Energy, Plane-based Stereo and Tracking

Raquel Urtasun

TTI Chicago

March 5, 2013

More formally

- Any labeling can be uniquely represented by a partition of image pixels $\mathbf{P} = \{\mathcal{P}_l | l \in \mathcal{L}\}$, where $\mathcal{P}_l = \{p \in \mathcal{P} | f_p = l\}$ is a subset of pixels assigned label l.
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Example

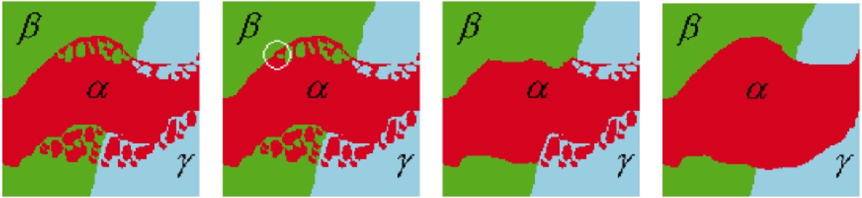

Figure: (a) Current partition (b) local move (c) $\alpha - \beta$-swap (d) α-expansion.

1. Start with an arbitrary labeling f
2. Set success := 0
3. For each pair of labels $\{\alpha, \beta\} \subset \mathcal{L}$
   3.1. Find $\hat{f} = \arg\min E(f')$ among f' within one α-β swap of f
   3.2. If $E(\hat{f}) < E(f)$, set f := $\hat{f}$ and success := 1
4. If success = 1 goto 2
5. Return f

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Finding optimal Swap move

- Given an input labeling f (partition $\mathcal{P}$) and a pair of labels α, β we want to find a labeling $\hat{f}$ that minimizes E over all labelings within one $\alpha - \beta$-swap of f.
- This is going to be done by computing a labeling corresponding to a minimum cut on a graph $\mathcal{G}_{\alpha\beta} = (\mathcal{V}_{\alpha\beta}, \mathcal{E}_{\alpha\beta})$.

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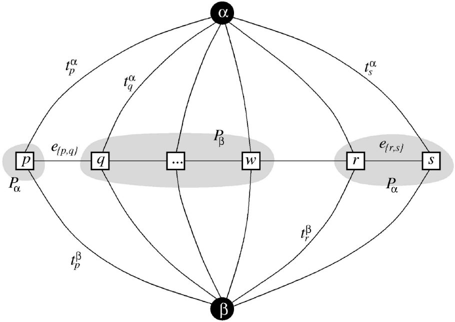
Graph Construction

- The set of vertices includes the two terminals α and β, as well as image pixels p in the sets $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ (i.e., $f_p \in \{\alpha, \beta\}$).
- Each pixel $p \in \mathcal{P}_{\alpha\beta}$ is connected to the terminals α and β, called t-links.
- Each set of pixels $p, q \in \mathcal{P}_{\alpha\beta}$ which are neighbors is connected by an edge $e_{p,q}$



edge	weight	for
t_p^α	$D_p(\alpha) + \sum_{q \in \mathcal{N}_p, q \notin \mathcal{P}_{\alpha\beta}} V(\alpha, f_q)$	$p \in \mathcal{P}_{\alpha\beta}$
t_p^β	$D_p(\beta) + \sum_{q \in \mathcal{N}_p, q \notin \mathcal{P}_{\alpha\beta}} V(\beta, f_q)$	$p \in \mathcal{P}_{\alpha\beta}$
$e_{\{p,q\}}$	$V(\alpha, \beta)$	$\{p,q\} \in \mathcal{N}$, $p,q \in \mathcal{P}_{\alpha\beta}$

Computing the Cut

- Any cut must have a single *t*-link not cut.
- This defines a labeling

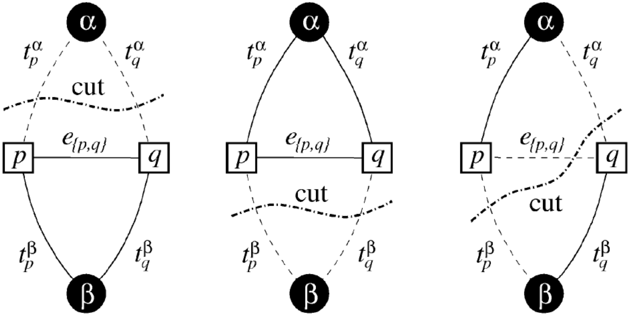
$$f_p^{\mathcal{C}} = \begin{cases} \alpha & \text{if } \; t_p^{\alpha} \in \mathcal{C} \;\; \text{for} \;\; p \in \mathcal{P}_{\alpha\beta} \\ \beta & \text{if } \; t_p^{\beta} \in \mathcal{C} \;\; \text{for} \;\; p \in \mathcal{P}_{\alpha\beta} \\ f_p & \text{for } \;\; p \in \mathcal{P}, \;\; p \notin \mathcal{P}_{\alpha\beta}. \end{cases}$$

- There is a one-to-one correspondences between a cut and a labeling.
- The energy of the cut is the energy of the labeling.
- See Boykov et al, "*fast approximate energy minimization via graph cuts*" PAMI 2001.

Properties

- For any cut, then

$$\begin{array}{lllll}
(a) & \textit{If} & t_p^\alpha, t_q^\alpha \in \mathcal{C} & \textit{then} & e_{\{p,q\}} \notin \mathcal{C}. \\
(b) & \textit{If} & t_p^\beta, t_q^\beta \in \mathcal{C} & \textit{then} & e_{\{p,q\}} \notin \mathcal{C}. \\
(c) & \textit{If} & t_p^\beta, t_q^\alpha \in \mathcal{C} & \textit{then} & e_{\{p,q\}} \in \mathcal{C}. \\
(d) & \textit{If} & t_p^\alpha, t_q^\beta \in \mathcal{C} & \textit{then} & e_{\{p,q\}} \in \mathcal{C}.
\end{array}$$



Finding the optimal α expansion

- Given an input labeling f (partition $\mathcal{P}$) and a label α we want to find a labeling $\hat{f}$ that minimizes E over all labelings within one α-expansion of f.
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Graph Construction

- The set of vertices includes the two terminals α and $\bar{\alpha}$, as well as all image pixels $p \in \mathcal{P}$.
- Additionally, for each pair of neighboring pixels p, q such that $f_p \neq f_q$ we create an auxiliary node $a_{p,q}$.

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- The set of edges is then

$$\mathcal{E}_\alpha = \left\{ \bigcup_{p \in \mathcal{P}} \{t_p^\alpha, t_p^{\bar{\alpha}}\}, \bigcup_{\substack{\{p,q\} \in \mathcal{N} \\ f_p \neq f_q}} \mathcal{E}_{\{p,q\}} \ , \bigcup_{\substack{\{p,q\} \in \mathcal{N} \\ f_p = f_q}} e_{\{p,q\}} \right\}$$

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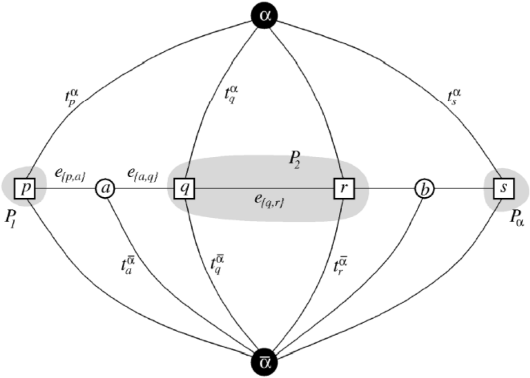

edge	weight	for
$t_p^{\bar\alpha}$	∞	$p \in \mathcal{P}_\alpha$
$t_p^{\bar\alpha}$	$D_p(f_p)$	$p \notin \mathcal{P}_\alpha$
t_p^{α}	$D_p(\alpha)$	$p \in \mathcal{P}$
$e_{\{p,a\}}$	$V(f_p, \alpha)$	
$e_{\{a,q\}}$	$V(\alpha, f_q)$	$\{p,q\} \in \mathcal{N},\ f_p \neq f_q$
$t_a^{\bar\alpha}$	$V(f_p, f_q)$	
$e_{\{p,q\}}$	$V(f_p, \alpha)$	$\{p,q\} \in \mathcal{N},\ f_p = f_q$

Properties

- There is a one-to-one correspondences between a cut and a labeling.

$$f_p^{\mathcal{C}} = \begin{cases} \alpha & \text{if} \quad t_p^{\alpha} \in \mathcal{C} \\ f_p & \text{if} \quad t_p^{\bar{\alpha}} \in \mathcal{C} \end{cases} \qquad \forall p \in \mathcal{P}.$$

- The energy of the cut is the energy of the labeling.
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Property 5.2. *If* $\{p,q\} \in \mathcal{N}$ *and* $f_p \neq f_q$, *then a minimum cut* $\mathcal{C}$ *on* $\mathcal{G}_\alpha$ *satisfies:*

$$\begin{array}{llll} (a) & \textit{If} \quad t_p^{\alpha}, t_q^{\alpha} \in \mathcal{C} & \textit{then} & \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = \emptyset. \\ (b) & \textit{If} \quad t_p^{\bar{\alpha}}, t_q^{\bar{\alpha}} \in \mathcal{C} & \textit{then} & \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = t_a^{\bar{\alpha}}. \\ (c) & \textit{If} \quad t_p^{\bar{\alpha}}, t_q^{\alpha} \in \mathcal{C} & \textit{then} & \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = e_{\{p,a\}}. \\ (d) & \textit{If} \quad t_p^{\alpha}, t_q^{\bar{\alpha}} \in \mathcal{C} & \textit{then} & \mathcal{C} \cap \mathcal{E}_{\{p,q\}} = e_{\{a,q\}}. \end{array}$$

Global Minimization Techniques

Ways to get an approximate solution typically

- Dynamic programming approximations
- Sampling
- Simulated annealing
- Graph-cuts: imposes restrictions on the type of pairwise cost functions
- Message passing: iterative algorithms that pass messages between nodes in the graph.

Now we can solve for the MAP (approximately) in general energies. We can solve for other problems than stereo

Let's look at data/bechmarks

Benchmarks

Two benchmarks with very different characteristics

(Middlebury)

(KITTI)

Middlebury Dataset

Middlebury Stereo Evaluation – Version 2

- Laboratory
- Lambertian

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Benchmarks for Stereo and metrics

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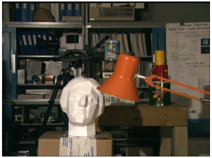


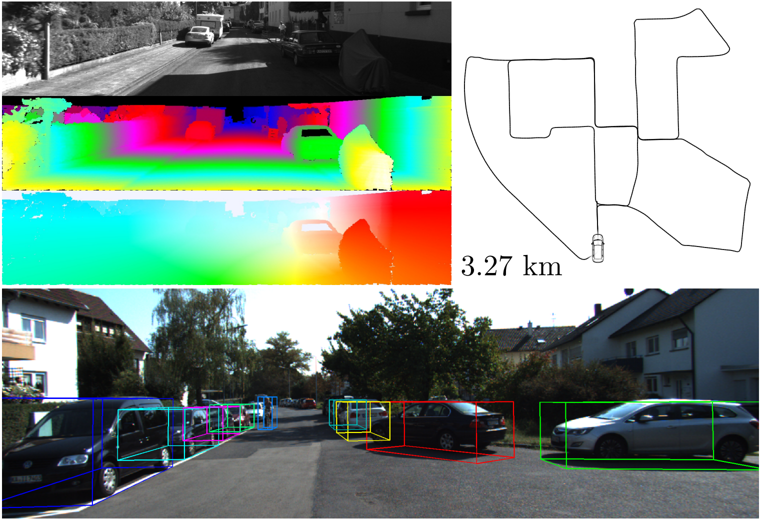
Error Threshold = 1 Algorithm	Avg.	Tsukuba ground truth			Venus ground truth			Teddy ground truth			Cones ground truth		
CoopRegion [41]	8.8	0.87 4	**1.16 1**	4.61 3	0.11 4	0.21 3	1.54 7	5.16 16	8.31 11	13.0 13	2.79 17	7.18 4	8.01 23
AdaptingBP [17]	9.0	1.11 19	1.37 7	5.79 19	0.10 3	0.21 4	1.44 5	4.22 8	7.06 6	11.8 9	2.48 7	7.92 11	7.32 10
ADCensus [94]	7.3	1.07 15	1.48 13	5.73 17	0.09 2	0.25 7	1.15 3	4.10 6	6.22 3	10.9 6	2.42 5	7.25 5	6.95 6
SurfaceStereo [79]	18.2	1.28 32	1.65 21	6.78 37	0.19 18	0.28 10	2.61 32	3.12 2	**5.10 1**	**8.65 1**	2.89 21	7.95 13	8.26 30
GC+SegmBorder [57]	27.1	1.47 45	1.82 32	7.86 58	0.19 19	0.31 12	2.44 26	4.25 9	5.55 2	10.9 7	4.99 77	**5.78 1**	8.66 37
WarpMat [55]	20.8	1.16 20	1.35 6	6.04 24	0.18 17	0.24 6	2.44 26	5.02 13	9.30 17	13.0 15	3.49 39	8.47 22	9.01 44
RDP [102]	12.5	0.97 10	1.39 9	5.00 9	0.21 23	0.38 19	1.89 13	4.84 10	9.94 19	12.6 11	2.53 8	7.69 8	7.38 11
RVbased [116]	11.6	0.95 9	1.42 11	4.98 8	0.11 6	0.29 11	**1.07 1**	5.98 21	11.6 31	15.4 27	2.35 3	7.61 6	6.81 5
OutlierConf [42]	12.9	0.88 5	1.43 12	4.74 5	0.18 16	0.26 9	2.40 22	5.01 12	9.12 16	12.8 12	2.78 16	8.57 23	6.99 7

- Best methods $<3\%$ errors (for all non-occluded regions)
- `http://vision.middlebury.edu/stereo/data/`

Benchmarks: KITTI Data Collection

- **Two stereo rigs** (1392 $\times$ 512 px, 54 cm base, 90° opening)
- **Velodyne** laser scanner, **GPS+IMU** localization
- **6 hours** at 10 frames per second!

The KITTI Vision Benchmark Suite



Novel Challenges

Fast guided cost-volume filtering (Rhemann et al., CVPR 2011)

Middlebury, Errors: 2.7%

- Error threshold: 1 px (Middlebury) / 3 px (KITTI)

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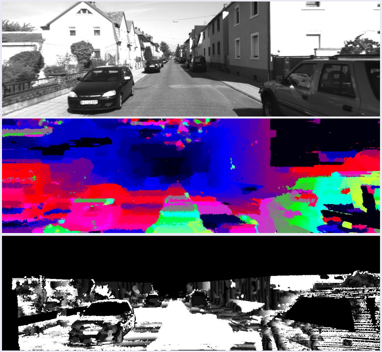
Middlebury, Errors: **2.7%**

KITTI, Errors: **46.3%**



- Error threshold: 1 px (Middlebury) / 3 px (KITTI)

So what is the difference?

Middlebury

KITTI

- Laboratory
- Lambertian

- Moving vehicle
- Specularities

Novel Challenges

So what is the difference?

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Novel Challenges

So what is the difference?

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- Laboratory
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- Moving vehicle
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- Sensor saturation
- Large label set

So what is the difference?

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- Laboratory
- Lambertian
- Rich in texture
- Medium-size label set
- Largely fronto-parallel

KITTI

- Moving vehicle
- Specularities
- Sensor saturation
- Large label set
- Strong slants

So what is the difference?

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Stereo Evaluation

Rank	Method	Setting	Out-Noc	Out-All	Avg-Noc	Avg-All	Density	Runtime	Environment	Compare
1	PCBP		**4.13 %**	**5.45 %**	**0.9 px**	**1.2 px**	100.00 %	5 min	4 cores @ 2.5 Ghz (Matlab + C/C++)	☐
Koichiro Yamaguchi, Tamir Hazan, David McAllester and Raquel Urtasun. Continuous Markov Random Fields for Robust Stereo Estimation. ECCV 2012.										
2	iSGM		5.16 %	7.19 %	1.2 px	2.1 px	94.70 %	8 s	2 cores @ 2.5 Ghz (C/C++)	☐
Simon Hermann and Reinhard Klette. Iterative Semi-Global Matching for Robust Driver Assistance Systems. ACCV 2012.										
3	SGM		5.83 %	7.08 %	1.2 px	1.3 px	85.80 %	3.7 s	1 core @ 3.0 Ghz (C/C++)	☐
Heiko Hirschmueller. Stereo Processing by Semi-Global Matching and Mutual Information. IEEE Transactions on Pattern Analysis and Machine Intelligence 2008.										
4	SNCC		6.27 %	7.33 %	1.4 px	1.5 px	100.00 %	0.27 s	1 core @ 3.0 Ghz (C/C++)	☐
N. Einecke and J. Eggert. A Two-Stage Correlation Method for Stereoscopic Depth Estimation. DICTA 2010.										
5	ITGV		6.31 %	7.40 %	1.3 px	1.5 px	100.00 %	7 s	1 core @ 3.0 Ghz (Matlab + C/C++)	☐
Rene Ranftl, Stefan Gehrig, Thomas Pock and Horst Bischof. Pushing the Limits of Stereo Using Variational Stereo Estimation. IEEE Intelligent Vehicles Symposium 2012.										
6	BSSM		7.50 %	8.89 %	1.4 px	1.6 px	94.87 %	20.7 s	1 core @ 3.5 Ghz (C/C++)	☐
Anonymous submission										
7	OCV-SGBM		7.64 %	9.13 %	1.8 px	2.0 px	86.50 %	1.1 s	1 core @ 2.5 Ghz (C/C++)	☐
Heiko Hirschmueller. Stereo processing by semiglobal matching and mutual information. PAMI 2008.										
8	ELAS		8.24 %	9.95 %	1.4 px	1.6 px	94.55 %	0.3 s	1 core @ 2.5 Ghz (C/C++)	☐
Andreas Geiger, Martin Roser and Raquel Urtasun. Efficient Large-Scale Stereo Matching. ACCV 2010.										
9	MS-DSI		10.68 %	12.11 %	1.9 px	2.2 px	100.00 %	10.3 s	>8 cores @ 2.5 Ghz (C/C++)	☐
Anonymous submission										
10	SDM		10.98 %	12.19 %	2.0 px	2.3 px	63.58 %	1 min	1 core @ 2.5 Ghz (C/C++)	☐
Jana Kostkova. Stratified dense matching for stereopsis in complex scenes. BMVC 2003.										
11	GCSF		12.06 %	13.26 %	1.9 px	2.1 px	60.77 %	2.4 s	1 core @ 2.5 Ghz (C/C++)	☐
Jan Cech, Jordi Sanchez-Riera and Radu P. Horaud. Scene Flow Estimation by Growing Correspondence Seeds. CVPR 2011.										
12	GCS		13.37 %	14.54 %	2.1 px	2.3 px	51.06 %	2.2 s	1 core @ 2.5 Ghz (C/C++)	☐
Jan Cech and Radim Sara. Efficient Sampling of Disparity Space for Fast And Accurate Matching. BenCOS 2007.										
13	CostFilter		19.96 %	21.05 %	5.0 px	5.4 px	100.00 %	4 min	1 core @ 2.5 Ghz (Matlab)	☐
Christoph Rhemann, Asmaa Hosni, Michael Bleyer, Carsten Rother and Margrit Gelautz. Fast Cost-Volume Filtering for Visual Correspondence and Beyond. CVPR 2011.										
14	OCV-BM		25.39 %	26.72 %	7.6 px	7.9 px	55.84 %	0.1 s	1 core @ 2.5 Ghz (C/C++)	☐
G. Bradski. The OpenCV Library. Dr. Dobb's Journal of Software Tools 2000.										
15	GC+occ		33.50 %	34.74 %	8.6 px	9.2 px	87.57 %	6 min	1 core @ 2.5 Ghz (C/C++)	☐
Vladimir Kolmogorov and Ramin Zabih. Computing Visual Correspondence with Occlusions using Graph Cuts. ICCV 2001.										

MRFs for stereo

Global methods: define a Markov random field over

- Pixel-level
- Fronto-parallel planes
- Slanted planes

Plane MRFs

- First segment an image into small regions, i.e., superpixels
- Assume that the 3D world is compose of small frontal/slanted planes

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- Good representation if the superpixels are small and respect boundaries

$$E(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \sum_i C(\mathbf{x}_i) + \sum_i \sum_{j \in \mathcal{N}_i} C(\mathbf{x}_i, \mathbf{x}_j)$$

with $\mathbf{x}_i \in \Re$ for the fronto-parallel planes, and $\mathbf{x}_i \in \Re^3$ for the slanted planes

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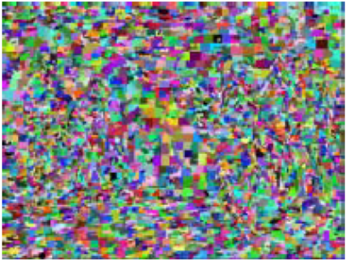
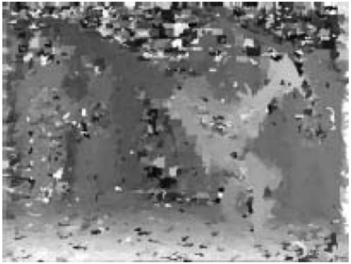
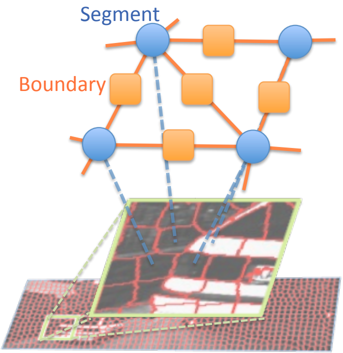
Slanted-plane MRFs





A more sophisticated occlusion model

- MRF on continuous variables (slanted planes) and discrete var. (boundary)
- Combines depth ordering (segmentation) and stereo



Superpixels (UCM [Arbelaez, et al. 2011] and SLIC [Achanta, et al. 2010])

Segment variable $\mathbf{y}_i = (\alpha_i, \beta_i, \gamma_i)$

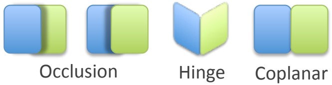
Slanted 3D plane of segment

Continuous variable

Boundary variable o_{ij}

Relationship between segments

4 states



Discrete variable

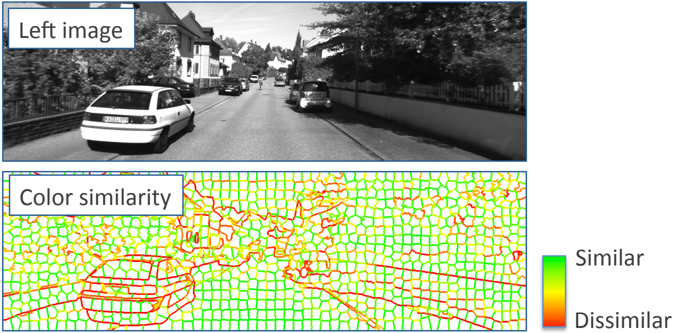
- Takes as input disparities computed by any local algorithm

Energy of PCBP-Stereo

- **y** the set of slanted 3D planes, **o** the set of discrete boundary variables

$$E(\mathbf{y}, \mathbf{o}) = E_{color}(\mathbf{o}) + E_{match}(\mathbf{y}, \mathbf{o}) + E_{compatibility}(\mathbf{y}, \mathbf{o}) + E_{junction}(\mathbf{o})$$

Similar color → Likely to be coplanar



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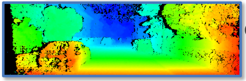
Agreement with result of input disparity map



Computed by any matching method
(Modified semi-global matching)

Truncated quadratic function $\phi_i^{\mathrm{TP}}(\mathbf{p},\mathbf{y}_i,K) = \min\left(|\mathcal{D}(\mathbf{p}) - \hat{d}_i(\mathbf{p},\mathbf{y}_i)|, K\right)^2$

Disparity map — Slanted plane

On boundary

"Occlusion" – Foreground segment owns boundary

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(1) Preference of boundary label (Coplanar > Hinge > Occlusion)

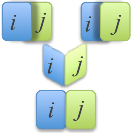
Impose penalty $\lambda_{\text{occ}} > \lambda_{\text{hinge}} > 0$

(2) Boundary labels ⟷ (match) Slanted planes

"Occlusion" ⟷ $\hat{d}_{\text{front}}(\mathbf{p}) > \hat{d}_{\text{back}}(\mathbf{p})$

"Hinge" ⟷ $\hat{d}_i(\mathbf{p}) = \hat{d}_j(\mathbf{p})$ on boundary

"Coplanar" ⟷ $\hat{d}_i(\mathbf{p}) = \hat{d}_j(\mathbf{p})$ in both segments



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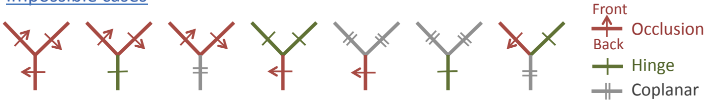
Occlusion boundary reasoning [Malik 1987]

Penalize impossible junctions

Impossible cases

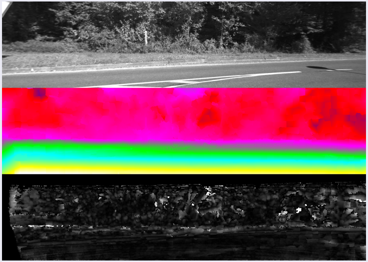


[K. Yamaguchi, T. Hazan, D. McAllester and R. Urtasun, ECCV12]

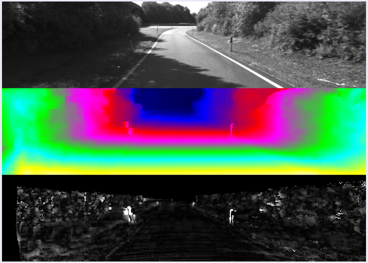
Easy Scenarios:

- Natural scenes, lots of texture, no objects
- A couple of errors at thin structures (poles)

Errors: $< 0.5\%$



Errors: $< 0.5\%$



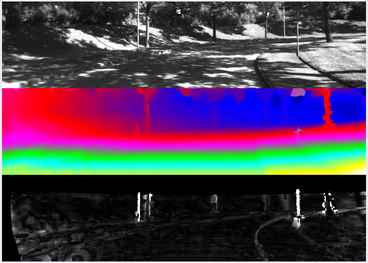
Stereo Evaluation

[K. Yamaguchi, T. Hazan, D. McAllester and R. Urtasun, ECCV12]

Easy Scenarios:

- Shadows help the disambiguation process
- Errors at thin structures and far away textureless regions

Errors: $< 0.5\%$



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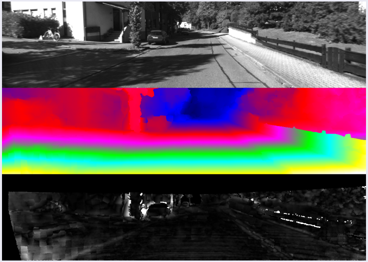


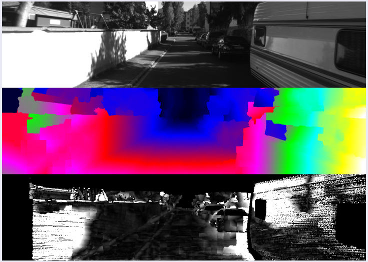
Stereo Evaluation

[K. Yamaguchi, T. Hazan, D. McAllester and R. Urtasun, ECCV12]

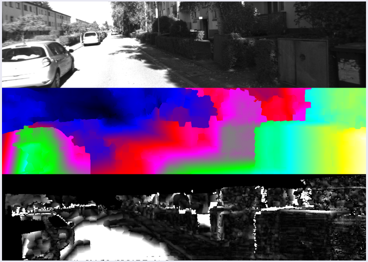
Hard Scenarios:

- Textureless or saturated areas
- Ambiguous reflections

Errors: 22.1%



Errors: 17.4%



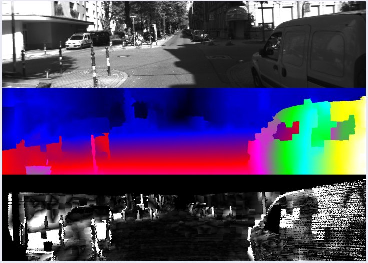
Stereo Evaluation

[K. Yamaguchi, T. Hazan, D. McAllester and R. Urtasun, ECCV12]

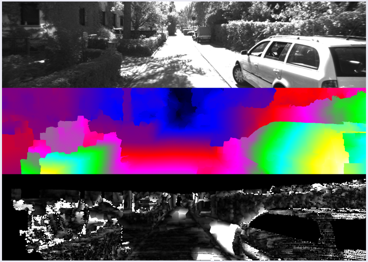
Hard Scenarios:

- Depth discontinuities / complicated geometries

Errors: 11.2%



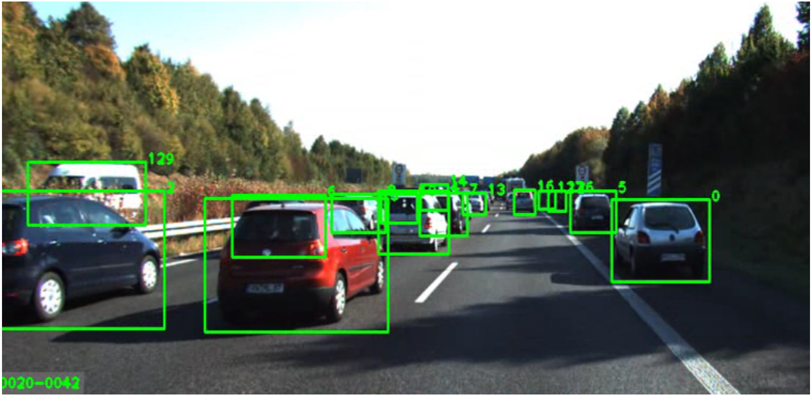
Errors: 10.5%



A different view on tracking

Tracking as a graph minimization

- **Goal**: Given a set of detections in video, link the detections into tracks
- Discover which detections are of the same object, and how many objects there are



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Notation and Problem Definition

- Let $\mathcal{X} = \{\mathbf{x}_i\}$ be a set of object observations
- Each $\mathbf{x}_i$ is detection response $\mathbf{x}_i = (x_i, s_i, a_i, t_i)$, where x_i is the position, s_i is the scale, a_i is the appearance and t_i is the time step (frame index)

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- $P(\mathcal{T}_k)$ can be modeled as a Markov chain, with initialization probability P_{ent}, termination probability P_{exit}, and transition probability $P_{link}(\mathbf{x}_{k_{i+1}}|\mathbf{x}_{k_i})$

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$$\mathcal{T}^* = \arg\max_{\mathcal{T}} \prod_i P(\mathbf{x}_i|\mathcal{T}) \prod_{\mathcal{T}_k \in \mathcal{T}} P(\mathcal{T}_k)$$
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Useful definitions

- To couple the non-overlap constraints with the objective function we define 0-1 indicator variables

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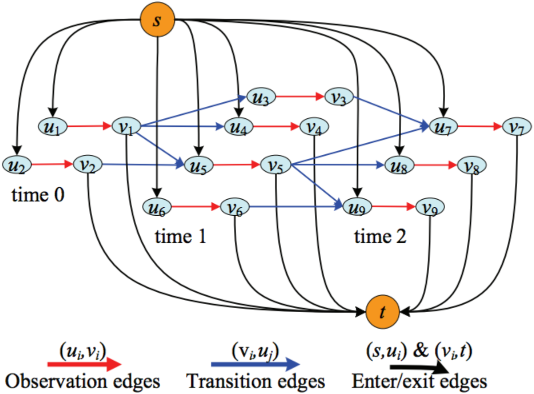

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- For a given $f(G)$, the minimal cost can be solved for in polynomial time by a min-cost flow algorithm

  - Construct the graph $G(V, E, C, f)$ from observation set $\mathcal{X}$
  - Start with empty flow
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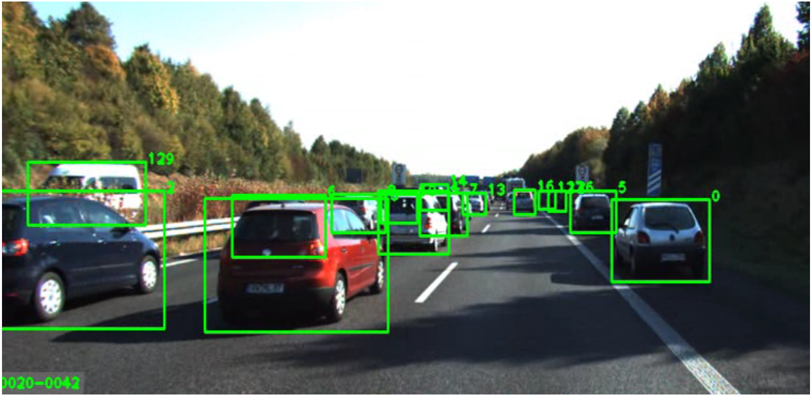
Tracking Results

[L. Zhang, Y. Li and R. Nevatia, CVPR08]



- What are the problems with this approach?

Grouping

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Techniques we will see

- K-means style clustering, e.g., SLIC superpixels
- Normalized cuts
- Graph-based superpixels
- Wathershed transform
- Mean-shift

Simple K-means

- Find three clusters in this data

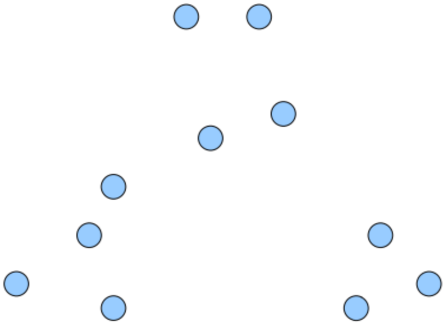

Figure: From M. Tappen

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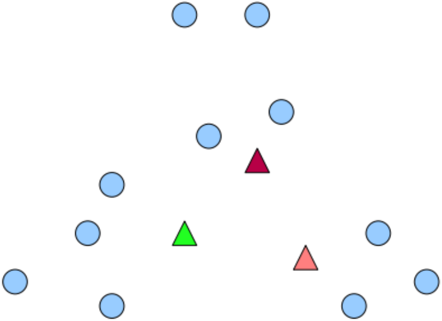

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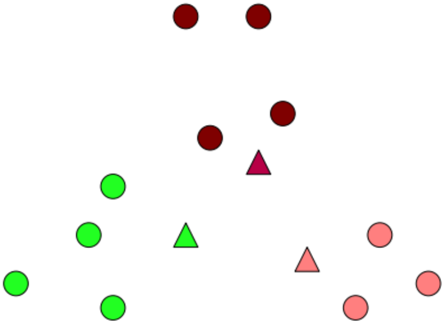

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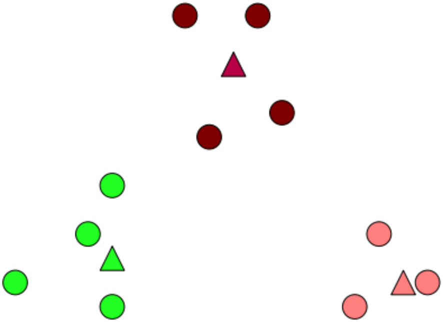

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K-means style algorithms

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K-means style algorithms

- We would like to encode
  - Super-pixels have regular shape
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Results

[R. Achanta and A. Shaji and K. Smith and A. Lucchi and P. Fua and S. Susstrunk, PAMI12]

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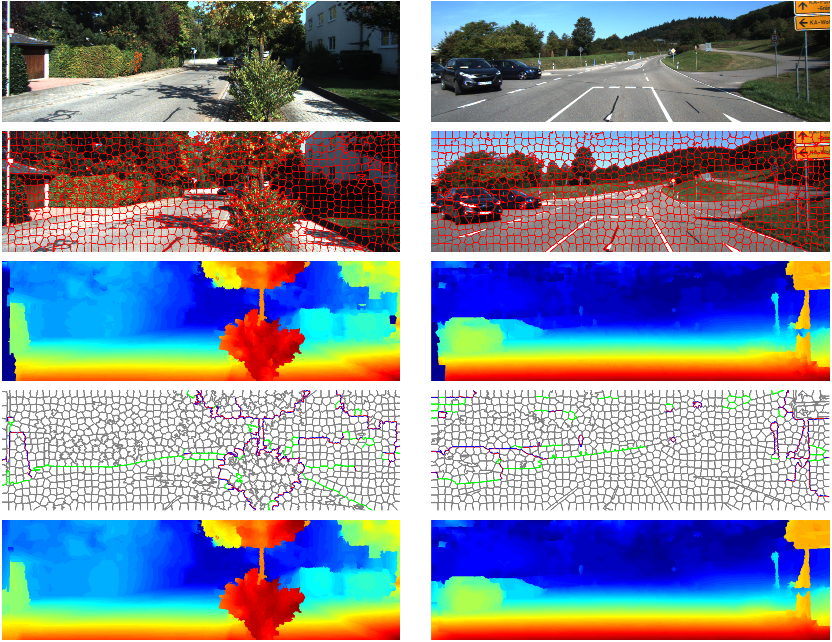

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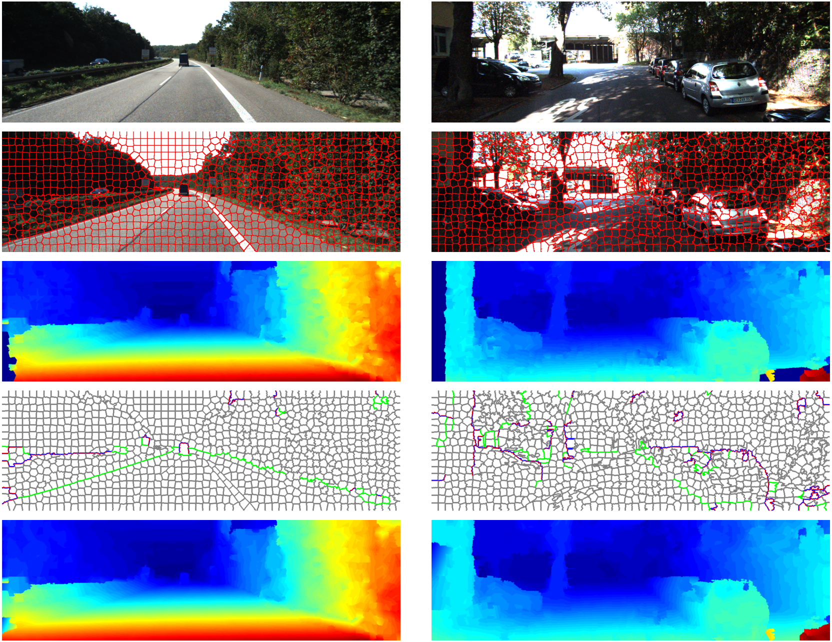

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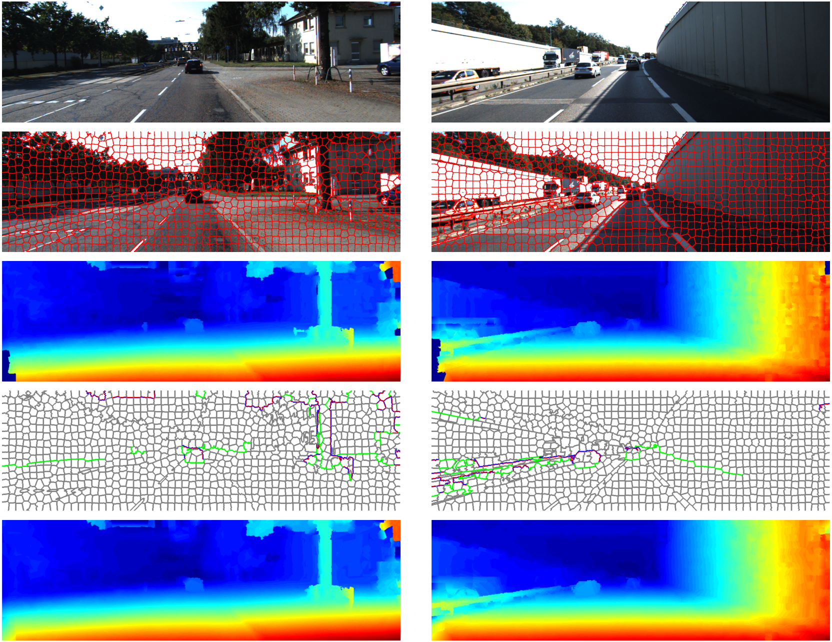
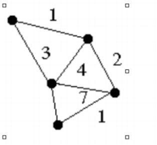


Techniques we will see

- K-means style clustering, e.g., SLIC superpixels
- Normalized cuts
- Graph-based superpixels
- Wathershed transform
- Mean-shift

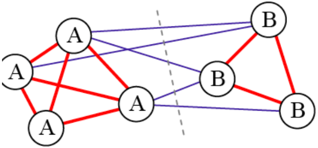
Segmentation as a mincut problem



$$\begin{bmatrix} 0 & 1 & 3 & \infty & \infty \\ 1 & 0 & 4 & \infty & 2 \\ 3 & 4 & 0 & 6 & 7 \\ \infty & \infty & 6 & 0 & 1 \\ \infty & 2 & 7 & 1 & 0 \end{bmatrix}$$

Weight Matrix: W

- Examines the **affinities** (similarities) between nearby pixels and tries to separate groups that are connected with weak affinities.



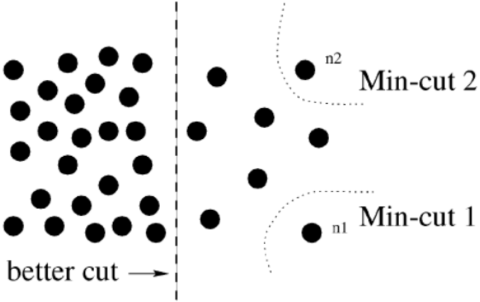
- The cut separate the nodes into two groups

Minimun Cuts

- The cut between two groups A and B is defined as the sum of all the weights being cut

$$cut(A, B) = \sum_{i \in A, j \in B} w_{i,j}$$

- Problem: Results in small cuts that isolates single pixels



- We need to normalize somehow

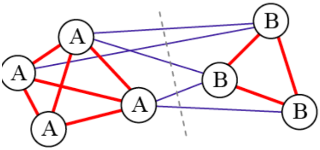
Normalized Cuts

[J. Shi and J. Malik, PAMI00]

- Better measure is the normalized cuts

$$N_{cut}(A,B) = \frac{cut(A,B)}{assoc(A,V)} + \frac{cut(A,B)}{assoc(B,V)}$$

with $assoc(A,A) = \sum_{i \in A, j \in A} w_{ij}$ is the association term within a cluster and $Assoc(A,V) = assoc(A,A) + cut(A,B)$ is the sum of all the weights associated with nodes in A.



	A	B	sum
A	$assoc(A,A)$	$cut(A,B)$	$assoc(A,V)$
B	$cut(B,A)$	$assoc(B,B)$	$assoc(B,V)$
sum	$assoc(A,V)$	$assoc(B,v)$	

- We want minimize the disassociation between the groups and maximize the association within the groups

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- Minimizing this **Rayleigh quotient** is equivalent to solving the generalized eigenvalue system

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Algorithm

[J. Shi and J. Malik, PAMI00]

1. Given an image or image sequence, set up a weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and set the weight on the edge connecting two nodes to be a measure of the similarity between the two nodes.
2. Solve $(\mathbf{D} - \mathbf{W})\boldsymbol{x} = \lambda \mathbf{D}\boldsymbol{x}$ for eigenvectors with the smallest eigenvalues.
3. Use the eigenvector with the second smallest eigenvalue to bipartition the graph.
4. Decide if the current partition should be subdivided and recursively repartition the segmented parts if necessary.

Examples

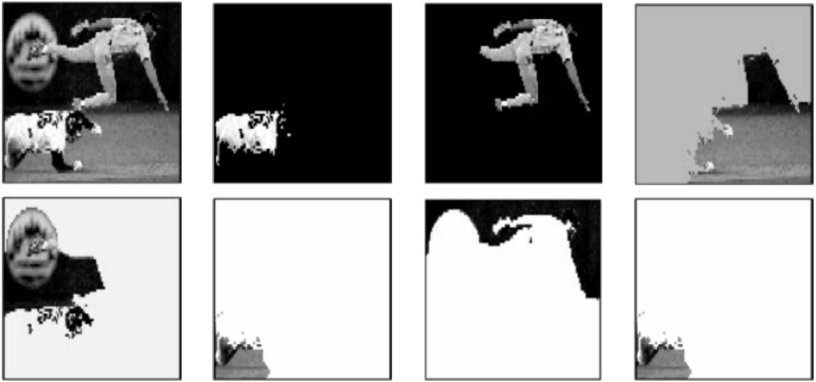

Figure: Shi and Malik N-Cuts