CSC 411: Lecture 14: Principal Components Analysis & Autoencoders

Richard Zemel, Raquel Urtasun and Sanja Fidler

University of Toronto

Today

- Dimensionality Reduction
- PCA
- Autoencoders

Mixture models and Distributed Representations

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- Can be several binary/discrete variables, or continuous

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- How can we find these dimensions from the data?

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- Data is assumed to be continuous:
  - linear relationship between data and the learned representation

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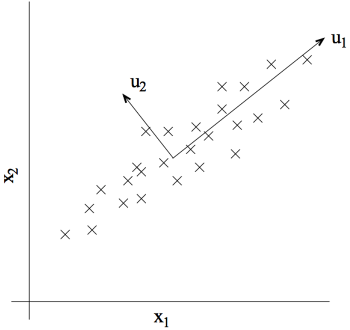
- Structure of data vectors is encoded in sample covariance



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- Assemble these eigenvectors into a $D \times M$ matrix U
- We can now express D-dimensional vectors $\mathbf{x}$ by projecting them to M-dimensional $\mathbf{z}$

$$\mathbf{z} = U^T \mathbf{x}$$

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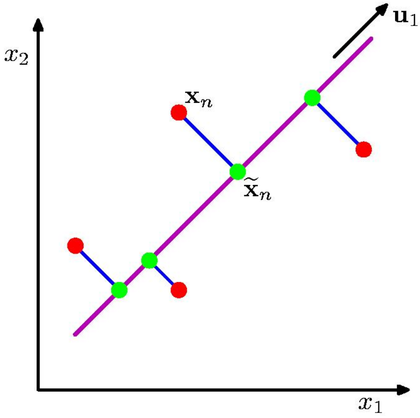
  2. Project each input vector $\mathbf{x}$ into this subspace, e.g.,

$$z_j = \mathbf{u}_j^T\mathbf{x}; \qquad \mathbf{z} = U_{1:M}^T\mathbf{x}$$

Two Derivations of PCA

- Two views/derivations:
  - Maximize variance (scatter of green points)
  - Minimize error (red-green distance per datapoint)



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where

$$\tilde{\mathbf{x}}^{(n)} = \sum_{j=1}^{M} z_j^{(n)} \mathbf{u}_j + \sum_{j=M+1}^{D} b_j \mathbf{u}_j$$

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- Objective minimized when first M components are the eigenvectors with the maximal eigenvalues

$$z_j^{(n)} = \mathbf{u}_j^T \mathbf{x}^{(n)}; \quad b_j = \bar{\mathbf{x}}^T \mathbf{u}_j$$

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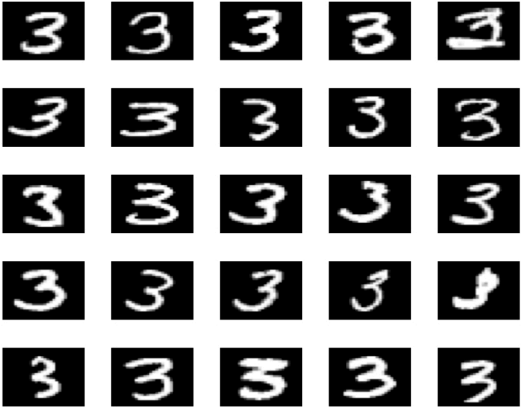
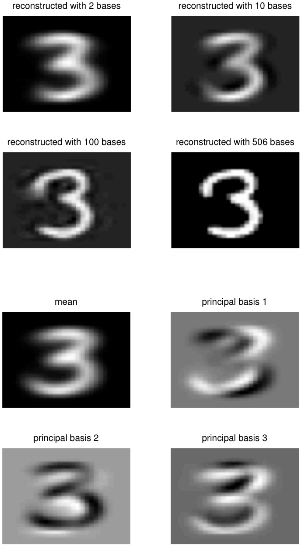
  - PCA with 3 components obtains 79% accuracy on face/non-face discrimination on test data vs. 76.8% for GMM with 84 states
- Can also be good for visualization

Applying PCA to faces: Learned basis

Applying PCA to digits





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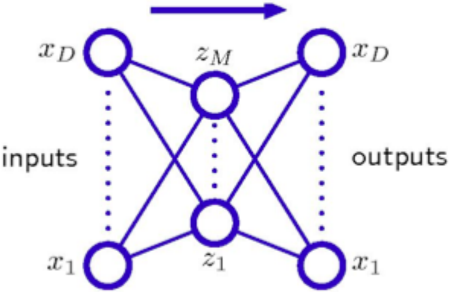

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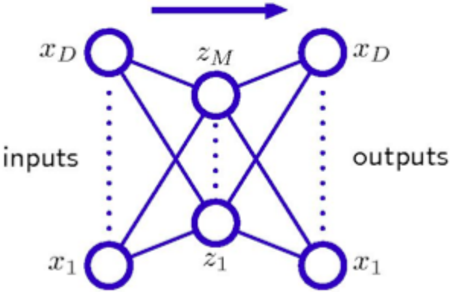

- The goal is to minimize reconstruction error

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- In other words, the optimal solution is PCA.

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- Some subtleties but in general this is an accurate description

Comparing Reconstructions