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# INTEGRALITY GAPS OF $2 - o(1)$ FOR VERTEX COVER SDPs IN THE LOVÁSZ–SCHRIJVER HIERARCHY*

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**Abstract.** Linear and semidefinite programming are highly successful approaches for obtaining good approximations for NP-hard optimization problems. For example, breakthrough approximation algorithms for MAX CUT and SPARSEST CUT use semidefinite programming. Perhaps the most prominent NP-hard problem whose exact approximation factor is still unresolved is VERTEX COVER. Probabilistically checkable proof (PCP)-based techniques of Dinur and Safra [*Ann. of Math.* (2), 162 (2005), pp. 439–486] show that it is not possible to achieve a factor better than 1.36; on the other hand no known algorithm does better than the factor of 2 achieved by the simple greedy algorithm. There is a widespread belief that semidefinite programming (SDP) techniques are the most promising methods available for improving upon this factor of 2. Following a line of study initiated by Arora et al. [*Theory Comput.*, 2 (2006), pp. 19–51], our aim is to show that a large family of linear programming (LP)- and SDP-based algorithms fail to produce an approximation for VERTEX COVER better than 2. Lovász and Schrijver [*SIAM J. Optim.*, 1 (1991), pp. 166–190] introduced the systems $LS$ and $LS_+$ for systematically tightening LP and SDP relaxations, respectively, over many rounds. These systems naturally capture large classes of LP and SDP relaxations; indeed, $LS_+$ captures the celebrated SDP-based algorithms for MAX CUT and SPARSEST CUT mentioned above. We rule out polynomial-time SDP-based $2 - \Omega(1)$ approximations for VERTEX COVER using $LS_+$. In particular, for every $\epsilon > 0$ we prove an integrality gap of $2 - \epsilon$ for VERTEX COVER SDPs obtained by tightening the standard LP relaxation with $\Omega(\sqrt{\log n / \log \log n})$ rounds of $LS_+$. While tight integrality gaps were known for VERTEX COVER in the weaker $LS$ system [G. Schoenebeck, L. Trevisan, and M. Tulsiani, *Proceedings of the* 39*th Annual ACM Symposium on Theory of Computing*, ACM Press, New York, 2007, pp. 302–310], previous results did not rule out a $2 - \Omega(1)$ approximation after even two rounds of $LS_+$.

**Key words.** vertex cover, integrality gap, Lovász–Schrijver semidefinite programming hierarchy

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## 1. Introduction.

A *vertex cover* in a graph $G = (V, E)$ is a set $S \subseteq V$ such that every edge $e \in E$ intersects $S$ in at least one endpoint. The minimum VERTEX COVER problem asks what size the minimum vertex cover in $G$ is. Determining how well we can approximate VERTEX COVER is one of the outstanding open problems in the complexity of approximation: while VERTEX COVER has a trivial 2-approximation algorithm, no better approximation algorithms are known.

This contrasts with the situation for another famous problem MAX CUT: for many years, no approximation algorithm was known that could yield better than a $(0.5 + o(1))$-approximation (the trivial randomized algorithm gives a 0.5-approximation) until the seminal paper of Goemans and Williamson [14], which used semidefinite programming (SDP) to obtain a 0.878-approximation algorithm. Since then SDP has yielded breakthrough approximation algorithms for various NP-hard optimization problems and has arguably become our most powerful tool for designing approxi-

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mation algorithms. Consequently, SDP is believed (see Lovász [23], for instance) to be the most promising technique for attacking the Vertex Cover problem.

However, Goemans and Kleinberg [13] showed in 1995 that the standard SDP for Vertex Cover has an integrality gap of $2-\epsilon$ for every $\epsilon > 0$. Subsequently, Charikar [6] showed that the integrality gap remains $2-\epsilon$, even when the standard SDP is strengthened with additional triangle inequality constraints. Hatami, Magen, and Markakis [16] strengthened this further, showing that no better approximation is obtained, even when the so-called pentagonal inequality constraints are added to the SDP. The possibility of reducing the integrality gap by adding hypermetric inequalities, which includes the triangle and pentagonal inequalities, of support at most $O(\sqrt{\log n/\log\log n})$ was eliminated in [11]. Further, it was recently shown [12] that even if all hypermetric inequalities are used, the integrality gap remains $2-o(1)$.

Indeed, the state of the art is such that SDP-based algorithms for Vertex Cover must settle for competing in "how big" the "little oh" term is in the $2-o(1)$ factor. Halperin [15] gives a $(2-\log\log\Delta/\log\Delta)$-approximation, where $\Delta$ is the maximal degree of the graph. The best approximation algorithm currently known for arbitrary graphs is due to Karakostas [18] who obtains a $(2-\Omega(1/\sqrt{\log n}))$-approximation algorithm using a stronger SDP relaxation.

Nevertheless, it is consistent with the known hardness results for Vertex Cover that there could be some other SDP with integrality gap, say, 1.4. In particular, the best probabilistically checkable proof (PCP)-based hardness result known (Dinur and Safra [8]) shows only that a 1.36-approximation of Vertex Cover is NP-hard. Only by assuming Khot's unique games conjecture [19] do we get a tight $2-o(1)$ inapproximability result [20]. However, determining the validity of the unique games conjecture (or directly improving on [8]) remains a difficult open problem.

To get a better picture of the approximability of Vertex Cover (especially in light of the inability to resolve the issue with PCP-based methods), Arora et al. [3] suggested the following approach: rule out good approximations by large families of algorithms. One such family is the class of relaxations for Vertex Cover in the Lovász–Schrijver "lift-and-project" hierarchies. Lovász and Schrijver [24] define procedures $LS$ and $LS_+$ for systematically tightening linear and semidefinite relaxations, respectively, over many rounds. These procedures are often called *lift-and-project* procedures. Important algorithmic properties of $LS$ and $LS_+$ are (a) $n$ rounds of even the weaker $LS$ procedure suffice to obtain exact solutions, and (b) we can optimize a linear function over the $r$th tightening of the $LS$ and $LS_+$ relaxations in $n^{O(r)}$ time (provided the original relaxation had a polynomial-time separation oracle).

Many celebrated SDP-based algorithms, including the seminal Max Cut algorithm of Goemans and Williamson [14] and the Arora–Rao–Vazirani algorithm [4] for Sparsest Cut, can be derived using a constant number of rounds of $LS_+$. Thus proving inapproximability results for $LS_+$-based algorithms rules out one of the most promising classes of algorithms that we currently have for obtaining $2-\Omega(1)$ approximations for Vertex Cover. Furthermore, unlike PCP-based results, we emphasize that such results do not rely on any complexity theoretic assumptions.

Arora et al. [3] obtained the first result along these lines for Vertex Cover showing that $\Omega(\log n)$ rounds of the weaker $LS$ procedure have an integrality gap of $2-\epsilon$ for every $\epsilon > 0$. Tourlakis [28] proved an integrality gap of $1.5-\epsilon$, for Vertex Cover for $\Omega(\log^2 n)$ rounds of $LS$. Subsequently, a beautiful result by Schoenebeck, Trevisan, and Tulsiani [27] showed that the integrality gap is $2-\epsilon$, even after $\Omega(n)$ rounds of $LS$. Several related results about the performance of lift-and-project systems for Vertex Cover have appeared after the conference version of the current paper was published. For instance, a tight integrality gap was proven for the Sherali–Adams hierarchy (a lift-and-project hierarchy which is stronger than the $LS$ but incomparable to the $LS_+$ hierarchy) by Charikar, Makarychev, and Makarychev [7] for $n^{\delta(\epsilon)}$ rounds. Very recently, integrality gaps of $7/6 - \epsilon$ [25] and 1.36 [29] were obtained for the powerful Lasserre hierarchy for $\Omega(n)$ and $\Omega(n^{\delta})$ rounds, respectively. Interestingly, all these results are incomparable, meaning that none of them is stronger than the other (see section 5 for comparative discussion). Unfortunately, the integrality gap instances used in previous works cannot be used to prove a tight integrality gap for even one round of $LS_+$.

The only known integrality gaps for Vertex Cover $LS_+$ relaxations prior to the current paper were proved by Schoenebeck, Trevisan, and Tulsiani [26] who showed that the integrality gap remains 7/6 for $\Omega(n)$ rounds of $LS_+$. The graphs they use are obtained using the standard Feige et al. [9] reduction from max-3xor to Vertex Cover. Such instances cannot prove stronger integrality gaps for $LS_+$ since their integrality gaps are *at most* 7/6 after one round of $LS_+$.

To summarize, previously known results do not preclude a polynomial time $2 - \Omega(1)$ approximation algorithm for Vertex Cover using $LS_+$ tightenings. In particular, showing a $2-\epsilon$ integrality gap for even two rounds of $LS_+$ remained a challenging open problem (Charikar's construction [6] does imply a $2 - \epsilon$ gap for one round).

In this paper we rule out such approximations. Our starting point is the graph families used to show tight integrality gaps for various Vertex Cover SDPs in [13, 6, 16] and recently in [11, 12] (similar graphs were used by Alon and Kahale [2] in independent work contemporaneous with [13] studying the Lovász theta function). We briefly describe these graphs. The vertex set is $\{-1, 1\}^m$, and two vertices are adjacent if their Hamming distance is exactly $(1-\gamma)m$. A result of Frankl and Rödl [10] bounds from above the size of any independent set in such graphs by $m(2 - \Omega(\gamma^2))^m$. Hence, for constant $\gamma > 0$ (or even $\gamma$ a slowly vanishing function of $m$) any vertex cover has size $(1 - o(1))|V|$. Of course for $\gamma = 0$ these graphs are just perfect matchings on $2^m$ vertices. The cleverness of the construction lies in how a minuscule increase in $\gamma$ dramatically changes the independent set size while not appreciably altering the "geometry" of the graph (and hence not appreciably increasing the SDP value from the perfect matching case—the SDP value depends continuously on the geometry of the graph solution).

We use this graph family to show that for every $\epsilon > 0$, $\Omega(\sqrt{\log n/ \log\log n})$ rounds of $LS_+$ have an integrality gap of $2 - \epsilon$ for Vertex Cover. Our main theorem also implies that the integrality gap remains at least $2 - O(\sqrt{\log\log n/ \log n})$ after $O(1)$ rounds of $LS_+$. Hence, the approximation ratio achieved by Karakostas's [18] algorithm is essentially tight for "polynomial" time $LS_+$ relaxations. Our main technical tool is the construction of a sequence of tensoring operations on vectors. These operations have the property that inner products on the set of tensored vectors are a polynomial function of the inner products of the original vectors. These extend similar tensoring operations used by Charikar [6] (and implicit in earlier work by Kahn and Kalai [17]). However, our application calls for more complicated polynomials, and moreover the polynomials (and hence the tensored vectors) change as the induction unwinds in our lower bound argument (details in section 3).

**Organization of the paper.** Section 2 contains all necessary definitions including a description of $LS_+$. Section 3 outlines our approach, while section 4 contains the proof of our main result. Section 5 discusses relevant lower bounds, limitations of our approach, and poses some open problems.

**2. Definitions, notation, and tools.**

**2.1. Standard SDPs for Vertex cover.** The standard way to formulate Vertex Cover for a graph $G = (V, E)$ as a quadratic integer program is

$$\begin{array}{lll} \min & \sum_{i \in V} (1 + x_0 x_i)/2 & \\ \text{such that (s.t.)} & (x_0 - x_i)(x_0 - x_j) = 0 & \forall ij \in E, \\ & x_i \in \{-1, 1\} & \forall i \in \{0\} \cup V. \end{array}$$

The set of vertices $i$ for which $x_i = x_0$ gives a minimal vertex cover. This quadratic program leads to the SDP relaxation:

$$\begin{array}{lll} \min & \sum_{i \in V} (1 + \mathbf{v}_0 \cdot \mathbf{v}_i)/2 & \\ \text{s.t.} & (\mathbf{v}_0 - \mathbf{v}_i) \cdot (\mathbf{v}_0 - \mathbf{v}_j) = 0 & \forall ij \in E, \\ & \|\mathbf{v}_i\| = 1 & \forall i \in \{0\} \cup V. \end{array} \tag{1}$$

Note the relation of this SDP to the Lóvasz theta function [22, 13] for the Maximum Independent Set problem: Given a graph the Maximum Independent Set problem requires one to find an independent set, namely, a subset of nonadjacent vertices, of maximum size. Clearly the complement of any independent set is a vertex cover and vice versa. Therefore, if we replace $\mathbf{v}_i$ by $\mathbf{v}_0 - \mathbf{v}_i$ in SDP (1), we get a relaxation for Independent Set. The optimal value of this relaxation is the well-known Lóvasz theta function.

We can strengthen the relaxation (1) by adding the vector analogues of constraints valid for the original quadratic integer program. Examples of such constraints are the triangle and "extended" triangle inequalities (respectively)

$$(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{v}_i - \mathbf{v}_k) \geq 0 \quad \forall i, j, k \in \{0\} \cup V, \tag{2}$$

$$(\mathbf{v}_i \pm \mathbf{v}_j) \cdot (\mathbf{v}_i \pm \mathbf{v}_k) \geq 0 \quad \forall i, j, k \in \{0\} \cup V. \tag{3}$$

The SDP relaxation (1) was studied in [13]. The SDP tightened using (2) was studied in [6], while the SDP tightened using (2) *and* (3) (as well as the so-called pentagonal inequalities) was studied in [16]. Further tightenings of (1) by stronger families of valid inequalities were studied in [11, 12].

**2.2. The Lovász–Schrijver lift-and-project system.** A convex cone is a set $K \subseteq \mathcal{R}^{n+1}$ such that for every $\mathbf{y}, \mathbf{z} \in K$ and for every $\alpha, \beta \geq 0$, $\alpha\mathbf{y} + \beta\mathbf{z} \in K$. Let $\mathbf{e}_i$ denote the vector with 1 in coordinate $i$ and 0 everywhere else. Hence, $Y\mathbf{e}_i$ denotes the $i$th column of a matrix $Y$.

For a convex cone $K \subseteq \mathbb{R}^{n+1}$ let $M_+(K) \subseteq \mathbb{R}^{(n+1)\times(n+1)}$ consist of all symmetric $(n+1) \times (n+1)$ matrices $Y$ such that

1. for all $i = 0, 1, \ldots, n$, $Y_{0i} = Y_{ii}$.
2. for all $i = 0, 1, \ldots, n$, $Y\mathbf{e}_i, Y\mathbf{e}_0 - Y\mathbf{e}_i \in K$.
3. $Y$ is positive semidefinite (PSD).

We then define $N_+(K) = \{Y\mathbf{e}_0 : Y \in M_+(K)\} \subseteq \mathbb{R}^{n+1}$. That is, a vector $\mathbf{y} = (y_0, \ldots, y_n)$ is in $N_+(K)$ if there exists $Y \in M_+(K)$ such that $Y\mathbf{e}_0 = \mathbf{y}$ in which case $Y$ is called a *protection matrix for* $\mathbf{y}$. Define $N_+^k(K)$ inductively by setting $N_+^0(K) = K$ and $N_+^k(K) = N_+(N_+^{k-1}(K))$.

Let $G = (V, E)$ be a graph and assume that $V = \{1, \ldots, n\}$. The VERTEX COVER convex cone for $G$, $VC(G)$, is the set of vectors $\mathbf{y} \in \mathbb{R}^{n+1}$ such that

$$y_i + y_j \geq y_0 \quad \forall\, ij \in E, \tag{4}$$
$$y_0 \geq y_i \geq 0 \quad \forall\, i \in V, \tag{5}$$
$$y_0 \geq 0.$$

Constraints (4) are called the *edge constraints*, and constraints (5) are called the *box constraints*.

The value of the VERTEX COVER relaxation arising from $k$ rounds of $LS_+$ is the solution of

$$\begin{array}{ll} \min & \displaystyle\sum_{i=1}^{n} y_i \\ \text{s.t.} & (y_0, \ldots, y_n) \in N_+^k(VC(G)) \text{ and } y_0 = 1. \end{array}$$

The *integrality gap* of this relaxation (for $n$-vertex graphs) is the largest ratio between the minimum vertex cover size of $G$ and the optimum in the above program, over all $n$-vertex graphs $G$.

To get an idea of the power of $LS_+$, we note first that the relaxation $N_+(VC(G))$ is at least as strong as the standard SDP relaxation for VERTEX COVER since the Cholesky decomposition of any matrix $Y \in M_+(VC(G))$ satisfies (under an affine transformation) SDP (1). In fact, it even satisfies the triangle inequalities (2) for the case $i = 0$. On the other hand, one can show that adding both the standard and "extended" triangle inequalities (constraints (2) and (3), respectively) to the standard VERTEX COVER SDP results in a relaxation at least as strong as $N_+(VC(G))$. Indeed, we will (implicitly) exploit the latter fact when constructing SDP solutions for our lower bound.

**2.3. Vectors and tensoring.** We will use $\mathbf{0}$ to denote the all-0 vector. Given two vectors $\mathbf{x}, \mathbf{y} \in \{-1, 1\}^n$, their *Hamming distance* $d_H(\mathbf{x}, \mathbf{y})$ is $|\{i \in [n] : x_i \neq y_i\}|$. For two vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$ denote by $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n+m}$ the vector whose projection on the first $n$ coordinates is $\mathbf{u}$ and on the last $m$ coordinates is $\mathbf{v}$.

Recall that the tensor product $\mathbf{u} \otimes \mathbf{v}$ of vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$ is the vector in $\mathbb{R}^{nm}$ indexed by ordered pairs from $n \times m$ and that assumes the value $\mathbf{u}_i \mathbf{v}_j$ at coordinate $(i, j)$. Define $\mathbf{u}^{\otimes d}$ to be the vector in $\mathbb{R}^{n^d}$ obtained by tensoring $\mathbf{u}$ with itself $d$ times.

DEFINITION 1. *Let $P(x) = c_1 x^{t_1} + \cdots + c_q x^{t_q}$ be a polynomial with nonnegative coefficients. Then we define $T_P$ to be the function that maps a vector $\mathbf{u}$ to the vector $T_P(\mathbf{u}) = (\sqrt{c_1} u^{\otimes t_1}, \ldots, \sqrt{c_q} u^{\otimes t_q})$.*

FACT 1. *For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, $T_P(\mathbf{u}) \cdot T_P(\mathbf{v}) = P(\mathbf{u} \cdot \mathbf{v})$.*

**2.4. Frankl–Rödl graphs.**

DEFINITION 2. *Fix $\gamma$, $0 \leq \gamma \leq 1$, and an integer $m \geq 1$. The* Frankl–Rödl graph *$G_m^\gamma$ is the graph with vertices $\{-1, 1\}^m$ and where two vertices $i, j \in \{-1, 1\}^m$ are adjacent if $d_H(i, j) = (1 - \gamma)m$.*

Relatives of the following lemma appear in [10] in various guises, but it seems as if the exact statement that we will use requires a further small step which we sketch in Appendix A. The key difference with variants in [10] is that we explicitly allow $\gamma$ to be a function of $m$.

LEMMA 1. *Let $m$ be an integer, and let $\gamma = \gamma(m) > 0$ be a sufficiently small number so that $\gamma \cdot m$ is an even integer. Then there are no independent sets in $G_m^\gamma$ of size larger than $(m+1)2^m(1-\gamma^2/64)^m$.*

**2.5. Saturated vectors.** In general, our lower bounds will be proved by arguing about vectors whose coordinates are either 0/1 or take at most one other fixed value. The following definition formalizes this.

DEFINITION 3. *A vector $\mathbf{y} \in [0,1]^{n+1}$ is an $\epsilon$-*vector *if $y_0 = 1$ and $y_i \in \left\{0, \frac{1}{2}+\epsilon, 1\right\}$ for all $1 \le i \le n$.*

Note that $\epsilon$-vectors have the property that the sum of any two non-0/1 coordinates is $1+2\epsilon$. A weaker condition on vectors in $[0,1]^{n+1}$ would be to only require that the sum of any two non-0/1 coordinates is at least $1+2\epsilon$. Such vectors were used in [27], and the following definition is adapted from their paper.

DEFINITION 4 (see [27]). *Let $G = (V,E)$ be a graph. A vector $\mathbf{y} \in VC(G)$ is $\epsilon$-*saturated *if for every edge $ij \in E$ such that $y_i$ and $y_j$ are both not integral, $y_i + y_j \ge 1 + 2\epsilon$.*

Saturated vectors have the following important property proved in [27] (we include a proof in Appendix B for completeness).

LEMMA 2 (see [27]). *Let $G = (V,E)$ be any graph and suppose $\mathbf{x} \in VC(G)$ is $\epsilon$-saturated. Then $\mathbf{x}$ is a convex combination of $\epsilon$-vectors in $VC(G)$.*

The lemma essentially says that proving lower bounds for $\epsilon$-saturated vectors reduces to proving lower bounds for $\epsilon$-vectors. This will be crucial for our arguments since we know only how to find protection matrices for $\epsilon$-vectors. We remark that our definition for saturation is slightly different than the one in [27], as there they only require that *one* of $y_i$ or $y_j$ in Definition 4 be nonintegral. Consequently, Lemma 2 becomes somewhat stronger to accommodate this difference, but the additional argument for this strengthening is trivial (see Appendix B).

**3. Overview of the proof.** We start with a Frankl–Rödl graph $G = G_m^\gamma$ and denote by $n = 2^m$ the size of $G$. We will show that the point $\mathbf{x} = (1, 1/2+\epsilon, \ldots, 1/2+\epsilon)$ is contained in the polytope or more accurately in the convex body, defined after $\Omega(\sqrt{\log n / \log\log n})$ rounds of $LS_+$. This clearly gives us our desired tight integrality gap.

The standard way to prove that a certain point $\mathbf{x}$ is in the polytope resulting from $r$ rounds of $LS_+$ (hereafter, the "$r$th polytope") is as follows: 1. Exhibit a symmetric PSD "protection" matrix $Y$ for $\mathbf{x}$ such that the diagonal and first column of $Y$ equal $\mathbf{x}$. 2. Show inductively that the vectors $Ye_i$ and $Y(\mathbf{e}_0 - \mathbf{e}_i)$ are in the $(r-1)$st polytope. By the definition of $LS_+$ it will then follow that $\mathbf{x}$ is in the $r$th polytope.

To define a protection matrix for $\mathbf{x}$ we will start with the canonical set of vectors associated with the vertices of $G$, namely, the normalized versions of the vectors $\{-1,1\}^m$ (these vectors were also the starting point for [13, 6, 16] and later for [11, 12]). These vectors have the appealing property that the inner product of vectors associated with two vertices $i$ and $j$ is solely a function of their Hamming distance $d_H(i,j)$. Observe that this property will not be compromised by applying the $T_P$ tensoring transformation to the vectors. Indeed, we will use this tensoring transformation with a specific polynomial $P$ to obtain a new set of tensored vectors and then define our candidate protection matrix to be essentially the Gram matrix of these vectors. (Note that Charikar [6] also uses a tensor transformation to prove his integrality gap for the SDP with triangle inequalities.)

A consequence of the observation above is that the values on the diagonal of the Gram matrix are all identical. So this protection matrix recipe works only for vectors like $\mathbf{x}$ where all fractional values are the same. In fact, for technical reasons which we do not get into in this outline, this recipe produces valid protection matrices only when $\mathbf{x}$ is a $\rho$-vector for some $0 < \rho < 1/2$.

To continue our inductive argument we would in turn like to use the same recipe to find candidate protection matrices for each of the $2n$ vectors $Y\mathbf{e}_i$ and $Y(\mathbf{e}_0 - \mathbf{e}_i)$ (or, more accurately, for the projections of these vectors onto the hyperplane $x_0 = 1$). The problem is that while these $2n$ vectors may indeed be in the $(r-1)$st polytope, they may not be $\rho$-vectors. (This is because the entries $Y_{ij}$ of $Y\mathbf{e}_i$ are a polynomial function of $d_H(i, j)$, and the latter is distributed like a binomial distribution when $i$ is fixed.) So the recipe cannot be used without extra work.

To remedy the situation, we will apply a "correction" phase as follows. (Note that "correction" phases of some sort or another can be found in many previous works [3, 1, 5, 28, 26, 27].) We will construct the tensored vectors so that the vectors $Y\mathbf{e}_i$, $Y(\mathbf{e}_0 - \mathbf{e}_i)$ have high saturation. We will then use Lemma 2 to express these vectors as convex combinations of $\rho'$-vectors from $VC(G)$ for some $\rho' > 0$ (this is the "correction" part). We then carry on the induction with these $\rho'$-vectors to show that they lie in the $(r-1)$st polytope. Convexity then implies that the vectors $Y\mathbf{e}_i$, $Y(\mathbf{e}_0 - \mathbf{e}_i)$ are also in the $(r-1)$st polytope.

To summarize, we start with a vector $\mathbf{x} = (1, 1/2 + \epsilon_0, \ldots, 1/2 + \epsilon_0)$, $\epsilon_0 = \epsilon$, and after one round we need to show that the $2n$ vectors $Y\mathbf{e}_i$, $Y(\mathbf{e}_0 - \mathbf{e}_i)$ corresponding to $\mathbf{x}$'s protection matrix $Y$ have large saturation $\epsilon_1$; then we continue with vectors with fractional values $1/2 + \epsilon_1$, and so on. In this process, the obvious objective is to make the sequence $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$ as slowly decreasing as possible, thereby making it last for many rounds before it becomes negative (which amounts to negative saturation, and hence the corresponding vectors are not in $VC(G)$ at all). We will show that for each round $i$, we can ensure that $\epsilon_i = \epsilon_{i-1} - O(\gamma)$. Thus for arbitrarily small initial $\epsilon_0$, we get an induction chain of length $\Omega(\epsilon_0/\gamma)$.

The engine of this process and our main technical tool are the tensor-inducing polynomials. Along with the sequence of decreasing saturation values, we shall have a sequence of polynomials with positive coefficients, $P_0, P_1, P_2, \ldots$, where $P_i$ depends on $\epsilon_i$ and determines $\epsilon_{i+1}$. The choice of this sequence is at the heart of the matter. The nonnegativity requirement on the coefficients is what makes this a challenging task. Charikar [6] used a polynomial designed to produce vectors that satisfy the triangle inequality. This polynomial is the sum of a linear term and a degree $O(1/\gamma)$ monomial that unfortunately produces a poor saturation and hence cannot be used to proceed beyond one round of $LS_+$. In particular, the saturation it provides is about $1/m \ll \gamma$. The problem is intrinsic: consider the vector $Y(\mathbf{e}_0 - \mathbf{e}_i)$ for some fixed $i$. It is easy to see that no matter which polynomial we use, edges incident to vertex $i$ will have no slack at all in $Y(\mathbf{e}_0 - \mathbf{e}_i)$. Such an edge $ij$ will not in itself affect the saturation, as its vertices will have integral values; however, the continuous nature of the construction means that nearby edges $i'j'$ will not have integral values since their values will correspond to evaluating the polynomial at points only slightly different than those for $ij$. But then, to ensure that $i'j'$ has good saturation, our polynomial must vary a lot between the cases corresponding to $ij$ and $i'j'$. This calls for a polynomial with a very large derivative and hence one with very-high-degree $d \gg m$; in contrast, the polynomial that Charikar uses has degree independent of $m$.

## 4. Main theorem.

LEMMA 3. *Let $m$ be a sufficiently large integer and $\gamma > 0$. Let $n = 2^m$, and let $\epsilon$ be a sufficiently small constant such that $\epsilon > 5\gamma$. Suppose in addition that $\mathbf{y} \in \mathbb{R}^{n+1}$*

*is an $\epsilon$-vector in $VC(G_m^\gamma)$. Then there exists a protection matrix $Y$ for $\mathbf{y}$ such that for all $i$ with $0 < y_i < 1$, $Ye_i/y_i$ and $Y(e_0 - e_i)/(1 - y_i)$ are convex combinations of $(\epsilon - 6\gamma)$-vectors that lie in $VC(G_m^\gamma)$. In particular, $\mathbf{y} \in N_+(VC(G_m^\gamma))$.*

Given Lemma 3, we can prove our main theorem from which the integrality gaps for $LS_+$ stated in the introduction immediately follow.

THEOREM 1. *Let $m$ be sufficiently large, and fix $\gamma \geq 12\sqrt{(\log m)/m}$ such that $\gamma m$ and $1/\gamma$ are both even. Let $\epsilon$ be a sufficiently small constant such that $\epsilon > 5\gamma$. Let $n = 2^m$, and let $r = \lfloor \frac{\epsilon}{6\gamma} \rfloor - 1$. Then the integrality gap of $N_+^r(VC(G_m^\gamma))$ is at least $2 - 4\epsilon - 2/m$.*

*Proof.* Let $\mathbf{y} = (1, \frac{1}{2} + \epsilon, \ldots, \frac{1}{2} + \epsilon) \in \mathbb{R}^{n+1}$. Clearly $\mathbf{y} \in VC(G_m^\gamma)$. A simple inductive argument using Lemma 3 then implies that $\mathbf{y} \in N_+^r(VC(G_m^\gamma))$.

On the other hand, Lemma 1 implies that the largest independent set in $G_m^\gamma$ has size at most

$$(m+1)2^m e^{-\frac{\gamma^2 m}{64}} \leq \frac{(m+1)2^m}{e^{\frac{144}{64}\log m}} \leq \frac{2^m}{m}.$$

Hence, the integrality gap for $N_+^r(VC(G_m^\gamma))$ is at least $\frac{2^m - 2^m/m}{n(\frac{1}{2}+\epsilon)} = \frac{2(1-1/m)}{1+2\epsilon} \geq 2 - 4\epsilon - \frac{2}{m}$. ∎

**4.1. Proof of Lemma 3.** Fix $m$ and $\gamma$ and consider $G = G_m^\gamma$. Denote the vertices $V$ of $G$ as vectors $\mathbf{w}_i \in \{-1, 1\}^m$, $1 \leq i \leq 2^m$, and for each vector $\mathbf{w}_i \in V$ define $\mathbf{u}_i = \frac{1}{\sqrt{m}}\mathbf{w}_i$. Note that $\|\mathbf{u}_i\| = 1$ for all $i \in V$ and $\mathbf{u}_i \cdot \mathbf{u}_j = 2\gamma - 1$ for all $ij \in E$. Moreover, $-1 \leq \mathbf{u}_i \cdot \mathbf{u}_j \leq 1 - \frac{2}{m}$ for all $1 \leq i < j \leq 2^m$.

Given a polynomial $P$ with nonnegative coefficients, we will now define a procedure that takes the vectors $\{\mathbf{u}_i\}$, applies the tensoring operation $T_P$ from section 2.3 to obtain a new set of vectors, and then applies a linear transformation to the resulting vectors. The Gram matrix of the vectors resulting from this procedure will be called $Y(P, \mathbf{y})$. Our goal will be to pick $P$ so that $Y(P, \mathbf{y})$ is a protection matrix for $\mathbf{y}$.

First, define $\mathbf{v}_0 = (1, 0, \ldots, 0)$. For each vertex $1 \leq i \leq 2^m$ define

$$\mathbf{v}_i = \begin{cases} \mathbf{v}_0 & \text{if } y_i = 1, \\ \mathbf{0} & \text{if } y_i = 0, \\ (\frac{1}{2} + \epsilon, \frac{\sqrt{1-4\epsilon^2}}{2} \cdot T_P(\mathbf{u}_i)) & \text{if } y_i = \frac{1}{2} + \epsilon. \end{cases} \tag{6}$$

Let $Y(P, \mathbf{y}) \in \mathbb{R}^{(n+1)\times(n+1)}$ be the PSD matrix defined by $Y(P, \mathbf{y})_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$. We define a class of polynomials and show that for any polynomial $P$ in this class, $Y(P, \mathbf{y})$ is a protection matrix for $\mathbf{y}$.

DEFINITION 5. *A polynomial $P(x)$ is called* $(\gamma, \epsilon, m)$-useful *if it satisfies the following conditions:*

1. *$P$ has only nonnegative coefficients,*
2. *$P(1) = 1$,*
3. *$P(x) \geq P(2\gamma - 1) = -\frac{1-2\epsilon}{1+2\epsilon}$ for all $x \in [-1, 1]$,*
4. *For all $i \in \{1, \ldots, 2^m\}$ and all $jk \in E$,*

$$-\frac{4\epsilon}{1-2\epsilon} \leq P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k) \leq \frac{4\epsilon}{1+2\epsilon}. \tag{7}$$

CLAIM 1. *If $P$ is $(\gamma, \epsilon, m)$-useful, then $Y = Y(P, \mathbf{y}) \in M_+(VC(G))$. In particular, $Y$ is a protection matrix for $\mathbf{y}$, and hence $\mathbf{y} \in N_+(VC(G))$.*

*Proof.* Since $Y$ is PSD by definition, to show that $Y$ is a protection matrix for $\mathbf{y}$ it suffices to show that (A) for all $0 \le i \le n$, $Y_{i0} = Y_{ii} = y_i$, and (B) for all $1 \le i \le n$, $Y\mathbf{e}_i, Y(\mathbf{e}_0 - \mathbf{e}_i) \in VC(G)$.

Consider (A) first. Clearly $Y_{i0} = Y_{ii} = y_i$ whenever $y_i \in \{0, 1\}$. In particular, note that $Y_{00} = 1$. So assume that $y_i = 1/2 + \epsilon$. Clearly $Y_{i0} = \frac{1}{2} + \epsilon$. So consider $Y_{ii}$, which equals

$$\mathbf{v}_i \cdot \mathbf{v}_i = \left(\frac{1}{2} + \epsilon\right)^2 + \frac{1 - 4\epsilon^2}{4} T_P(\mathbf{u}_i) \cdot T_P(\mathbf{u}_i) = \frac{1}{4} + \epsilon + \epsilon^2 + \frac{1 - 4\epsilon^2}{4} P(\mathbf{u}_i \cdot \mathbf{u}_i) = \frac{1}{2} + \epsilon, \tag{8}$$

where the last equality follows from the fact that the $\mathbf{u}_i$ are unit vectors and $P(1) = 1$.

Now consider (B). We must show that for $1 \le i \le n$, $Y\mathbf{e}_i$ and $Y(\mathbf{e}_0 - \mathbf{e}_i)$ both satisfy the edge constraints (4) and the box constraints (5). Note that if $y_i \in \{0, 1\}$, then $\{Y\mathbf{e}_i, Y(\mathbf{e}_0 - \mathbf{e}_i)\} = \{\mathbf{0}, Y\mathbf{e}_0\} \subseteq VC(G)$, and these constraints are trivially satisfied. So assume $y_i = \frac{1}{2} + \epsilon$.

The box constraints require for all $1 \le j \le n$ that $0 \le Y_{ij} \le Y_{i0}$ and $0 \le Y_{0j} - Y_{ij} \le Y_{00} - Y_{i0}$. Equivalently, for all $1 \le j \le n$,

$$Y_{i0} + Y_{j0} - Y_{00} \le Y_{ij} \le Y_{i0}. \tag{9}$$

On the other hand, the edge constraints require for all $1 \le i \le n$ and all $jk \in E$ that

$$Y_{ij} + Y_{ik} \ge Y_{i0}, \tag{10}$$

$$(Y_{0j} - Y_{ij}) + (Y_{0k} - Y_{ik}) \ge Y_{00} - Y_{i0}. \tag{11}$$

Since (9) holds when $y_i \in \{0, 1\}$, by symmetry it also holds if $y_j \in \{0, 1\}$. So assume $y_j = \frac{1}{2} + \epsilon$. We first show that the right inequality in (9) holds. Fix $j \in \{1, \ldots, n\}$. Note that since $y_i = y_j = \frac{1}{2} + \epsilon$, it follows from (8) that $\|\mathbf{v}_i\| = \|\mathbf{v}_j\|$. So, $Y_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j \le \|\mathbf{v}_i\|^2 = Y_{ii} = Y_{i0}$.

Now consider the left inequality in (9). We have that

$$\begin{aligned} Y_{ij} + Y_{00} - Y_{i0} - Y_{j0} &= Y_{ij} - 2\epsilon \\ &= \left[\frac{1}{4} + \epsilon + \epsilon^2 + \frac{1 - 4\epsilon^2}{4} T_P(\mathbf{u}_i) \cdot T_P(\mathbf{u}_j)\right] - 2\epsilon \\ &= \frac{1}{4} - \epsilon + \epsilon^2 + \frac{1 - 4\epsilon^2}{4} P(\mathbf{u}_i \cdot \mathbf{u}_j) \\ &\ge 0, \end{aligned}$$

where the last inequality follows by property 3 of a $(\gamma, \epsilon, m)$-useful polynomial and the fact that the $\mathbf{u}_i$ are unit vectors. So (9) holds.

Now consider the remaining constraints. Fix $j, k \in \{0, 1, \ldots, 2^m\}$. Using constraints (9), the fact that $Y_{ii} = Y_{i0}$ for all $i$, and the fact that $\mathbf{y}$ is an $\epsilon$-vector in $VC(G)$, it is easy to verify that constraints (10) and (11) hold whenever one of $y_j$ or $y_k$ are integral. So assume $y_j = y_k = \frac{1}{2} + \epsilon$.

Constraint (10) then holds if the following is at least 1:

$$\begin{aligned} \frac{Y_{ij} + Y_{ik}}{Y_{i0}} &= 2\left(\frac{1}{2} + \epsilon\right) + \frac{1 - 2\epsilon}{2} T_P(\mathbf{u}_i) \cdot (T_P(\mathbf{u}_j) + T_P(\mathbf{u}_k)) \\ &= 1 + 2\epsilon + \frac{1 - 2\epsilon}{2} (P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k)). \end{aligned} \tag{12}$$

Similarly, (11) holds if the following is at least 1:

$$\frac{(Y_{0j} - Y_{ij}) + (Y_{0k} - Y_{ik})}{Y_{00} - Y_{i0}} = 1 + 2\epsilon - \frac{1+2\epsilon}{2}(P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k)). \tag{13}$$

But by property 4 of a $(\gamma, \epsilon, m)$-useful polynomial, for all $1 \le i \le n$ and all $jk \in E$, (12) and (13) are indeed both at least 1. The claim follows. ∎

By Lemma 2, to complete the proof of Lemma 3 it suffices to show that there exists a $(\gamma, \epsilon, m)$-useful polynomial $P$ such that if $Y = Y(P, \mathbf{y})$, then for all $i$ such that $y_i = \frac{1}{2} + \epsilon$ the vectors $Y\mathbf{e}_i/y_i$ and $Y(\mathbf{e}_0 - \mathbf{e}_i)/(1 - y_i)$ are $(\epsilon - 6\gamma)$-saturated. (The vectors $Y\mathbf{e}_i/y_i$ and $Y(\mathbf{e}_0 - \mathbf{e}_i)/(1 - y_i)$ are the "normalized" versions of $Ye_i$ and $Y(\mathbf{e}_0 - \mathbf{e}_i)$, i.e., their projections onto the hyperplane $x_0 = 1$.)

To that end, let us first compute the saturation of these vectors for an arbitrary but fixed $(\gamma, \epsilon, m)$-useful polynomial $P$. Fix $i$ such that $y_i = \frac{1}{2} + \epsilon$ and consider $Y\mathbf{e}_i/y_i$. Let $I = \{i\} \cup \{j : y_j \in \{0, 1\}\}$. Then the saturation of $Y\mathbf{e}_i/y_i$ is at least

$$\begin{aligned}
&\min_{j,k \notin I, jk \in E} \frac{1}{2}((Y_{ij} + Y_{ik})/y_i - 1) \\
&= \min_{j,k \notin I, jk \in E} \left[\epsilon + \frac{1-2\epsilon}{4}(P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k))\right] \\
&\ge \min_{j,k \neq i, jk \in E} \left[\epsilon + \frac{1-2\epsilon}{4}(P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k))\right], \qquad (14)
\end{aligned}$$

where the equality follows by (12) and the fact that $y_j, y_k \notin \{0, 1\}$. Similarly, the saturation of $Y(\mathbf{e}_0 - \mathbf{e}_i)/(1 - y_i)$ is at least

$$\begin{aligned}
&\min_{j,k \notin I, jk \in E} \frac{1}{2}\left(\frac{(Y_{0j} - Y_{ij}) + (Y_{0k} - Y_{ik})}{1 - y_i} - 1\right) \\
&= \min_{j,k \notin I, jk \in E} \left[\epsilon - \frac{1+2\epsilon}{4}(P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k))\right] \\
&\ge \min_{j,k \neq i, jk \in E} \left[\epsilon - \frac{1+2\epsilon}{4}(P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k))\right], \qquad (15)
\end{aligned}$$

where the equality follows by (13) and the fact that $y_j, y_k \notin \{0, 1\}$.

Lemma 3 now follows from the following lemma proved in section 4.2, which shows that $(\gamma, \epsilon, m)$-useful polynomials of the type we require do in fact exist.

LEMMA 4. *Let $m$ be an integer and $\gamma$ a sufficiently small positive real such that $\frac{m}{2\gamma}$ and $\frac{1}{2\gamma}$ are even integers and $m$ is significantly larger than $\frac{1}{\gamma}$. Suppose $\epsilon > 5\gamma$. Then there exists a $(\gamma, \epsilon, m)$-useful polynomial $P$ such that for all $i, j, k \in \{-1, 1\}^m$ where $j, k \neq i$ and $jk \in E$,*

$$|P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k)| \le 20\gamma. \tag{16}$$

**4.2. Proof of Lemma 4: Constructing $(\gamma, \epsilon, m)$-useful polynomials.** In this section we prove Lemma 4. Before giving the proof, we motivate the construction of the required polynomial $P$ and discuss how we arrived at the correct definition.

Requirement 2 of Definition 5, namely, that $P(1) = 1$, ensures that $\mathbf{v}_0 \cdot \mathbf{v}_i = \mathbf{v}_i^2$ and implies that the sum of the monomial coefficients of $P$ are 1. In other words, the required polynomial $P$ will be a weighted average of the "pure" monomials $x^s$, $s \in \{0, 1, \ldots\}$. As was discussed before, the Frankl–Rödl graph $G_m^\gamma$ with $\gamma = 0$ is just a perfect matching on the vertices of the cube. A useful observation is that for $G_m^0$, Lemma 4 can be proved by taking $\epsilon = 0$ and $P(x) = x$. (Of course, this graph is useless for proving lower bounds since a bipartite graph exhibits no integrality gap.) From a geometric standpoint, viewing the vertices as the set $\{-1, 1\}^n \subset \mathbb{R}^n$, the graph $G_m^\gamma$ is a small perturbation of $G_m^0$. Therefore, intuitively, the polynomial $P$ we will use to prove the lemma should be a small perturbation of the polynomial $x$. That is, in the weighted average of pure monomials making up $P$, most of the weight should be placed on the linear part.

The evolution in previous works of integrality gap constructions can be viewed as an evolution of the underlying tensoring polynomials $P$ those constructions were based on. Indeed, while not described quite in this language (as well as not addressing the lift-and-project framework) one can view the integrality gap constructions for the standard SDP of Goemans and Kleinberg [13] as a solution using the tensoring polynomial $P(x) = x$ and the SDP considered by Charikar [6] (whose SDP is in fact equivalent to one round of $LS+$) as using a polynomial $P$ with weight roughly $1 - \gamma$ on $x$ and roughly $\gamma$ on the monomial $x^{1/\gamma}$. Charikar's considerably more complicated construction is in fact a result of satisfying requirement 3 of Definition 5, which necessitates that the polynomial will attain its minimum at $2\gamma - 1$.

For our purposes, we start with two parameters $\epsilon$ and $\gamma$, with $\epsilon \gg \gamma$, and we need to satisfy requirement 3 of Definition 5 as well as inequality (16). A key observation is that by using any polynomial with small (say, $O(1/\gamma)$) degree, such as that of Charikar, we get that $P(1 - 1/m)$ is almost the same as $P(1)$. But then we will not satisfy inequality (16) for triplets $i, j, k$ such that $ij \in E$ and $d_H(i, k) = 1$, since for such a triplet we will have

$$\begin{aligned} P(\mathbf{u}_i \cdot \mathbf{u}_j) + P(\mathbf{u}_i \cdot \mathbf{u}_k) &= P(2\gamma - 1) + P(1 - 2/m) \\ &= -(1 - 2\epsilon)/(1 + 2\epsilon) + 1 - (P(1) - P(1 - 2/m)) \\ &\approx 4\epsilon \\ &\gg \gamma. \end{aligned}$$

We rectify this by ensuring that $P(x) < P(1) - \Theta(\epsilon)$ for all $x \leq 1 - 1/m$ and by designing $P$ so that $P(1)$ "jumps" to 1 (as required by requirement 2 of Definition 5). This can be achieved by adding a very-high-order term to the weighted combination making up $P$.

So to summarize, to satisfy the conditions of Lemma 4, our polynomial $P$ will be a weighted average of a linear monomial, a midorder monomial, and a very-high-order monomial, with most of the weight on the linear monomial. Indeed, all three components will be seen in the proof of Lemma 5 below.

Here is another, possibly simpler, way to think of the type of transformation we need to apply to the vector solution for $G_m^0$ to obtain an integrality gap solution for $G_m^\gamma$. Take a polynomial à la Charikar's which satisfies inequality (16) but violates requirement 3 of Definition 5 in that $P(2\gamma - 1) = 1 - \Theta(\gamma)$. The trick is to rescale all its values *other than its value at* 1 by $1 - \epsilon$ to achieve requirement 3, while not compromising on inequality (16). This can be done by rescaling the vectors by $\sqrt{1 - \epsilon}$ and adding a new orthogonal component to all vectors involved unique to each vertex, scaled by $\sqrt{\epsilon}$. This would have achieved the same effect, namely, the inner product of any two different vectors would be always bounded by $1 - \epsilon$. Notice that in this way we don't actually produce a polynomial $P$ that satisfies inequality (16), but rather directly produce vectors that exhibit the right behavior. We therefore opt to use the first approach that is expressible completely in the language of polynomials.

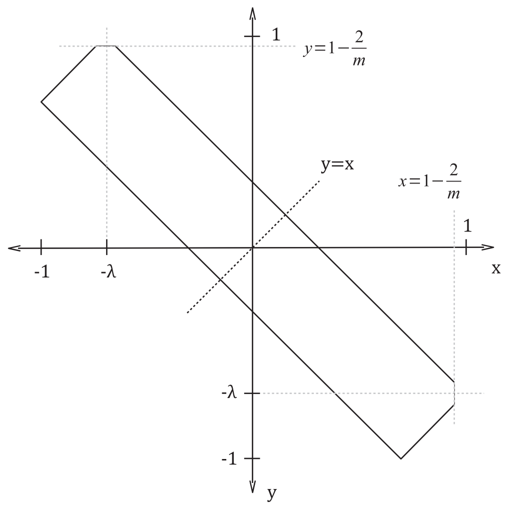


FIG. 1. *The domain $R$. The value $-\lambda$ represents the critical inner product $-1+2\gamma$.*

We now move to the proof of the lemma: Fix $\epsilon$ and $\gamma$ as in the statement of the Lemma 4. Let $R$ be the subset of $\mathbb{R}^2$ that consists of all $(x, y) \in [-1, 1]^2$ for which $|x+y| \le 2\gamma$, $|x-y| \le 2(1-\gamma)$, $x < 1 - \frac{1}{m}$, and $y < 1 - \frac{1}{m}$ (see Figure 1).

CLAIM 2. *To prove the lemma it suffices to find a polynomial $P$ with nonnegative coefficients such that $P(1) = 1$, for all $x \in [-1, 1]$ $P(x) \ge P(2\gamma-1) = (2\epsilon-1)/(2\epsilon+1)$ and such that*

$$|P(x) + P(y)| \le 20\gamma \quad \forall (x, y) \in R. \tag{17}$$

*Proof.* By definition, $P$ satisfies the first three properties of a $(\gamma, \epsilon, m)$-useful polynomial.

Next recall that the vectors $\mathbf{u}_i$ satisfy the property $-1 \le \mathbf{u}_i \cdot \mathbf{u}_j \le 1 - \frac{2}{m}$ for all $1 \le i \ne j \le 2^m$. Further, if $jk \in E$ and $i \ne j, k$, then since $\mathbf{u}_j + \mathbf{u}_k$ is supported on $\gamma m$ coordinates on which it assumes values $\pm 2/\sqrt{m}$, we get that

$$|\mathbf{u}_i \cdot \mathbf{u}_j + \mathbf{u}_i \cdot \mathbf{u}_k| = |\mathbf{u}_i \cdot (\mathbf{u}_j + \mathbf{u}_k)| \le 2\gamma.$$

Similarly, $|\mathbf{u}_i \cdot \mathbf{u}_j - \mathbf{u}_i \cdot \mathbf{u}_k| \le 2(1-\gamma)$. Hence, $\{(\mathbf{u}_i \cdot \mathbf{u}_j, \mathbf{u}_i \cdot \mathbf{u}_k) : j, k \ne i \text{ and } jk \in E\} \subseteq R$. So (17) implies (16). Moreover, since $5\gamma < \epsilon$, it implies property 4 of a $(\gamma, \epsilon, m)$-useful polynomial in all cases except when $i = k$. However, in that case we have

$$P(\mathbf{u}_i \cdot \mathbf{u}_i) + P(\mathbf{u}_i \cdot \mathbf{u}_j) = P(1) + P(2\gamma - 1) = 1 + \frac{2\epsilon - 1}{2\epsilon + 1} = \frac{4\epsilon}{1 + 2\epsilon},$$

and hence property 4 holds in that case too. ∎

Lemma 4 now follows from the following technical lemma.

LEMMA 5. *Let $m$ be an integer and $\gamma$ a sufficiently small positive real such that $\frac{1}{\gamma}$ is an even integer and $m$ is significantly larger than $\frac{1}{\gamma}$. Let $\epsilon > 3\gamma$ be sufficiently small. Then there exists a polynomial $P$ satisfying the conditions in Claim* 2.

*Proof.* Let $P(x) = \Delta(x+1)x^{\frac{2m}{\gamma}} + cx^{\frac{1}{\gamma}} + (1 - c - 2\Delta)x$, where $c$, $\Delta$ are positive constants we will define below so that $P$ satisfies the conditions of the lemma. Note that $P$ has a "high" degree component (i.e., $\Delta(x+1)x^{\frac{2m}{\gamma}}$) which vanishes at $-1$ as

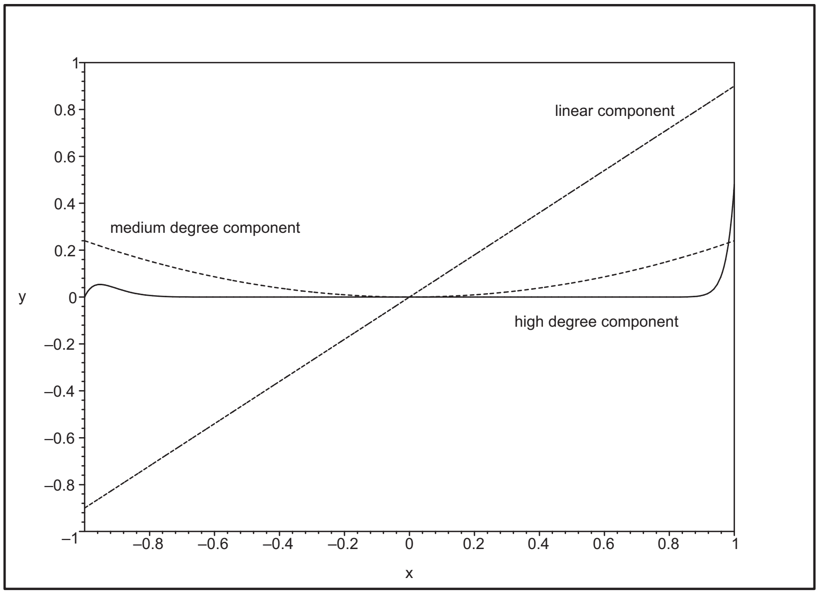


FIG. 2. *Relative behavior of the three components of $P$.*

well as a "medium" degree and a linear component (see Figure 2). Note also that $P(1) = 1$.

Necessary conditions for ensuring that $P(x) \geq P(2\gamma - 1) = (2\epsilon - 1)/(2\epsilon + 1)$ for $x \in [-1, 1]$ are that

$$\begin{cases} P'(2\gamma - 1) = 0, \\ P(2\gamma - 1) = \frac{2\epsilon - 1}{2\epsilon + 1}. \end{cases} \tag{18}$$

These two (linear) conditions immediately determine the values of $c$ and $\Delta$, though for our needs here, a rough estimation of $c$ and $\Delta$ with respect to $\gamma$ and $\epsilon$ will suffice. To that end, observe that when $x$ takes values close to $2\gamma - 1$, then the high order term $(x + 1)x^{\frac{2m}{\gamma}}$ in $P, P'$ is negligible compared to the other terms. Therefore, the following system is a good approximation of the conditions in (18):

$$\begin{cases} \left(\frac{1}{\gamma}(2\gamma - 1)^{\frac{1}{\gamma} - 1} - 1\right) c & -2\Delta = -1, \\ \left((2\gamma - 1)^{\frac{1}{\gamma}} + 1 - 2\gamma\right) c & +2(1 - 2\gamma)\Delta = \frac{4\epsilon}{2\epsilon + 1} - 2\gamma. \end{cases} \tag{19}$$

Recall here that $\epsilon$ is fixed, $1/\gamma$ is even, and that $\gamma = \sqrt{\log m/m}$ goes to 0 as $m$ grows. Taylor series then give that $(-1 + 2\gamma)^{1/\gamma} = e^{-2} - 2e^{-2}\gamma - O(\gamma^2)$. Given this estimation, it is easy to derive from (19) the following rough bounds that suffice for our analysis below:

$$\frac{2\epsilon}{1 + 2\epsilon} - 5\gamma < \Delta < 3\epsilon,$$
$$7\gamma < c < 8.5\gamma.$$

Note that since $\epsilon > 3\gamma$, these bounds ensure that $P$ has positive coefficients.

Next we verify that these bounds ensure that $P(x) \geq P(2\gamma - 1)$ for $x \in [-1, 1]$. Since $\frac{1}{\gamma}$ is even, $P''(x)$ is at least

$$\Delta \left(\frac{2m}{\gamma} + 1\right) \frac{2m}{\gamma} x^{\frac{2m}{\gamma}-1} + \Delta \left(\frac{2m}{\gamma} - 1\right) \frac{2m}{\gamma} x^{\frac{2m}{\gamma}-2}.$$

It is not hard to see then that $P''(x) \geq 0$ whenever $x \geq -1 + \frac{2\gamma}{2m+\gamma}$. So since $P'(2\gamma - 1) = 0$, it follows that $P(x) \geq P(2\gamma - 1)$ whenever $x \geq -1 + \frac{2\gamma}{2m+\gamma}$. It is more difficult to estimate $P''$ when $x < -1 + \frac{2\gamma}{2m+\gamma}$; instead, we will bound $P(x)$ directly for such $x$: our lower bounds for $c$ and $\Delta$ and the fact that $m$ is sufficiently large imply that for $x < -1 + \frac{2\gamma}{2m+\gamma}$,

$$\begin{aligned} P(x) &> c\left(1 - \frac{\gamma}{m}\right)^{\frac{1}{\gamma}} - (1 - c - 2\Delta) \\ &> -1 + 2\Delta + c + ce^{-\frac{1}{m}} \\ &> -\frac{1-2\epsilon}{1+2\epsilon} - 4\gamma + 0.9c \\ &> P(2\gamma - 1). \end{aligned}$$

Hence, $P(x) \geq P(2\gamma - 1)$ for every $x$ in $[-1, 1]$.

It remains to prove that $|P(x) + P(y)| \leq 20\gamma$ on $R$. Firstly, since $m \gg 1/\gamma$, we (very generously) have that $(x+1)x^{\frac{2m}{\gamma}} < \frac{\gamma}{6\epsilon}$ when $x \in [-1, 1 - \frac{1}{m}]$. Secondly, $|x^{\frac{1}{\gamma}} + y^{\frac{1}{\gamma}}| \leq 2$ over $R$. Finally, by the definition of $R$, we have that $|x + y| \leq 2\gamma$ for all $(x, y) \in R$. Hence, for all $(x, y) \in R$, the expression $|P(x) + P(y)|$ is bounded from above by

$$\Delta \left|(x+1)x^{\frac{2m}{\gamma}} + (y+1)y^{\frac{2m}{\gamma}}\right| + c\left|x^{\frac{1}{\gamma}} + y^{\frac{1}{\gamma}}\right| + (1 - c - 2\Delta)|x + y|.$$

These three terms are at most $\gamma$, $17\gamma$, and $2\gamma$, respectively, implying that $|P(x) + P(y)| \leq 20\gamma$. ∎

**5. Discussion and open problems.** It is well known that lift-and-project systems derive all local linear constraints after appropriately many number of rounds. More precisely, after $r$ rounds all valid linear inequalities with support at most $r$ are derived. For systems with a PSD constraint, it is natural to ask whether a given lift-and-project system derives after some number of rounds all valid linear inequalities on inner products of support at most $r$. In other words, if integral solutions to the original system satisfy certain quadratic constraints, under what conditions can the lift-and-project system derive the corresponding constraints on vectors? For Vertex Cover, the triangle inequality is the quintessential example of such a quadratic inequality. More generally, integral Vertex Cover solutions satisfy any $\ell_1$ inequality, namely, any inequality on vector distances with respect to the $\ell_2^2$ norm that is valid for all distance functions that can be represented in some $\ell_1$ space. It was shown [11] that for most graphs, the triangle inequality and, in general, any pure hypermetric inequality is not produced by the $LS_+$ system when applied to the standard linear relaxation for Vertex Cover. In particular, this is true for the Frankl–Rödl graph instances we consider in the current work. This clearly exposes a serious weakness of the system. It was, however, shown [12] that the SDP solution in the current work

Table 1

*Known lower bounds for* Vertex Cover *in the various linear and SDP systems. We remind that the NP-hardness is* 1.36*, while the unique games conjecture (UGC)-hardness is* $2-\epsilon$.

| | LS | SA | LS$_+$ | LA | LA |
|---|---|---|---|---|---|
| Integrality gap | $2-\epsilon$ | $2-\epsilon$ | $2-\epsilon$ | 1.36 | $7/6-\epsilon$ |
| $\sharp$ of rounds | $\Omega(n)$ | $\Omega(n^{\delta(\epsilon)})$ | $\Omega(n)$ | $\Omega(n^{\delta})$ | $\Omega(n)$ |

does in fact satisfy *all* hypermetrics[1] of small support, of which the triangle inequality is a special case. In contrast we do not know if the same holds true for local $\ell_1$ inequalities that are not hypermetric inequalities.

Lasserre [21] describes a lift-and-project system that does indeed eventually derive *all* valid vector inequalities. In light of [12] the most interesting inequalities derived by the Lasserre system for Vertex Cover are those $\ell_1$ inequalities that are not hypermetric inequalities. Currently, we do not know how to extend our arguments to prove lower bounds in the stronger Lasserre system.

In general, since $\ell_1$ constraints have proved powerful in tightening relaxations for problems such as Sparsest Cut [4], we believe that such an extension is of great importance. In Table 1 we summarize all currently known lower bounds for Vertex Cover in the various hierarchies. Interestingly, all lower bounds are incomparable, with the results for each system varying both in the size of the integrality gap and the number of rounds of lift-and-project tightenings. Any improvement to the results in Table 1 would be of great interest.

In particular, with respect to the result in the current paper, it would be very interesting to investigate how the integrality gap for Vertex Cover evolves beyond $\omega(\sqrt{\log n})$ rounds of $LS_+$. Note that our graph instances have odd girth (the length of the shortest odd length cycle) essentially $O(\frac{1}{\gamma}) \approx \sqrt{\log n}$ and that proving integrality gaps for Vertex Cover for more rounds than the odd girth proved quite challenging in the $LS$ context (see [28, 27]).

**Appendix A. Proof of Lemma 1.** The lemma we require is a fairly easy corollary of a lemma in [10] about sets avoiding intersections of a certain cardinality.

Lemma 6 (Theorem 1.4 in [10]). *Let $\eta$ be a sufficiently small number and $m$ an integer. Also, let $\mathcal{F}$ and $\mathcal{G}$ be two set families over the universe $[m]$ so that $|F\cap G| \neq \lfloor m\eta \rfloor$ for every $F\in\mathcal{F}$, $G\in\mathcal{G}$. Then $4^{-m}|\mathcal{F}||\mathcal{G}| \le (1-\eta^2/4)^m$.*

Our first step is to observe that Lemma 1 is equivalent to showing an identical bound to Lemma 6 on the size of a set-family over the universe $[m]$ such that every two sets have symmetric distance different than $(1-\gamma)m$. Indeed, consider the correspondence between points in $\{-1,1\}^m$ and subsets of $[m]$ in which the $\pm1$ vectors are the characteristic vectors of the corresponding sets. In this correspondence the Hamming distance between points is the symmetric difference between the sets they represent. Finally, an independent set in $G_m^\gamma$ is a set of points in $\{-1,1\}^m$ so that no two have Hamming distance $(1-\gamma)m$.

Let $\mathcal{A}$ be a family of sets so that the symmetric difference between any two sets in $\mathcal{A}$ is not $(1-\gamma)m$ and consider its partition

$$A_k = \{S\in\mathcal{A} : |S| = k\}.$$

Let $w\in\{0,1,\ldots,m\}$ be such that $A_w$ is the largest family among the $A_k$. We may assume here that $w\le m/2$, as otherwise we can work with the complementary vectors.

[1]Hypermetric inequalities are a natural countable class of $\ell_1$ inequalities.

Next we observe that for two sets $S, T \in A_w$ we have that

$$(1-\gamma)m \neq |S \triangle T| = |S| + |T| - 2|S \cap T| = 2(w - |S \cap T|),$$

and therefore

$$|S \cap T| \neq w - (1-\gamma)m/2. \tag{20}$$

We proceed now to bound $|\mathcal{A}|$. For this we will examine two cases for the value of $w$.

*Case* 1. First assume that $|w/m - 1/2| \geq \gamma/4$. In this case the naive bound $|A_w| \leq \binom{m}{w}$ suffices. Indeed, using Chernoff bound for the estimation of binomial coefficients we get

$$\begin{aligned}\binom{m}{(m/2)(1-\gamma/4)} &\leq 2^m \exp(-(m\gamma/2)^2/(4m)) \\ &= 2^m \exp(-m\gamma^2/64),\end{aligned}$$

and therefore $|\mathcal{A}| \leq (m+1) \cdot 2^m (1-\gamma^2/64)^m$.

*Case* 2. In the more interesting case in which $w > \frac{m}{2}(1-\gamma/2)$ it follows from inequality (20) and the fact that $w \leq m/2$ that $A_w$ is a family that avoids intersections of size $\eta m$, where $\gamma/4 < \eta \leq \gamma/2$. We now apply Lemma 6 with $\mathcal{F} = \mathcal{G} = A_w$ to get

$$|A_w| \leq 2^m(1-\eta^2/4)^{m/2} \leq 2^m(1-\gamma^2/64)^{m/2} \leq 2^m \exp(-m\gamma^2/32).$$

As before $|\mathcal{A}| \leq (m+1) \cdot 2^m \exp(-m\gamma^2/32)$, which completes the proof of Lemma 1. ∎

It is interesting to note that the above estimate is nearly tight: consider the family $\mathcal{A}$ of all sets of cardinality less than $(1-\gamma)m/2$. Clearly this family avoids symmetric differences of cardinality $(1-\gamma)m$. Now $|\mathcal{A}| = \sum_{j<\frac{m}{2}(1-\gamma)} \binom{m}{j}$, which is at least $\frac{\gamma m}{2}\binom{m}{\frac{m}{2}(1-2\gamma)}$. The last expression can be further bounded from below by

$$\frac{\gamma m}{2} 2^{mH(1/2-\gamma)} \sim \frac{\gamma m}{2} 2^{m(1-\gamma^2/4)} = 2^m \frac{\gamma m}{2} 2^{-\gamma^2 m/4}.$$

So for $|\mathcal{A}|$ to be $o(2^m)$ we must have that $\gamma m 2^{-\gamma^2 m/4} = o(1)$, and so $\gamma = \Omega(\sqrt{\log m/m})$.

**Appendix B. Proof of Lemma 2.** For completeness, we include in this section a proof of the lemma by Schoenebeck, Trevisan, and Tulsiani [27] (Lemma 2 here) for expressing an $\epsilon$-saturated vector as a convex combination of $\epsilon$-vectors.

*Proof*. Partition $V$ as follows: Let $V_- = \{i \in V : x_i < 1/2 + \epsilon\}, V_+ = \{i \in V : x_i > 1/2 + \epsilon\}, V_0 = \{i \in V : x_i = 1/2 + \epsilon\}$. Let $r(0) = 0$, and for all $i \in V$ let

$$r(i) = \begin{cases} 1 - \frac{x_i}{1/2+\epsilon}, & i \in V_-, \\ 1, & i \in V_0, \\ 1 - \frac{1-x_i}{1/2-\epsilon}, & i \in V_+, \end{cases}$$

setting at the end the maximum of the $r(i)$'s equal to 1. Note that since $\mathbf{x}$ is $\epsilon$-saturated, whenever $ij \in E$ and $i \in V_-$, we must have $j \in V_+$. Moreover, for such a pair we must have that $r(j) \geq r(i)$ because

$$\begin{aligned} r(j) - r(i) &= 1 - \frac{1 - x_j}{1/2 - \epsilon} - \left(1 - \frac{x_i}{1/2 + \epsilon}\right) \\ &= \frac{x_i}{1/2 + \epsilon} - \frac{1 - x_j}{1/2 - \epsilon} \\ &= \frac{x_i(1/2 - \epsilon) - (1 - x_j)(1/2 + \epsilon)}{(1/2 + \epsilon)(1/2 - \epsilon)} \\ &= \frac{x_i + x_j - (1 + 2\epsilon)}{2(1/4 - \epsilon^2)} + \frac{\epsilon(x_j - x_i)}{1/4 - \epsilon^2} \\ &> 0, \end{aligned}$$

where the last inequality follows from the fact that $\mathbf{x}$ is $\epsilon$-saturated.

Reorder the $r(i)$'s so that $0 = r(i_0) \leq r(i_1) \leq \cdots \leq r(i_{|V|})$. For each $t = 1, \ldots, |V|$, let $\mathbf{x}^{(t)}$ be the $\epsilon$-vector where

$$x_i^{(t)} = \begin{cases} 0, & i \in V_- \text{ and } r(i) \geq r(i_t), \\ 1, & i \in V_+ \text{ and } r(i) \geq r(i_t), \\ \frac{1}{2} + \epsilon, & \text{otherwise.} \end{cases}$$

We claim these vectors are in $VC(G)$. To see why consider an edge $ij$. The constraint $x_i^{(t)} + x_j^{(t)} \geq 1$ is satisfied unless at least one of $x_i^{(t)}$ and $x_j^{(t)}$ is 0. However, if $x_i^{(t)} = 0$, then $i \in V_-$ and $r(i) \geq r(i_t)$. So the feasibility of $\mathbf{x}$ implies $j \in V_+$, and hence $r(j) \geq r(i_t)$. So $x_j^{(t)} = 1$, and the constraint is satisfied.

It remains to argue that $\mathbf{x}$ is in the convex hull of the $\mathbf{x}^{(t)}$'s. To that end, we define a distribution $\mathcal{D}$ over the vectors $\mathbf{x}^{(t)}$ such that $\mathbf{x}^{(t)}$ is assigned the probability $r(i_t) - r(i_{t-1})$. It is easy to verify now that $\mathrm{E}_t[x_j^{(t)}] = x_j$ for all $j \in V$. ∎

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