mathbf{x}_i \in \mathbf{T}_k \\ \beta_i & \text{otherwise.} \end{cases}$$

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Useful definitions

- To couple the non-overlap constraints with the objective function we define 0-1 indicator variables

$$f_{en,i} = \begin{cases} 1 & \text{if } \exists \mathcal{T}_k \in \mathcal{T}, \mathcal{T}_k \text{ starts from } \mathbf{x}_i \\ 0 & \text{otherwise.} \end{cases}$$

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Min-cost flow problem

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- This can be obtained as

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- Which can be reformulated as

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- What are the relationships between the costs and the probabilities we had before?

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Mapping to Min cost-flow network

- This can be mapped into a cost-flow network $G(\mathcal{X})$ with source s and sink t

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- The constraint is equivalent to the flow conservation constraint
- The objective is the cost of the flow in G.

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$$\begin{aligned} \min_{\mathcal{T}} \quad & \sum_i C_{en,i} f_{en,i} + \sum_{i,j} C_{i,j} f_{i,j} + \sum_i C_{ex,i} f_{ex,i} + \sum_i C_i f_i \\ s.t. \quad & f_{en,i} + \sum_j f_{j,i} = f_i = f_{ex,i} + \sum_j f_{i,j} \qquad \forall i \end{aligned}$$

- For every observation $\mathbf{x}_i \in \mathcal{X}$ create two nodes u_i, v_i, and an arc with cost $c(u_i, v_j) = C_i$ and flow f_i.
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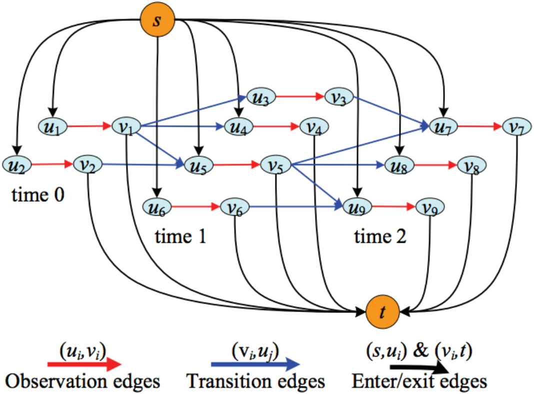

How is to optimize the objective

- For a given $f(G)$, the minimal cost can be solved for in polynomial time by a min-cost flow algorithm

  - Construct the graph $G(V, E, C, f)$ from observation set $\mathcal{X}$
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    - Augment $f(G)$ by one.
    - Find the min cost flow by the algorithm of [12].
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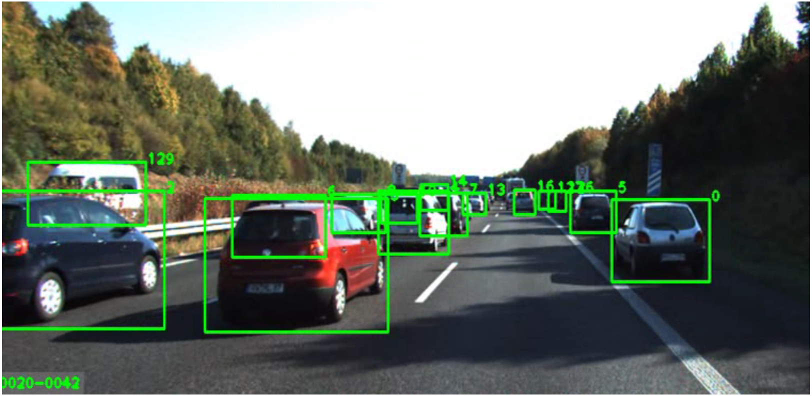
Tracking Results

[L. Zhang, Y. Li and R. Nevatia, CVPR08]



- What are the problems with this approach?

Grouping

Gestalt "Theory"

There exist a variety of factors in grouping

- **Proximity:** Tokens that are nearby tend to be grouped together

- **Similarity:** Similar tokens tend to be grouped together

Gestalt "Theory"

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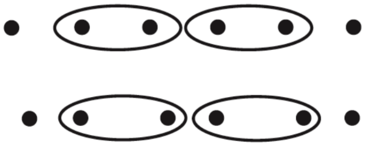
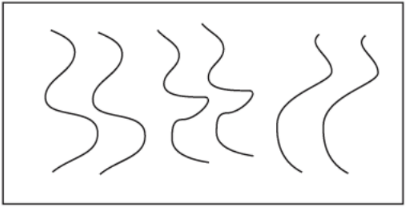
- **Common fate:** Tokens with coherent motion tend to be grouped together
- **Common region:** Tokens that lie inside the same closed region tend to be group together



- **Parallelism:** Parallel curves or tokens tend to be grouped together



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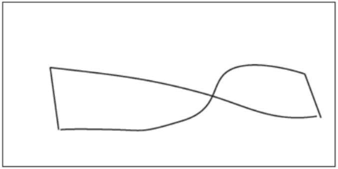
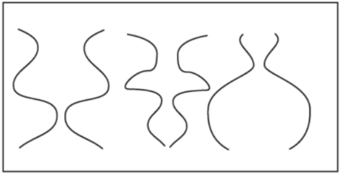
- **Closure:** Tokens or curves that tend to lead to closed curves tend to be close together



- **Symmetry:** Curves that lead to symmetric groups are typically grouped together



Gestalt "Theory"

There exist a variety of factors in grouping

- **Continuity:** Tokens than lead to continuous (with a relax notion of continuity) curves tend to be grouped
- **Familiar Configuration:** Tokens that, when grouped, lead to a familiar object tend to be grouped together

Effects of Grouping

- Grouping makes you see hallucinate contours

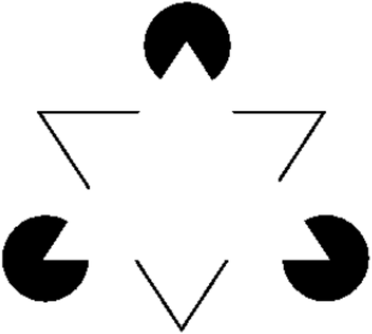


Figure: Kanizsa Triangle

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Motivation of clustering

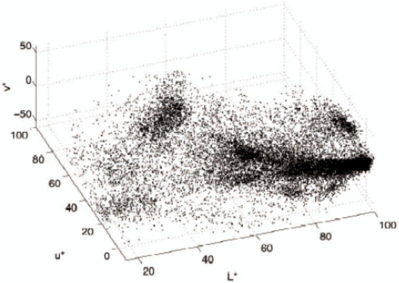

Figure: Illustration from Comanciu and Meer

Example of grouping techniques

- K-means style clustering, e.g., SLIC superpixels
- Normalized cuts
- Graph-based superpixels
- Mean-shift
- Watershed transform

Simple K-means

- Find three clusters in this data

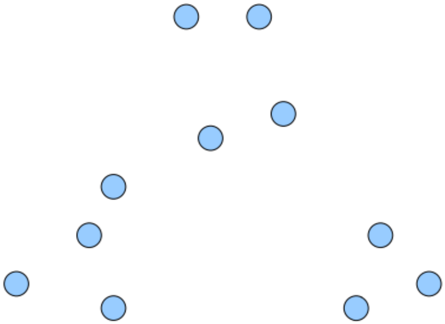

Figure: From M. Tappen

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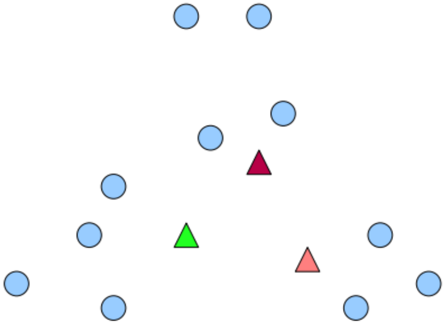

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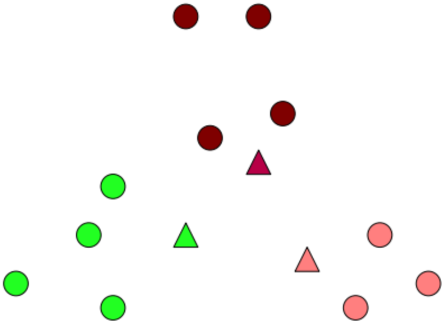

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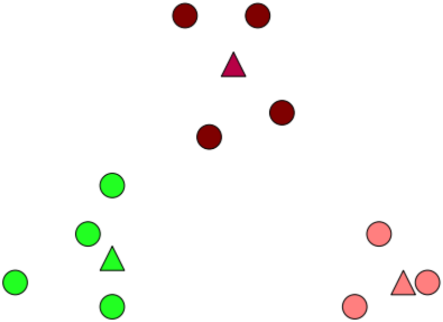

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Results

[R. Achanta and A. Shaji and K. Smith and A. Lucchi and P. Fua and S. Susstrunk, PAMI12]

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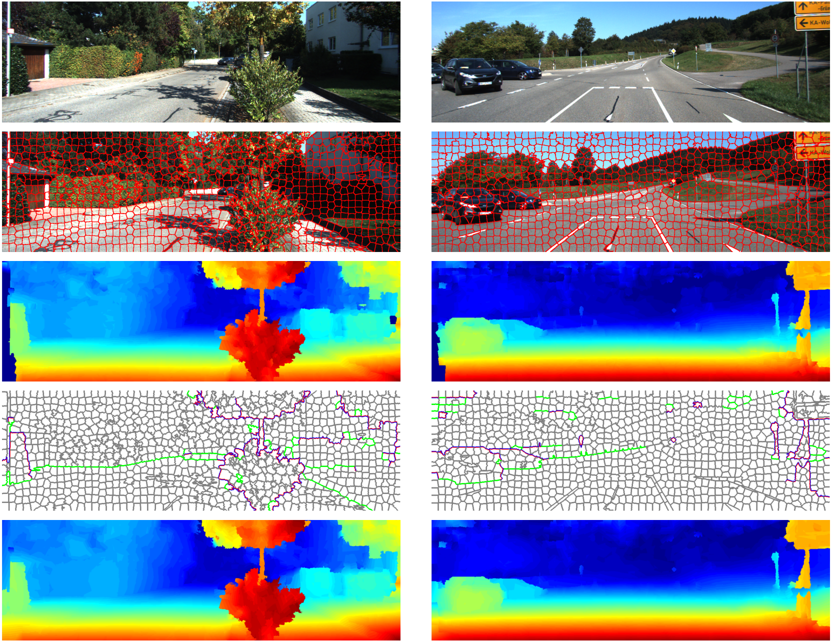
[K. Yamaguchi, D. McAllester and R. Urtasun, CVPR13]



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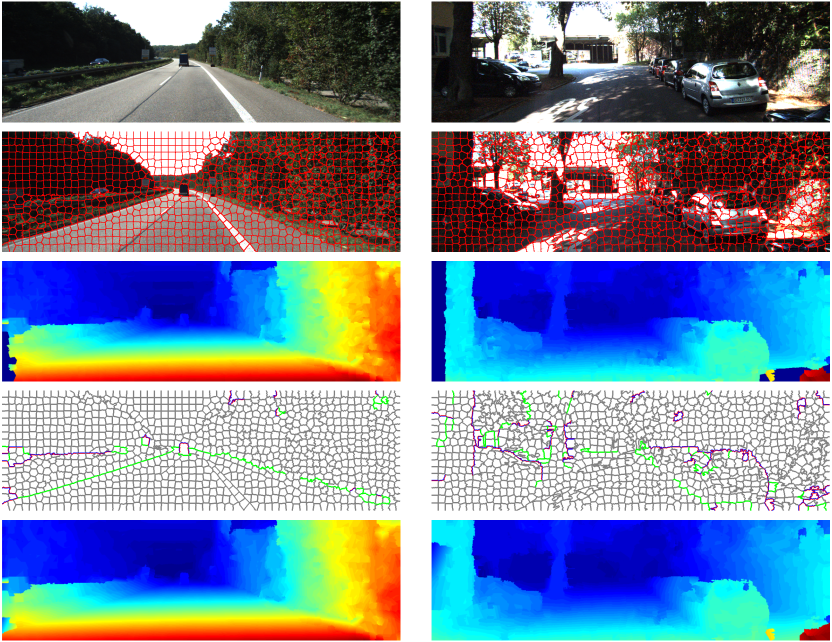

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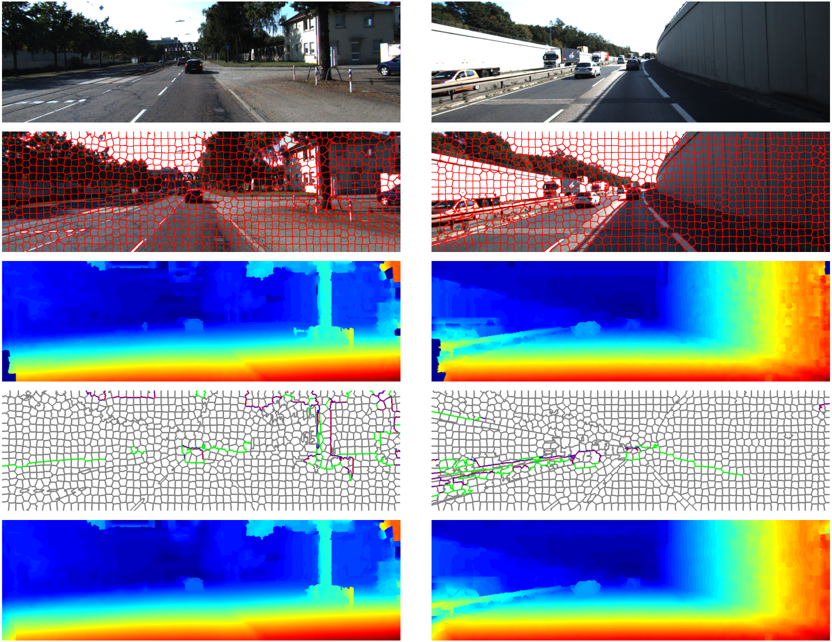

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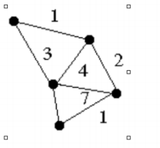
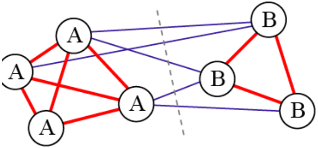


$$\begin{bmatrix} 0 & 1 & 3 & \infty & \infty \\ 1 & 0 & 4 & \infty & 2 \\ 3 & 4 & 0 & 6 & 7 \\ \infty & \infty & 6 & 0 & 1 \\ \infty & 2 & 7 & 1 & 0 \end{bmatrix}$$

Weight Matrix: W

- Examines the **affinities** (similarities) between nearby pixels and tries to separate groups that are connected with weak affinities.



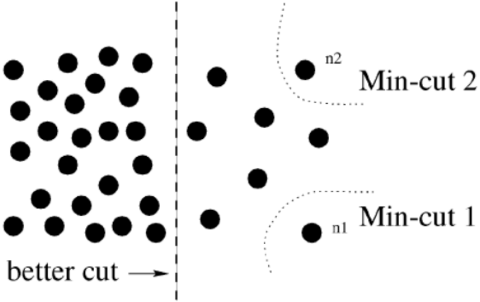
- The cut separate the nodes into two groups

Minimun Cuts

- The cut between two groups A and B is defined as the sum of all the weights being cut

$$cut(A, B) = \sum_{i \in A, j \in B} w_{i,j}$$

- Problem: Results in small cuts that isolates single pixels



- We need to normalize somehow

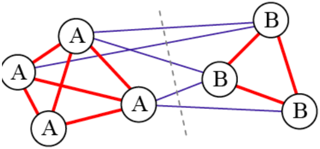
Normalized Cuts

[J. Shi and J. Malik, PAMI00]

- Better measure is the normalized cuts

$$N_{cut}(A,B) = \frac{cut(A,B)}{assoc(A,V)} + \frac{cut(A,B)}{assoc(B,V)}$$

with $assoc(A,A) = \sum_{i \in A, j \in A} w_{ij}$ is the association term within a cluster and $Assoc(A,V) = assoc(A,A) + cut(A,B)$ is the sum of all the weights associated with nodes in A.



	A	B	sum
A	$assoc(A,A)$	$cut(A,B)$	$assoc(A,V)$
B	$cut(B,A)$	$assoc(B,B)$	$assoc(B,V)$
sum	$assoc(A,V)$	$assoc(B,v)$	

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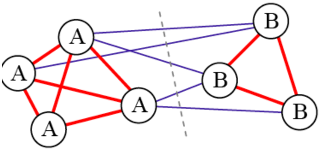

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Algorithm

[J. Shi and J. Malik, PAMI00]

1. Given an image or image sequence, set up a weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and set the weight on the edge connecting two nodes to be a measure of the similarity between the two nodes.
2. Solve $(\mathbf{D} - \mathbf{W})\boldsymbol{x} = \lambda \mathbf{D}\boldsymbol{x}$ for eigenvectors with the smallest eigenvalues.
3. Use the eigenvector with the second smallest eigenvalue to bipartition the graph.
4. Decide if the current partition should be subdivided and recursively repartition the segmented parts if necessary.

Examples

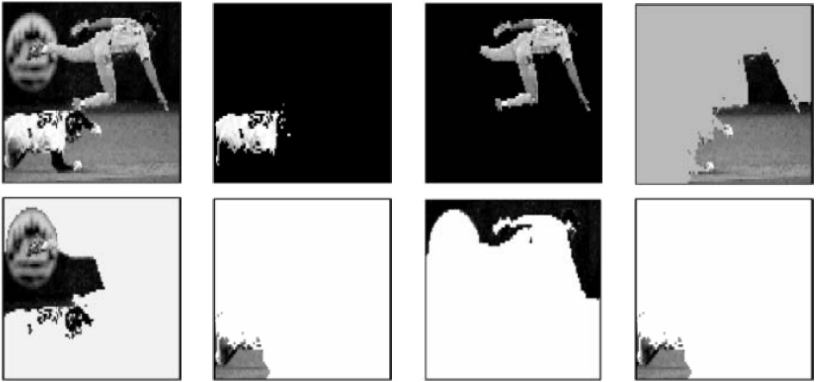

Figure: Shi and Malik N-Cuts

Example of grouping techniques

- K-means style clustering, e.g., SLIC superpixels
- Normalized cuts
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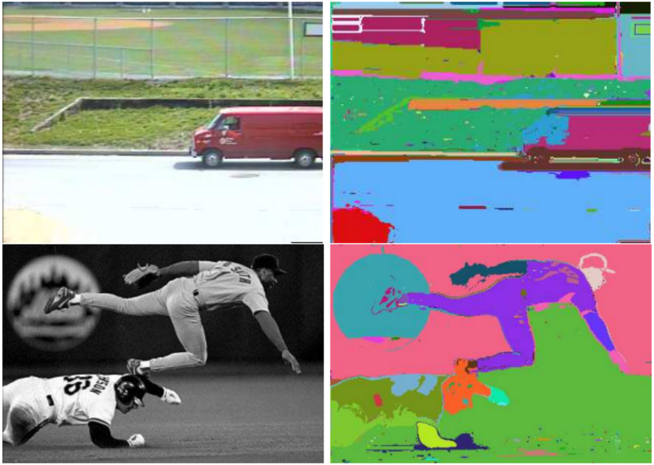
Results

[P. Felzenszwald and D. Huttenlocher, IJCV04]



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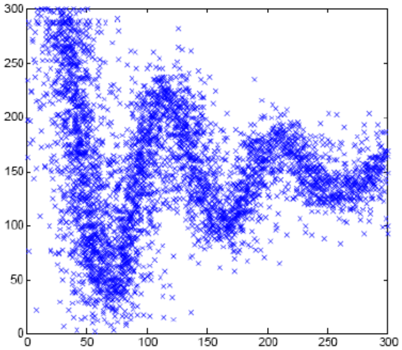
Basics of Kernel Density Estimation

- We have a bunch of points drawn from some distribution
- What's the distribution that generated these points?



[Source: M. Tappen]

Parametric vs Non-Parametric

- We can fit a parametric distribution, e.g., mixture of Gaussians
- KDE idea: Use the data to define the distribution

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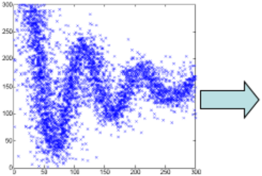

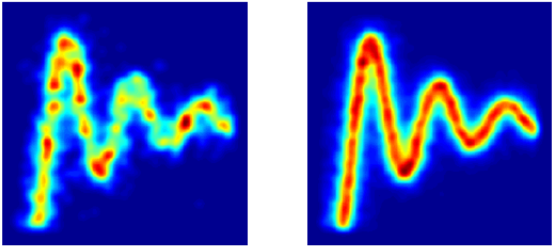

(a) 2000 Samples (b) 20000 Samples

Figure 2-2: Kernel density estimates of the density function shown in Figure 2-1(a). Figure (a) shows the estimate found with a relatively small number of samples. It is uneven and does not approximate the true density well. (b) With more samples, the estimate of the density improves significantly.

[Source: M. Tappen]

KDE

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$$\hat{f}(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} K_H(\mathbf{x} - \mathbf{x}_i)$$

with $\mathbf{x}_i$ the points, and $K_H(\mathbf{x} - \mathbf{x}_i)$ the kernel

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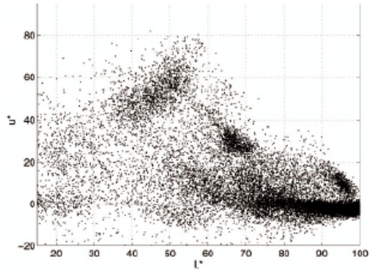
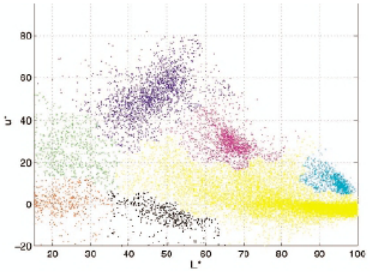


u*
80
60
40
20
0
−20
20
30
40
50
60
70
80
90
100
L*

(a)



u*
80
60
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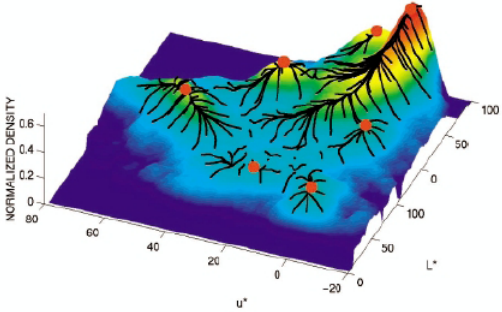
(b)

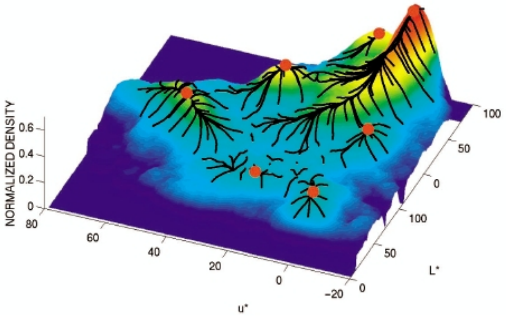


NORMALIZED DENSITY
0.6
0.4
0.2
0
80
60
40
20
0
−20
u*
100
50
0
100
50
0
L*

(c)

What is mean-shift

- The density will have peaks (also called modes)
- If we started at point and did gradient-ascent, we would end up at one of the modes
- Cluster based on which mode each point belongs to



[Source: M. Tappen]

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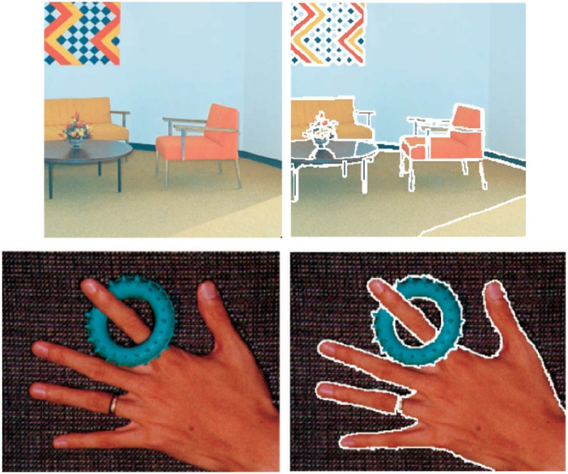

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