Inference (Proof) Procedures for FOL

(Chapter 9)

Proof Procedures

- Interestingly, proof procedures work by simply manipulating formulas. They do not know or care anything about interpretations.
- Nevertheless they respect the semantics of interpretations!
- We will develop a proof procedure for first-order logic called resolution.
  - Resolution is the mechanism used by PROLOG
  - Also used by automated theorem provers such as Prover9 (together with other techniques)

Properties of Proof Procedures

- Before presenting the details of resolution, we want to look at properties we would like to have in a (any) proof procedure.

- We write KB $\vdash$ f to indicate that f can be proved from KB (the proof procedure used is implicit).

Properties of Proof Procedures

- Soundness
  - $KB \vdash f \rightarrow KB \models f$

    i.e all conclusions arrived at via the proof procedure are correct: they are logical consequences.

- Completeness
  - $KB \models f \rightarrow KB \vdash f$

  i.e. every logical consequence can be generated by the proof procedure.

- Note proof procedures are computable, but they might have very high complexity in the worst case. So completeness is not necessarily achievable in practice.

Forward Chaining

- First-order inference mechanism (or proof procedure) that generalizes Modus Ponens:
  - From C1 ∧ … ∧ Cn → Cn+1 with C1, …, Cn in KB, we can infer Cn+1
  - We can chain this until we can infer some Ck such that each Ci is either in KB or is the result of Modus Ponens involving only prior Cm (m<i)
    - We then have that KB ⊢ Ck.
  - **Sound** (since it is only Generalized Modus Ponens) so we also have KB ⊨ Ck
  - **Complete**: only for rules that are in the above form
    → Need to study clausal form to make this more formal

Clausal form

Example: C1 ∧ … ∧ Cn → Cn+1

- Recall, this is equivalent to

¬ (C1 ∧ … ∧ Cn) ∨ Cn+1

¬ C1 ∨ … ∨ ¬ Cn ∨ Cn+1

- The last form is in clausal form: universally quantified conjunction of disjunctions = conjunctive normal form (CNF)
  - A literal is an atomic formula or the negation of an atomic formula, e.g. dog(fido), ¬cat(fido)
  - A clause is a disjunction of literals:
    - e.g. ¬owns(fido,fred) ∨ ¬dog(fido) ∨ person(fred)
    - We write
      (¬owns(fido,fred), ¬dog(fido), person(fred))
  - A clausal theory is a conjunction of clauses.

Clausal form (CNF)

- Most FOL proof procedures works with formulas expressed in clausal form.
  - Get rid of existential quantifiers
  - Only have to deal with universal quantification: much easier
- Important: every sentence in FOL can be converted into an inferentially equivalent CNF sentence:
  - Unsatisfiable iff the original sentence is unsatisfiable
- Forward chaining and refutation proofs (upcoming) use the clausal form.
  - Both use a single inference rule: resolution

Resolution Rule for Ground Clauses

- The resolution proof procedure consists of only **one simple rule** *(binary resolution)*:
  - From the two clauses
    - (P, Q1, Q2, …, Qk)
    - (¬P, R1, R2, …, Rn)
  - By "resolving" them, we infer the new clause
    - (Q1, Q2, …, Qk, R1, R2, …, Rn)
  - **Example**: ∀X,Y.largerThan(X,Y) → ¬fitsIn(X,Y)
    and fitsIn(clyde,cup)
  - As clauses: (¬largerThan(clyde,cup), ¬fitsIn(clyde,cup)
    and (fitsIn(clyde,cup))

    We infer by resolution: ¬largerThan(clyde,cup)

Resolution Proofs

- Logical consequences can be generated from the resolution rule in two ways:

1. **Forward Chaining inference** ("Consequence Finding")
   - Produce a sequence of consequences as resolution steps (Generalized modus ponens)
2. ***Our focus:*** **Refutation Proofs**
   - Show that the negation of the goal is inconsistent with the sentences of the KB by deriving the empty clause

Forward Chaining using Resolution

- If we have a sequence of clauses C1, C2, …, Ck such that each Ci is either in KB or is the result of a resolution step involving two prior clauses in the sequence.
- We then have that KB $\vdash$ Ck.
- **Sound** (since it is only Generalized Modus Ponens) so we also have KB $\models$ Ck
- **Complete** only for definite clause knowledge bases
  - definite clause: clause with exactly one positive literal
  - All rules with a single consequence and only positive antecedents,
    e.g. C1 $\wedge$ … $\wedge$ Cn $\rightarrow$ Cn+1
    have exactly one positive literal in clausal form:
    $\neg$ C1 v … v $\neg$ Cn v Cn+1

Prolog Programs are Clausal Theories

- Each clause in a Prolog program is Horn.
  - A Horn clause contains at most one positive literal.
  - The horn clause

    $\neg q1 \vee \neg q2 \vee \ldots \vee \neg qn \vee p$

    is equivalent to

    $q1 \wedge q2 \wedge \ldots \wedge qn \Rightarrow p$

    and is written as the following rule in Prolog:

    p :- q1 , q2 ,… ,qn

  (However Prolog also uses negation as failure!)
- Slightly more general than definite clauses
  - allows clauses with only negative literals = goal clause

Resolution Proof: Refutation proofs

- **Second proof procedure: Refutation proofs**
  - We determine if KB ⊢ f by showing that a contradiction can be generated from KB Λ ¬f.
    - ¬f is the goal clause
  - In this case a contradiction is an empty clause ().
  - We employ resolution to construct a sequence of clauses C1, C2, …, Cm such that
    - Ci is in KB Λ ¬f, or is the result of resolving two previous clauses in the sequence.
    - Cm = () i.e. its the empty clause.

Resolution Proof: Refutation proofs

- If we can find such a sequence C1, C2, …, Cm=(), we have that
  - KB $\vdash$ f.
  - Furthermore, this procedure is sound so
    - KB $\models$ f
- And the procedure is also complete so it is capable of finding a proof of any f that is a logical consequence of KB. I.e.
  - If KB $\models$ f then we can generate a refutation from KB $\wedge \neg$f

Resolution Proofs Example

Want to prove likes(clyde,peanuts) from:

1. (elephant(clyde), giraffe(clyde))
2. (¬elephant(clyde), likes(clyde,peanuts))
3. (¬giraffe(clyde), likes(clyde,leaves))
4. ¬likes(clyde,leaves)

Approach 1: Forward Chaining Proof:

- 3&4 → ¬giraffe(clyde) [5.]
- 5&1 → elephant(clyde) [6.]
- 6&2 → likes(clyde,peanuts) [7.] ✓

Resolution Proofs Example

Want to prove likes(clyde,peanuts) from:

1. (elephant(clyde), giraffe(clyde))
2. (¬elephant(clyde), likes(clyde,peanuts))
3. (¬giraffe(clyde), likes(clyde,leaves))
4. ¬likes(clyde,leaves)

Approach 2: Refutation Proof:

- ¬likes(clyde,peanuts) [5.]
- 5&2 → ¬elephant(clyde) [6.]
- 6&1 → giraffe(clyde) [7.]
- 7&3 → likes(clyde,leaves) [8.]
- 8&4 → () ✓

Resolution Proofs

- Proofs by refutation have the advantage (over forward chaining) that they are easier to find: They are more focused to the particular conclusion we are trying to reach.

- **To develop a complete resolution proof procedure for first-order logic we need:**
  1. A way of converting KB and f (the query) into clausal form.
  2. A way of doing resolution even when we have variables (unification).

We study those two mechanisms next.

Conversion to Clausal Form (C-T-C-F)

To convert the KB into clausal form we perform the following 8-step procedure:

1. **Eliminate Implications**.
2. **Move Negations inwards (and simplify ¬¬).**
3. **Standardize Variables.**
4. **Skolemize.**
5. **Convert to Prenix Form.**
6. **Distribute disjunctions over conjunctions**.
7. **Flatten nested conjunctions and disjunctions.**
8. **Convert to Clauses**.

Note: we expect the KB to consist only of sentences:

– Sentence = formula with no free (unquantified) variable

C-T-C-F: Eliminate implications

We use this example to show each step:

$$\forall X.p(X) \rightarrow \big(\quad \forall Y.p(Y) \rightarrow p(f(X,Y)) \\ \wedge \neg(\forall Y.\ \neg q(X,Y) \wedge p(Y)) \big)$$

1. Eliminate implications: A→B ➔ ¬A ∨ B

$$\forall X.\ \neg p(X) \\ \vee \big(\quad \forall Y.\neg p(Y) \vee p(f(X,Y)) \\ \wedge \neg(\forall Y.\ \neg q(X,Y) \wedge p(Y)) \big)$$

C-T-C-F: Move ¬ Inwards

$$\forall X.\ \neg p(X)$$
$$\quad v\ (\ \forall Y.\neg p(Y)\ v\ p(f(X,Y))$$
$$\qquad \wedge\ \neg(\forall Y.\ \neg q(X,Y) \wedge p(Y))\ )$$

2. **Move Negations Inwards (and simplify ¬¬).**

$$\forall X.\ \neg p(X)$$
$$\quad v\ (\ \forall Y.\neg p(Y)\ v\ p(f(X,Y))$$
$$\qquad \wedge\ \exists Y.\ q(X,Y)\ v\ \neg p(Y)\ )$$

C-T-C-F: : ¬ continue…

Rules for moving negations inwards

- DeMorgan laws:
  - ¬(A Λ B) ➔ ¬A v ¬B
  - ¬(A v B) ➔ ¬A Λ ¬B
- Quantifiers:
  - ¬∀X. f ➔ ∃X. ¬f
  - ¬∃X. f ➔ ∀X. ¬f
- Double Negation Law:
  - ¬¬A ➔ A

C-T-C-F: Standardize Variables

$$\forall X.\ \neg p(X) \vee \left(\forall Y.\neg p(Y) \vee p(f(X,Y)) \wedge \exists Y.q(X,Y) \vee \neg p(Y) \right)$$

3. **Standardize Variables (Rename variables so that each quantified variable is unique).**

$$\forall X.\ \neg p(X) \vee \left(\forall Y.(\neg p(Y) \vee p(f(X,Y)) \wedge \exists Z.q(X,Z) \vee \neg p(Z) \right)$$

C-T-C-F: Skolemize

$$\forall X.\ \neg p(X)$$
$$\quad v\ (\ \forall Y.\neg p(Y)\ v\ p(f(X,Y))$$
$$\qquad \wedge\ \exists Z.q(X,Z)\ v\ \neg p(Z)\ )$$

4. Skolemize (Remove existential quantifiers by introducing new function symbols).

$$\forall X.\ \neg p(X)$$
$$\quad v\ (\ \forall Y.\neg p(Y)\ v\ p(f(X,Y))$$
$$\qquad \wedge\ q(X,g(X))\ v\ \neg p(g(X))\ )$$

C-T-C-F: Skolemization continue…

Consider $\exists Y.\text{elephant}(Y) \wedge \text{friendly}(Y)$

- This asserts that there is some individual (binding for Y) that is both an elephant and friendly.

- To remove the existential, we **invent** a name for this individual, say ***a*.** This is a new constant symbol not equal to any previous constant symbols to obtain:

  $\text{elephant}(\mathbf{a}) \wedge \text{friendly}(\mathbf{a})$

- This is saying the same thing, since we do not know anything about the new constant **a**.

C-T-C-F: Skolemization continue

- It is essential that the introduced symbol “a” is **new.** Else we might know something else about “a” in KB.
- If we did know something else about “a” we would be asserting more than the existential.
- In original quantified formula we know nothing about the variable “Y”. Just what was being asserted by the existential formula.

C-T-C-F: Skolemization continue

Now consider $\forall X \exists Y.\ loves(X,Y)$.

- This formula claims that for every *X* there is some *Y* that X loves (perhaps a different Y for each X).

- Replacing the existential by a new constant won't work

  $\forall X.loves(X,\mathbf{a})$.

  Because this asserts that there is a **particular** individual "a" loved by every X.

- To properly convert existential quantifiers scoped by universal quantifiers we must use **functions** not just constants.

C-T-C-F: Skolemization continue

- We must use a function that mentions every universally quantified variable that scopes the existential.

- In this case *X* scopes *Y* so we must replace the existential *Y* by a function of *X*

$$\forall X.\ loves(X, g(X)).$$

where g is a **new** function symbol.

- This formula asserts that for every X there is some individual (given by g(X)) that X loves. g(X) can be different for each different binding of X.

C-T-C-F: Skolemization Examples

$\forall XYZ\ \exists W.\ r(X,Y,Z,W) \rightarrow \forall XYZ.\ r(X,Y,Z,h1(X,Y,Z))$

$\forall XY\ \exists W.\ r(X,Y,g(W)) \rightarrow \forall XY.\ r(X,Y,g(h2(X,Y)))$

$\forall XY\ \exists W\ \forall Z\ .\ r(X,Y,W) \wedge q(Z,W)$

$\rightarrow \forall XYZ\ .\ r(X,Y,h3(X,Y)) \wedge q(Z,h3(X,Y))$

C-T-C-F: Convert to prenix

$\forall X.\ \neg p(X)$

$\quad v\ (\ \forall Y.\neg p(Y)\ v\ p(f(X,Y))$

$\quad\quad \Lambda\ q(X,g(X))\ v\ \neg p(g(X))\ )$

5. **Convert to prenix form.** (Bring all quantifiers to the front – only universals, each with different name).

$\forall X \forall Y.\ \neg p(X)$

$\quad v\ (\neg p(Y)\ v\ p(f(X,Y))$

$\quad\quad \Lambda\ q(X,g(X))\ v\ \neg p(g(X))\ )$

C-T-C-F: disjunctions over conjunctions

$$\forall X \forall Y.\ \neg p(X) \vee (\neg p(Y) \vee p(f(X,Y)) \wedge q(X,g(X)) \vee \neg p(g(X)))$$

6. Disjunction over Conjunction (use distributive law): $A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$

$$\forall XY.\ \ \neg p(X) \vee \neg p(Y) \vee p(f(X,Y)) \wedge \neg p(X) \vee q(X,g(X)) \vee \neg p(g(X))$$

C-T-C-F: flatten & convert to clauses

7. Flatten nested conjunctions and disjunctions

(use associativity of v, Λ).

(A v (B v C)) ➔ (A v B v C)

8. Convert to Clauses

(remove quantifiers and break apart conjunctions).

$\forall XY.\quad (\neg p(X) \vee \neg p(Y) \vee p(f(X,Y))$

$\wedge\ (\neg p(X) \vee q(X,g(X)) \vee \neg p(g(X))$

a) $\neg p(X) \vee \neg p(Y) \vee p(f(X,Y))$

b) $\neg p(X) \vee q(X,g(X)) \vee \neg p(g(X))$

Resolution Proofs

- Proofs by refutation have the advantage that they are easier to find: They are more focused to the particular conclusion we are trying to reach.

- To develop a complete resolution proof procedure for first-order logic we need:
  1. A way of converting KB and the query into clausal form. ✓
  2. A way of doing resolution even when we have variables (unification).

We study those two mechanisms next.

Unification

- Ground clauses are clauses with no variables in them. For ground clauses we can use syntactic identity to detect when we have a P and ¬P pair.

- What about variables?
  Can the clauses
  - (P(john), Q(fred), R(X))
  - (¬P(Y), R(susan), R(Y))

  be resolved?

Unification.

- Intuitively, once reduced to clausal form, all remaining variables are universally quantified.
  Implicitly (¬P(Y), R(susan), R(Y)) represents clauses like
  - (¬P(fred), R(susan), R(fred))
  - (¬P(john), R(susan), R(john))
  - …
- So there is a "specialization" of this clause that can be resolved with (P(john), Q(fred), R(X))
  - we want to find such a specialization, but
  - want to **avoid to overspecialize**.

Unification: find a most general unifier.

- Consider: (¬p(X), s(X), q(fred)) and (p(Y), r(Y))
- Possible resolvants
  - (s(john), q(fred), r(john)) {Y=john, X=john}
  - (s(sally), q(fred), r(sally)) {Y=sally, X=sally}
  - (s(X), q(fred), r(X)) {Y=X}

- The last resolvant is "most-general", the other two are specializations of it.
- We want to keep the most general clause so that we can use it future resolution steps.

Unification: find a most general unifier.

- Unification is a mechanism for finding a "most general" matching: *Most general unifier.*

- First we consider substitutions.
  - A substitution is a finite set of equations of the form

    (V = t)

    where V is a variable and t is a term not containing V (t might contain other variables).

Substitutions.

- We can apply a substitution σ to a formula f to obtain a new formula $f\sigma$ by simultaneously replacing every variable mentioned in the left hand side of the substitution by the right hand side.

$$p(X,g(Y,Z))[X=Y,\ Y=f(a)] \rightarrow p(Y,g(f(a),Z))$$

- Note that substitutions are not applied sequentially, i.e., the first Y is not subsequently replaced by f(a).

Substitutions: Composition.

- We can compose two substitutions. θ and σ to obtain a new substitution $\theta\sigma$.

Let $\theta = \{X_1=s_1, X_2=s_2, \ldots, X_m=s_m\}$

$\sigma = \{Y_1=t_1, Y_2=t_2, \ldots, Y_k=t_k\}$

To compute their composition $\theta\sigma$

1. we apply σ to each RHS of θ and then add all equations of σ:
   $S = \{X_1=s_1\sigma, X_2=s_2\sigma, \ldots, X_m=s_m\sigma, Y_1=t_1, Y_2=t_2, \ldots, Y_k=t_k\}$
2. Delete identities, i.e., equations of the form V=V.
3. Delete equation $Y_i=s_i$ where Y_i is equal to one of the X_j in θ.
   Why?

The final set S is the composition $\theta\sigma$.

Composition Example.

$\theta = \{X=f(Y), Y=Z\}$, $\sigma = \{X=a, Y=b, Z=Y\}$

$\theta\sigma$

Substitutions.

- The empty substitution $\varepsilon = \{\}$ is also a substitution, and it acts as an identity under composition.
- More importantly substitutions when applied to formulas are associative:

$$(f\theta)\sigma = f(\theta\sigma)$$

- Composition is simply a way of converting the sequential application of a series of substitutions to a single simultaneous substitution.

Unifiers

- A unifier of two formulas f and g is a substitution σ that makes f and g syntactically identical.
- Not all formulas can be unified—substitutions only affect variables.

p(f(X),a) p(Y,f(w))

This pair cannot be unified as there is no way of making
a = f(w)
with a substitution.

MGU

- A substitution σ of two formulas f and g is a Most General Unifier (MGU) if

  1. σ is a unifier.
  2. For every other unifier θ of f and g there must exist a third substitution λ such that

     $$\theta = \sigma\lambda$$

  - This says that every other unifier is "more specialized" than σ.
  - The MGU of a pair of formulas f and g is unique up to renaming.

MGU

p(f(X),Z) p(Y,a)

- $\sigma = \{Y = f(a), X=a, Z=a\}$ is a unifier.

  $p(f(X),Z)\sigma =$
  $p(Y,a)\sigma =$

  But it is not an MGU.

- $\theta = \{Y=f(X), Z=a\}$ is an MGU.

  $p(f(X),Z)\ \theta =$
  $p(Y,a)\ \theta =$

MGU

p(f(X),Z) p(Y,a)

- σ is a specialization of θ: $\sigma = \theta\lambda$, where λ={X=a}

  θ = {Y=f(X), Z=a}
  σ = {Y = f(a), X=a, Z=a}
  → λ ={X=a}

MGU

We can compute MGUs.

- Intuitively we line up the two formulas and find the first sub-expression where they disagree.
  - The pair of subexpressions where they first disagree is called the disagreement set.
- The algorithm works by successively fixing disagreement sets until the two formulas become syntactically identical.

MGU

To find the MGU of two formulas f and g:

1. $k = 0$; $\sigma_0 = \{\}$; $S_0 = \{f,g\}$
2. If S_k contains an identical pair of formulas stop, and return σ_k as the MGU of f and g.
3. Else find the disagreement set $D_k=\{e_1,e_2\}$ of S_k
4. If e_1 = V a variable, and e_2 = t (a term not containing V) or vice-versa then let
   $\sigma_{k+1} = \sigma_k \{V=t\}$ (Compose the additional substitution)
   $S_{k+1} = S_k\{V=t\}$ (Apply the additional substitution)
   $k = k+1$
   GOTO 2
5. Else stop, f and g cannot be unified.

MGU Example 1

$$S_0 = \{p(f(a), g(X)) \;\; ; \;\; p(Y,Y)\}$$

MGU Example 2

$$S_0 = \{p(a,X,h(g(Z))) \;\; ; \;\; p(Z,h(Y),h(Y))\}$$

MGU Example 3

$S_0 = \{p(X,X) \;;\; p(Y,f(Y))\}$

Non-Ground Resolution

- Now we have everything we need in order to apply Resolution to arbitrary sentences in FOL.
- Resolution of non-ground clauses
  (i.e. with variables).
  From the two clauses
  $(L, Q1, Q2, \ldots, Qk)$
  $(\neg M, R1, R2, \ldots, Rn)$
  where there exists σ a MGU for L and M.

  We infer the new clause

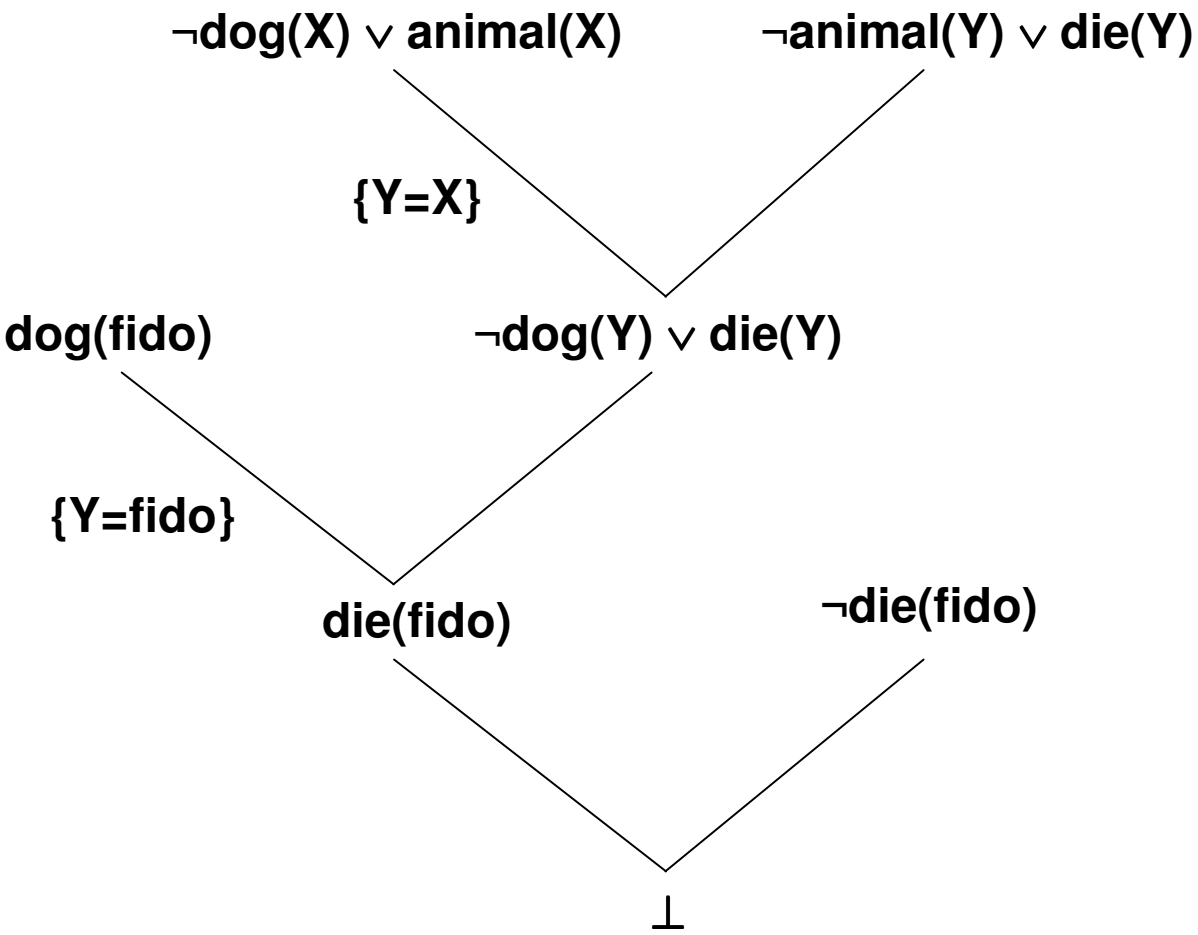
  $(Q1\sigma, \ldots, Qk\sigma, R1\sigma, \ldots, Rn\sigma)$

Non-Ground Resolution Example 1

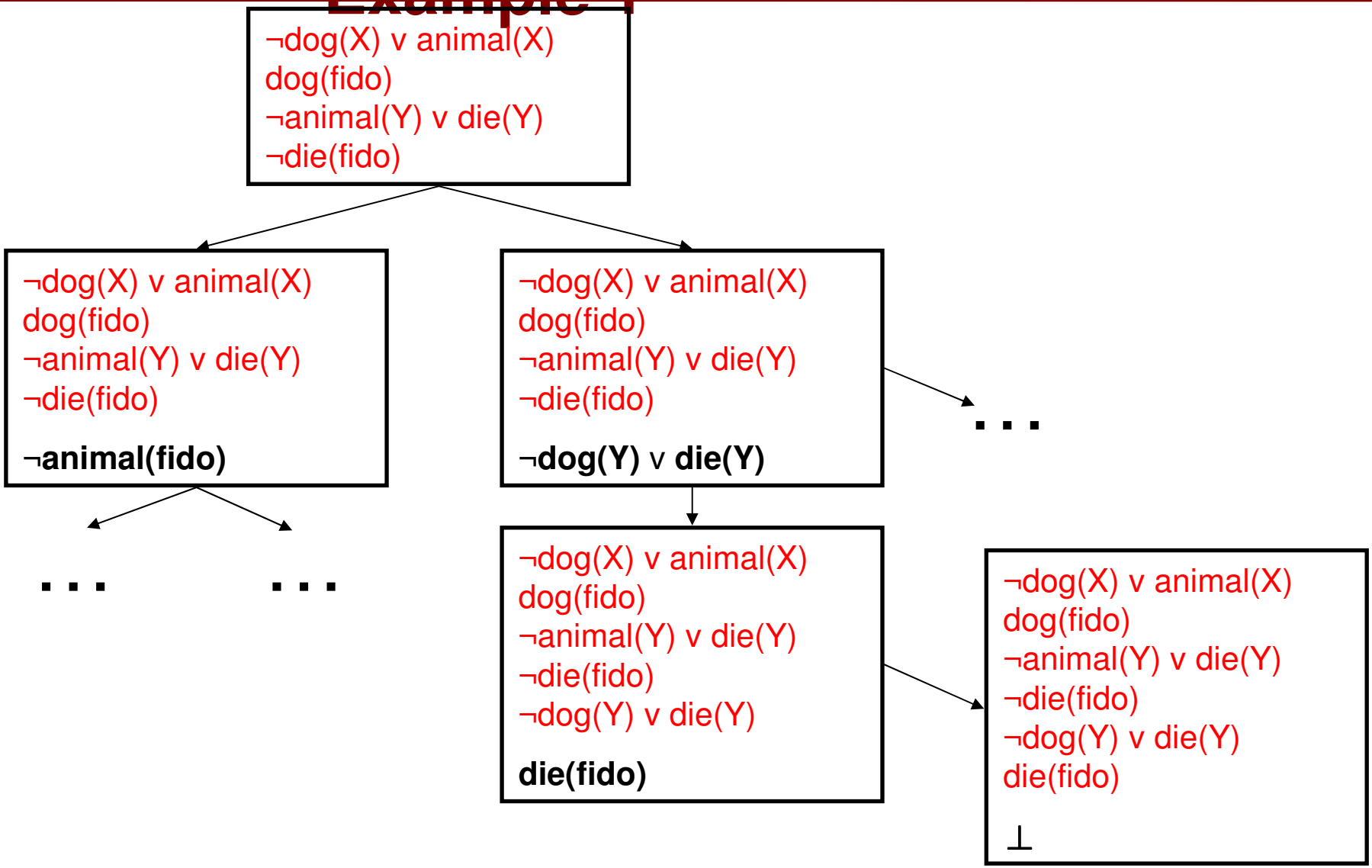
KB	Clausal Form
$\forall X\ (dog(X) \Rightarrow animal(X))$	$\neg dog(X) \vee animal(X)$
$dog(fido)$	$dog(fido)$
$\forall Y\ (animal(Y) \Rightarrow die(Y))$	$\neg animal(Y) \vee die(Y)$
Query (Goal)	*Negate the goal*
$die(fido)$	$\neg die(fido)$

Non-Ground Resolution Example 1

Resolution proof tree (with MGUs):



Resolution as Search Tree Example 1



Non-Ground Resolution: Notation.

1. (p(X), q(g(X)))
2. (r(a), q(Z), ¬p(a))

   L=p(X); M=p(a)
   σ = {X=a}

3. R[1a,2c]{X=a} (q(g(a)), r(a), q(Z))

The notation is important:

- "R" means resolution step.
- "1a" means the first (a-th) literal in the first clause i.e. p(X).
- "2c" means the third (c-th) literal in the second clause, ¬p(a).
  - 1a and 2c are the "clashing" literals.
- {X=a} is the substitution applied to make the clashing literals identical.

Resolution Proof Example

"Some patients like all doctors. No patient likes any quack. Therefore, no doctor is a quack."

Resolution Proof Step 1.
Pick symbols to represent these assertions.

p(X): X is a patient
d(X): X is a doctor
q(X): X is a quack
l(X,Y): X likes Y

Resolution Proof Example

Resolution Proof Step 2.

Convert each assertion to a first-order formula.

1. Some patients like all doctors.

F1.

Resolution Proof Example

1. No patient likes any quack

F2.

1. Therefore no doctor is a quack.

Query.

Resolution Proof Example

Resolution Proof Step 3.

Convert to Clausal form.

F1.

F2.

Negation of Query.

Resolution Proof Example

Resolution Proof Step 4.

Resolution Proof from the Clauses.

1. p(a)
2. (¬d(Y), l(a,Y))
3. (¬p(Z), ¬q(R), ¬l(Z,R))
4. d(b)
5. q(b)

Answer Extraction

- The previous example shows how we can answer true-false questions. With a bit more effort we can also answer “fill-in-the-blanks” questions (e.g., what is wrong with the car?).

- As in Prolog we use free variables in the query where we want to fill in the blanks. We simply need to keep track of the binding that these variables received in proving the query.
  - parent(art, jon) –is art one of jon’s parents?
  - parent(X, jon) -who is one of jon’s parents?

Answer Extraction

- A simple bookkeeping device is to use an predicate symbol answer(X,Y,…) to keep track of the bindings automatically.

- To answer the query parent(X,jon), we construct the clause

  (¬parent(X,jon), answer(X))

  - Negate the goal/query
  - Add answer predicate

- Now we perform resolution until we obtain a clause consisting of only answer literals (previously we stopped at empty clauses).

Answer Extraction: Example 1

1. father(art, jon)
2. father(bob,kim)
3. (¬father(Y,Z), parent(Y,Z))
   i.e. *all fathers are parents*
4. (¬parent(X,jon), answer(X))
   i.e. the query is: who is parent of jon?

Here is a resolution proof:

1. R[4,3b]{Y=X,Z=jon}
   (¬father(X,jon), answer(X))
2. R[5,1]{X=art} answer(art)

so art is parent of jon

Answer Extraction: Example 2

1. (father(art, jon), father(bob,jon) //either bob or art is parent of jon
2. father(bob,kim)
3. (¬father(Y,Z), parent(Y,Z)) //i.e. *all fathers are parents*
4. (¬ parent(X,jon), answer(X)) //i.e. query is parent(X,jon)

Here is a resolution proof:

1. R[4,3b]{Y=X,Z=jon} (¬father(X,jon), answer(X))
2. R[5,1a]{X=art} (father(bob,jon), answer(art))
3. R[6,3b] {Y=bob,Z=jon} (parent(bob,jon), answer(art))
4. R[7,4] {X=bob} (answer(bob), answer(art))

A disjunctive answer: either bob or art is parent of jon.

Factoring

1. (p(X), p(Y)) // ∀ X.∀ Y. ¬p(X) ➔ p(Y)
2. (¬p(V), ¬p(W)) // ∀ V.∀W. p(V) ➔ ¬p(W)

- These clauses are intuitively contradictory, but following the strict rules of resolution only we obtain:

3. R[1a,2a](X=V) (p(Y), ¬p(W))
   Renaming variables: (p(Q), ¬p(Z))
4. R[3b,1a](X=Z) (p(Y), p(Q))

No way of generating empty clause!

Factoring is needed to make resolution complete, without it resolution is incomplete!

Factoring

- If two or more literals of a clause C have an MGU θ, then Cθ with all duplicate literals removed is called a **factor** of C.
- C = (p(X), p(f(Y)), ¬q(X))
  θ = {X=f(Y)}
  Cθ = (p(f(Y)), p(f(Y)), ¬q(f(Y)))
  ➔ (p(f(Y)), ¬q(f(Y)) is a factor

Adding a factor of a clause can be a step of proof:

1. (p(X), p(Y))
2. (¬p(V), ¬p(W))
3. f[1ab]{X=Y} p(Y)

Prolog

- Prolog search mechanism *(without not and cut)* is simply an instance of resolution, except
  1. Clauses are Horn (only one positive literal)
     - Can use Unit Resolution: always resolve with a unit sentence (single literal) to get shorter clauses
       - Unit Resolution is complete for Horn clauses.
  2. Prolog uses a specific depth first strategy when searching for a proof. (Rules are used first mentioned first used, literals are resolved away left to right).

Automated Theorem Provers: Prover9

- Resolution used in automated theorem provers like Prover9
  - See: TPTP.org (Thousand Problems for Theorem Proving.
- Directly implementable:
  - Don't have to choose which inference rule to use next: there is only one!
  - Search strategies remove redundancy of preforming all posible deduction sequences.
- Have additional techniques to deal, i.e., with equality =:
  - Demodulation and Paramodulation
- Specialized algorithms for more efficient theorem proving with only equations (as in mathematical theorem proving)
  - e.g. E! or Waldmeister

Review: One Last Example!

Consider the following English description

- Whoever can read is literate.
- Dolphins are not literate.
- Flipper is an intelligent dolphin.

- Who is intelligent but cannot read.

Example: pick symbols & convert to first-order formula

- Whoever can read is literate.
  $\forall$ X. read(X) $\rightarrow$ lit(X)
- Dolphins are not literate.
  $\forall$ X. dolp(X) $\rightarrow \neg$ lit(X)
- Flipper is an intelligent dolphin
  dolp(flipper) $\wedge$ intell(flipper)

- Who is intelligent but cannot read?
  $\exists$ X. intell(X) $\wedge \neg$ read(X).

Example: convert to clausal form

- $\forall X.\ read(X) \rightarrow lit(X)$
  $(\neg read(X), lit(X))$
- Dolphins are not literate.
  $\forall X.\ dolp(X) \rightarrow \neg\ lit(X)$
  $(\neg dolp(X), \neg lit(X))$
- Flipper is an intelligent dolphin.
  $dolp(flipper)$
  $intell(flipper)$
- who are intelligent but cannot read?
  $\exists\ X.\ intell(X) \wedge \neg read(X).$
  Negated Query ➔ $\forall\ X.\ \neg\ intell(X) \vee read(X)$
  Clausal Form➔ $(\neg intell(X), read(X), answer(X))$

Example: do the resolution proof

1. (¬read(X), lit(X))
2. (¬dolp(X), ¬lit(X))
3. dolp(flip)
4. intell(flip)
5. (¬intell(X), read(X),answer(X))

1. R[5a,4] X=flip. (read(flip), answer(flip))
2. R[6,1a] X=flip. (lit(flip), answer(flip))
3. R[7,2b] X=flip. (¬dolp(flip), answer(flip))
4. R[8,3] answer(flip)

so flip is intelligent but cannot read!