Automatizability and Interpolation

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Review proof system definition

- A proof system for a language **L** is a polynomial time algorithm **V** s.t.
  - for all inputs **x**
    - **x**∈**L** iff ***there exists a string P s.t. V accepts input (x,P)***
    -
  - think of **P** as a *proof* that **x** is in **L** and **V** as a proof *verifier*

Complexity of proof systems

- **Defn:** The **complexity** a proof system **V** is a function **f:N→N** defined by

$$\mathbf{f(n)} = \max_{\mathbf{x}\in\mathbf{L},|\mathbf{x}|=\mathbf{n}} \ \min_{\mathbf{P}:\ \mathbf{V}\text{ accepts }(\mathbf{x},\mathbf{P})} |\mathbf{P}|$$

  - i.e. how large **P** has to be as a function of **|x|**
  - **V** is **polynomially-bounded** **iff** its complexity is a polynomial function of **n**

- Definition says **nothing** about how costly it is to find short proofs!
  - lower bounds are even stronger that way

Automatizability (sic)

- **Defn:** Given a proof system **V** for **L** and a function **$f: N \times N \to N$** we say that **V** is **f(n,S)-automatizable** iff there an algorithm **A_V** s.t.
  - given any input **x** with **$|x|=n$**, if **$x \in L$**, **A** outputs a proof **P** in **V** of this fact in time at most **f(n,S)** where **S** is the size of the shortest proof in **V** that **x** is in **L**

- We say that **V** is **automatizable** iff it is **f(n,S)**-automatizable for some f that is **$n^{O(1)}S^{O(1)}$**
  - **i.e., can find a proof in time polynomial in the size of the smallest one**

Width & Automatizability

- **Theorem** [BW]: **Every Davis-Putnam (DLL)/tree-like resolution proof of F of size S can be converted to one of width** $\lceil \log_2 S \rceil + w(F)$

- **Corollary** [CEI][BP][BW]: **Tree-like resolution is** $S^{O(\log n)}$**-automatizable**

- **Proof:** There are only $2^{\log S}\binom{n}{\log S} = n^{O(\log S)}$ clauses of length at most $\log S$. Run breadth-first resolution only deriving clauses of width $\log S$. Can keep space requirements down by making it a recursive search.

Width, Resolution, and PCR

- **Theorem** [BW] **Every resolution proof of F of size S can be converted to one of width**

$$O(\sqrt{n \log S}) + w(F)$$

- **Corollary: General resolution is** $2^{O(\sqrt{n \log S} \log n)}$**-automatizable**

- **Theorem: Tree-PCR and PCR are** $S^{O(\log n)}$**-automatizable and** $2^{O(\sqrt{n \log S} \log n)}$**-automatizable respectively**
  - **There are roughly n^d monomials of degree at most d & Groebner-basis like algorithm does linear algebra in that basis**

Interpolation

- Given formulas
  - **A(x,z)** in variables **x** and **z**
  - **B(y,z)** in variables **y** and **z**

- **Defn:** If $A(x,z) \lor B(y,z)$ is a tautology then an **interpolant C** is a function s.t.
  - **for any truth assignment ζ to z**
    - **$C(\zeta)=0$ implies $A(x,\zeta)$ is a tautology**
    - **$C(\zeta)=1$ implies $B(y,\zeta)$ is a tautology**

- Also dual form if $A(x,z) \land B(y,z)$ is unsatisfiable

Interpolation - origin of the name

- Given formulas
  - **A(x,z)** in free variables **x** and **z**
  - **B(y,z)** in free variables **y** and **z**

- **Theorem:** [Craig] If **A(x,z)**®**B(y,z)** is a tautology then there is an interpolant **C** with only free variables **z** such that **A(x,z)**®**C(z)** and **C(z)**®**B(y,z)**.

  - i.e. given Ø**A(x,z)**Ú**B(y,z)**: **C(z)**®**B(y,z)**, Ø**C(z)**®Ø**A(x,z)**

Feasible Interpolation

- **Defn:** Given a propositional proof system **V** and a function **$f:N \rightarrow N$** we say that **V** has **f-interpolation** iff **given an unsatisfiable formula of the form**
  $A(x,z) \wedge B(y,z)$ with proof size S in V
  there is a circuit of size at most $f(S)$ computing
  an interpolant C for $A(x,z) \wedge B(y,z)$;
  i.e. that says which of $A(x,z)$ or $B(y,z)$ is false

- **V has feasible interpolation iff f is polynomial**

- **V has monotone f-interpolation iff whenever the variables z occur only negatively in B and only positively in A, the circuit C is a monotone circuit.**

Automatizability & Interpolation

- **Lemma:** [Impagliazzo,BPR] If **V** is automatizable then **V** has feasible interpolation

- Proof: Let f be the polynomial function such that V is f-automatizable and $\mathbf{A_V}$ be the associated algorithm.
- Given unsatisfiable $\mathbf{A(x,z) \wedge B(y,z)}$ and an assignment z to **z**:

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  - **Run A_V on input $A(x,z) \wedge B(y,z)$ to a proof P of size $S' \le f(S)$ where S is the size of its optimal proof in V**

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  - **Run A_V on input $A(x,z)$ for $f(S')$ steps**
    - **if it finds a proof output 0**
    - **else output 1**

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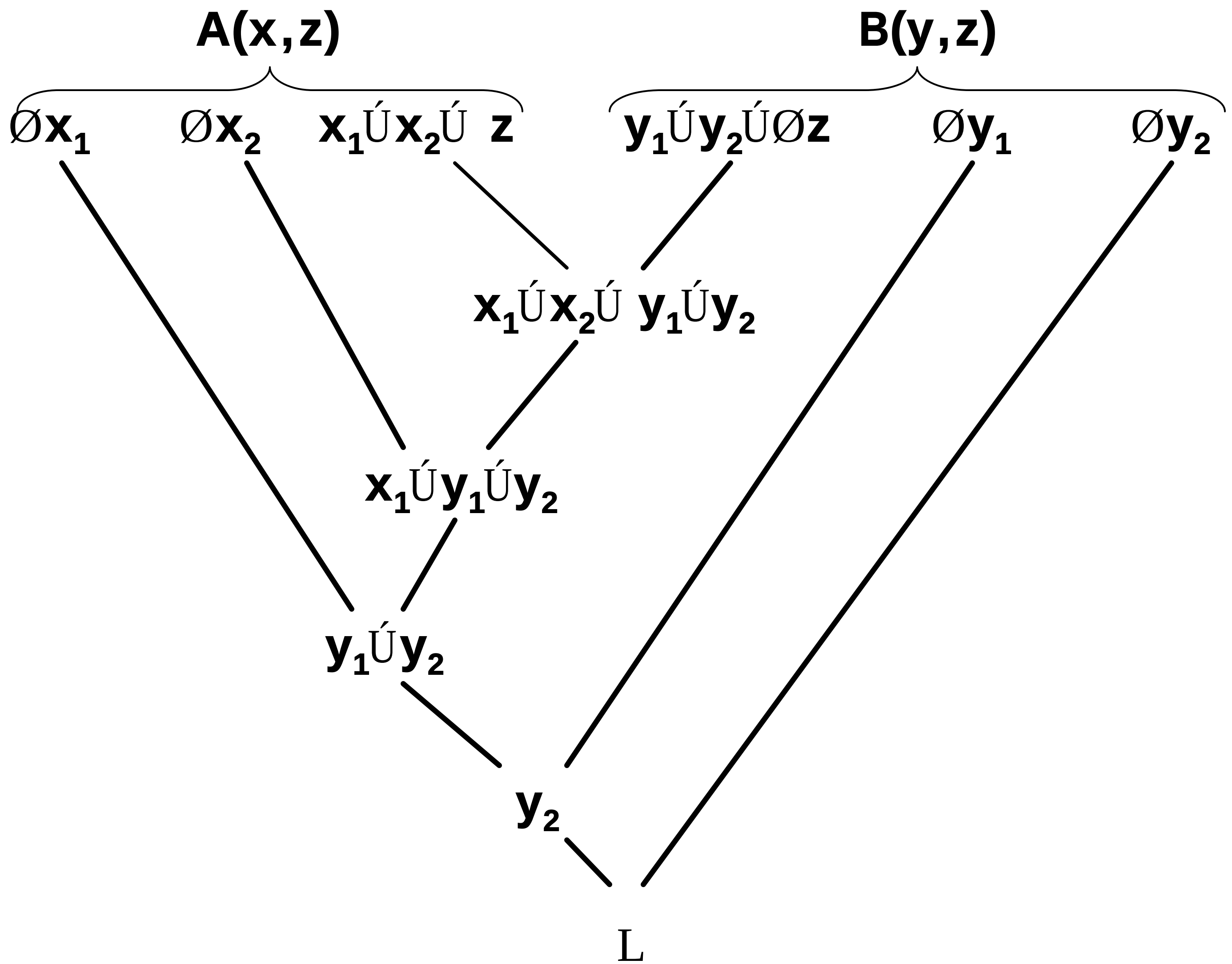
  - **Run A_V on input $A(x,\zeta)$ for $f(S')$ steps**
    - **if it finds a proof output 0**
    - **else output 1**
- **Note that if $B(y,\zeta)$ has satisfying assignment σ then plugging σ,ζ into the proof P yields a proof of size S' of unsatisfiability of $A(x,\zeta) \wedge B(\sigma,\zeta)$ which is $A(x,\zeta) \wedge 1$**

Interpolation and Resolution

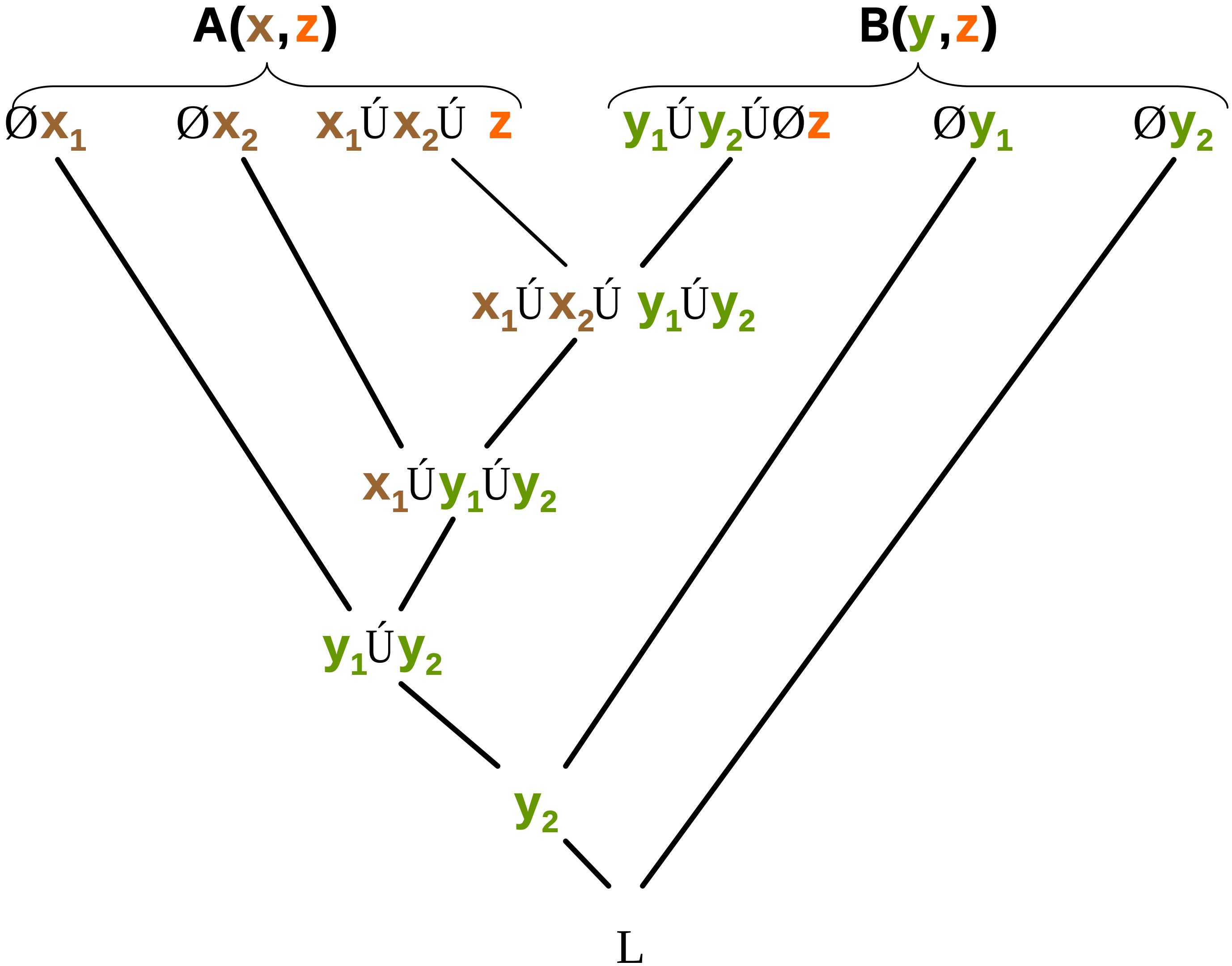
- **Theorem:** [Krajicek] Resolution has feasible (monotone) interpolation.

- **Proof idea:** structure of proof allows one to decide easily which clauses cause unsatisfiability

Interpolation for Resolution



Interpolation for Resolution



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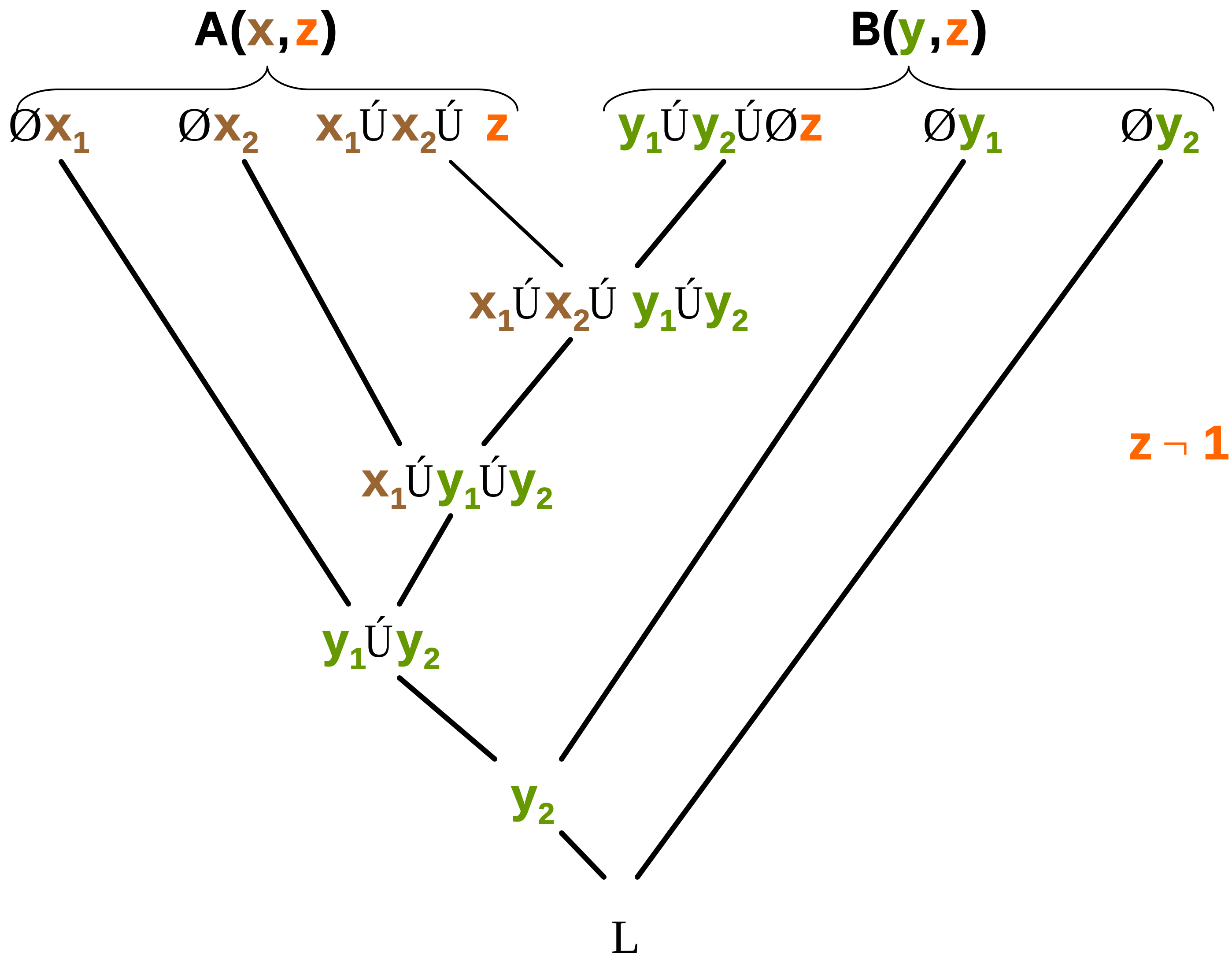

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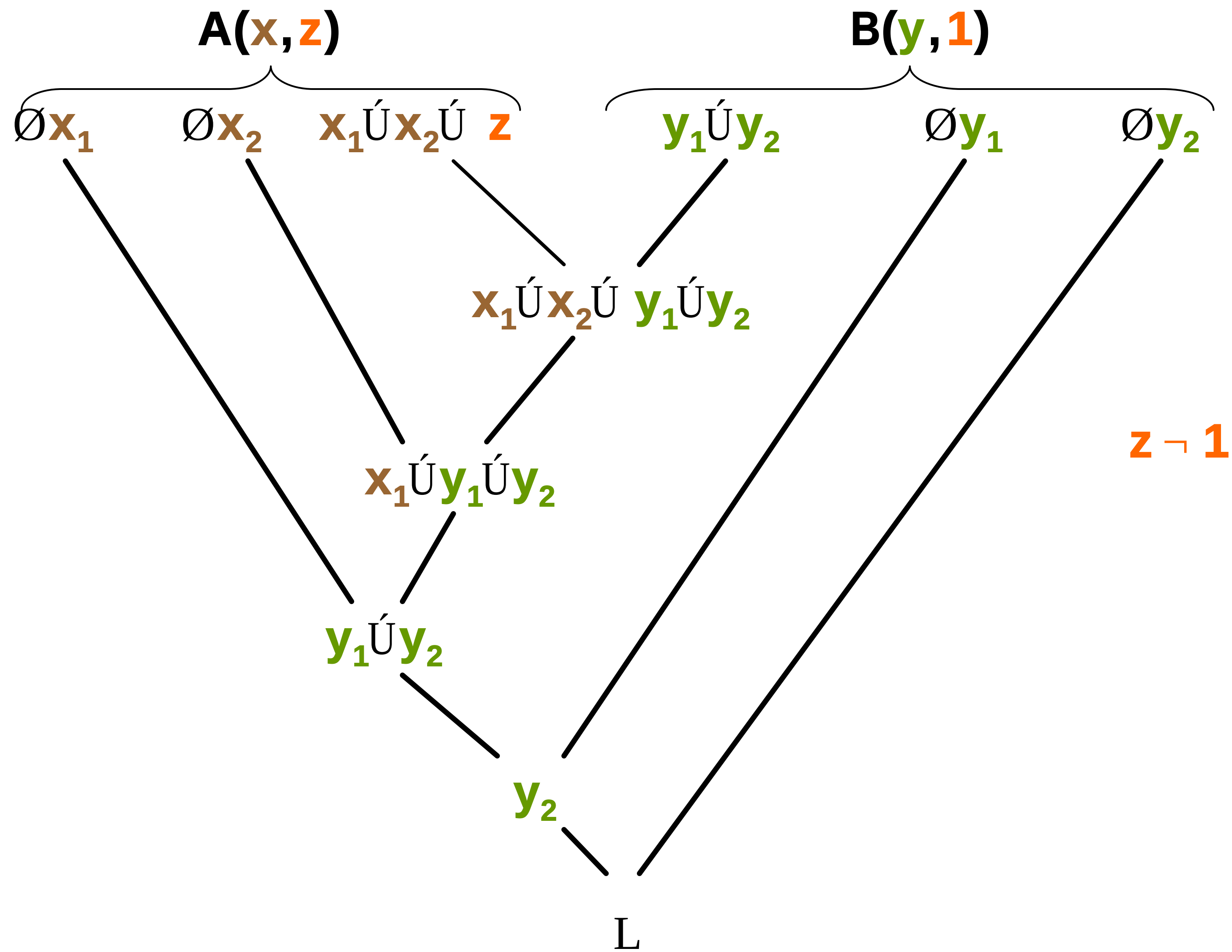
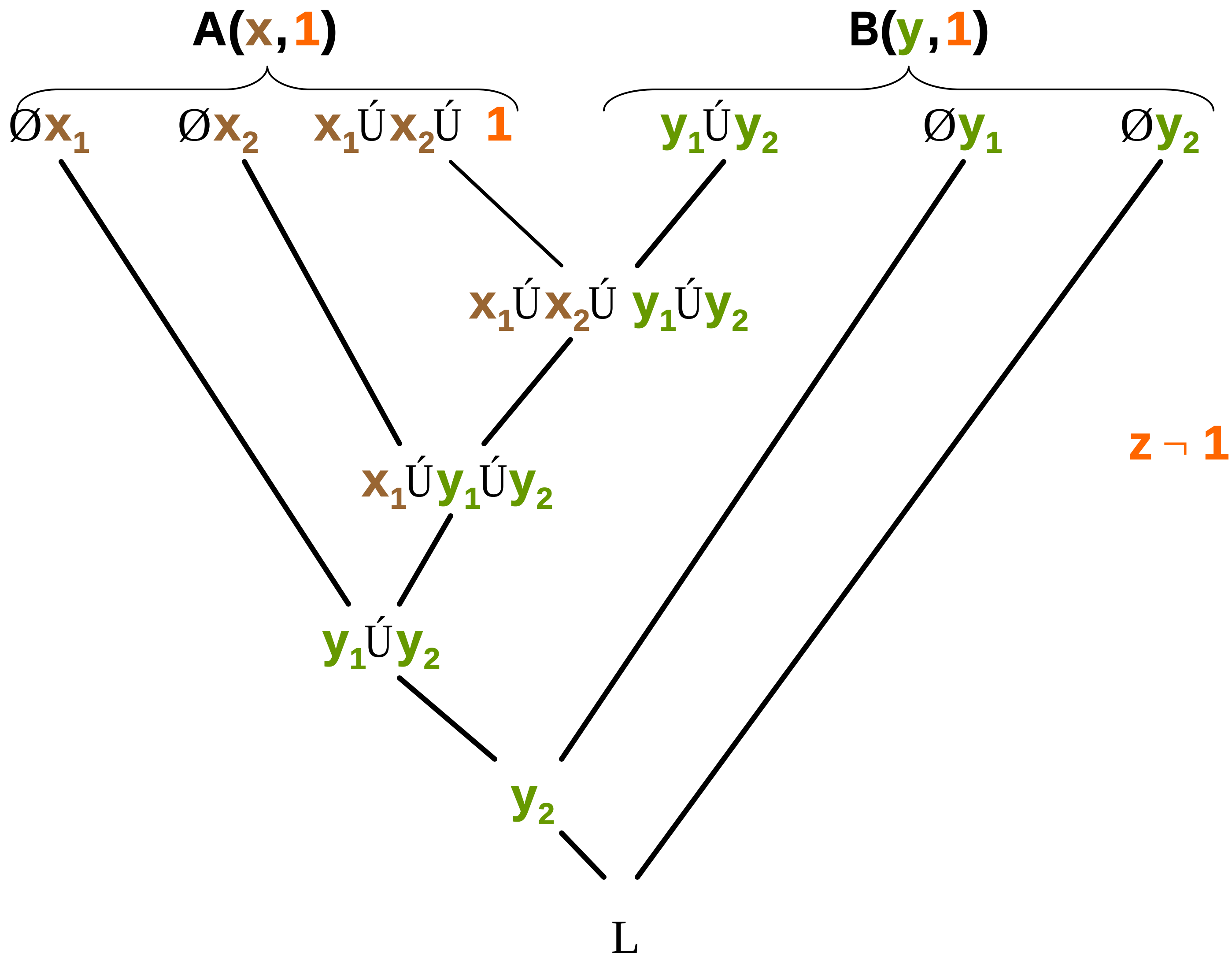


Interpolation for Resolution



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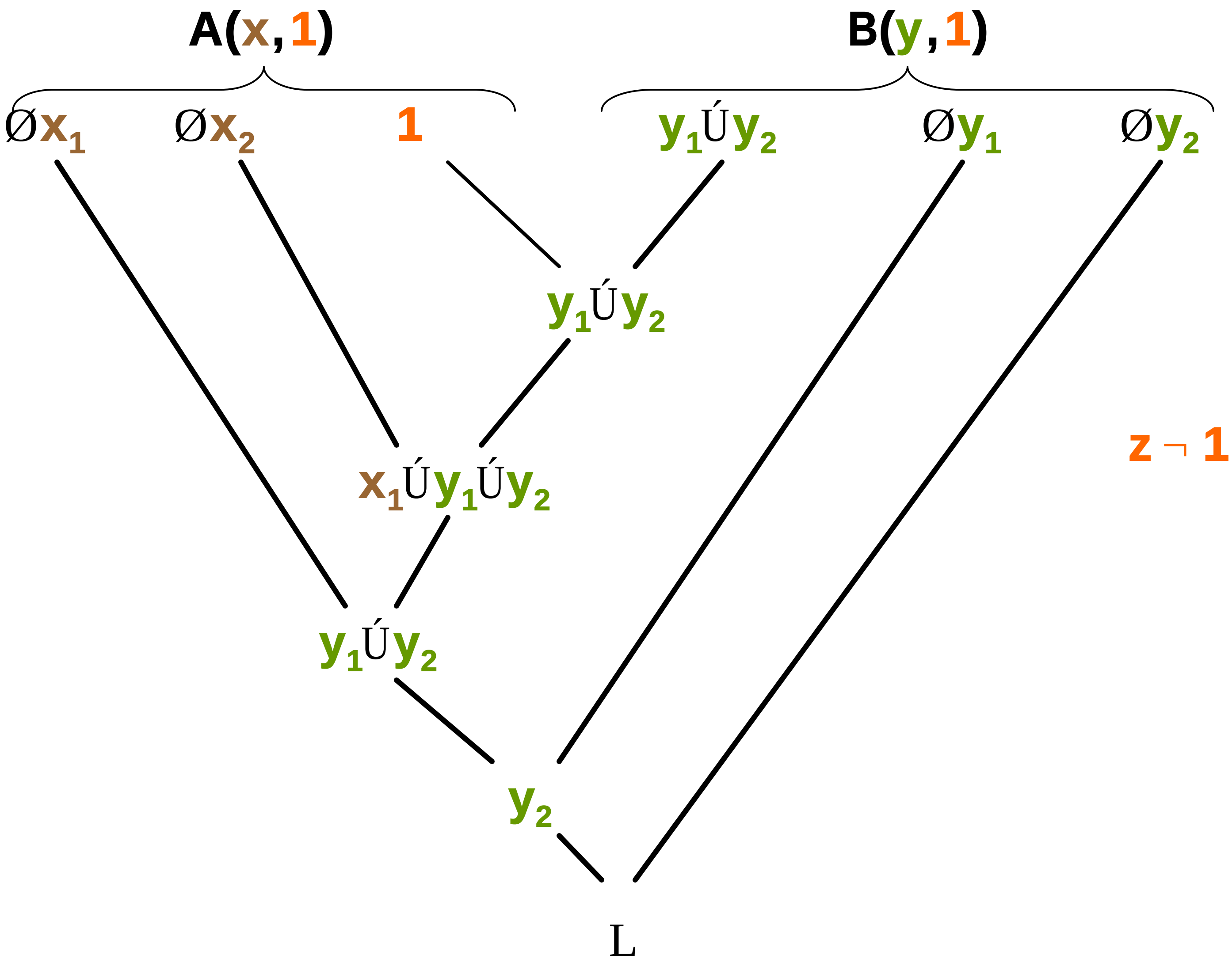
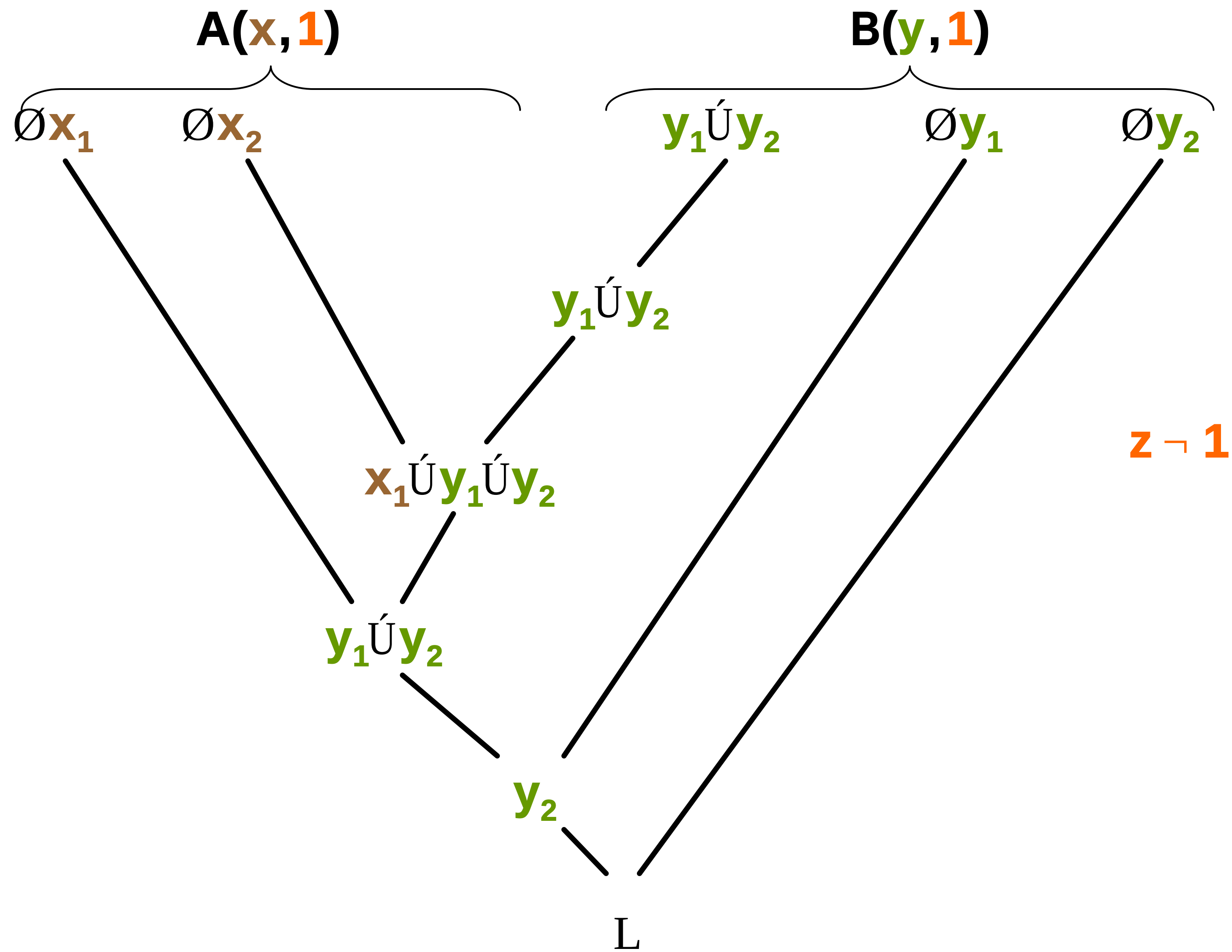
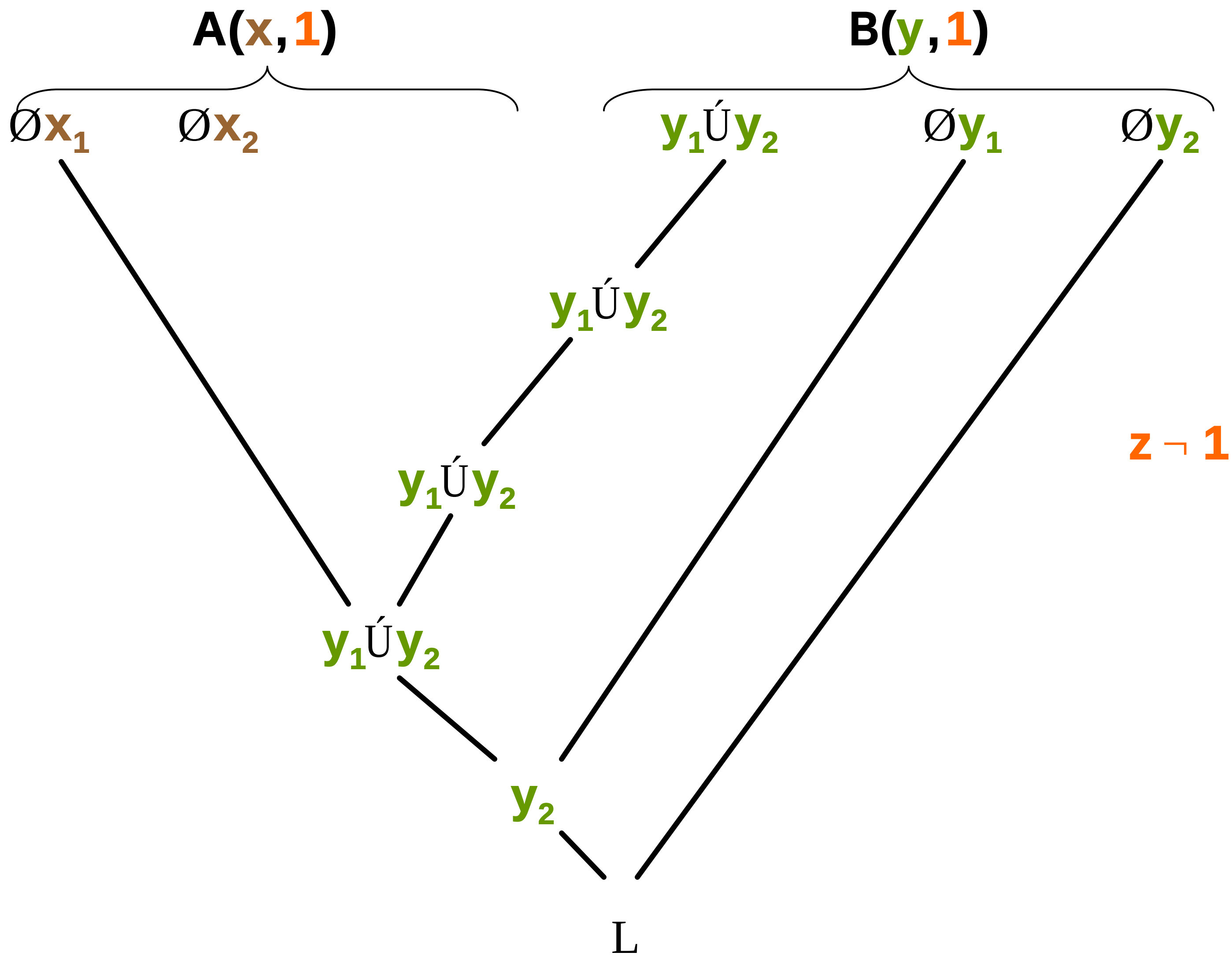


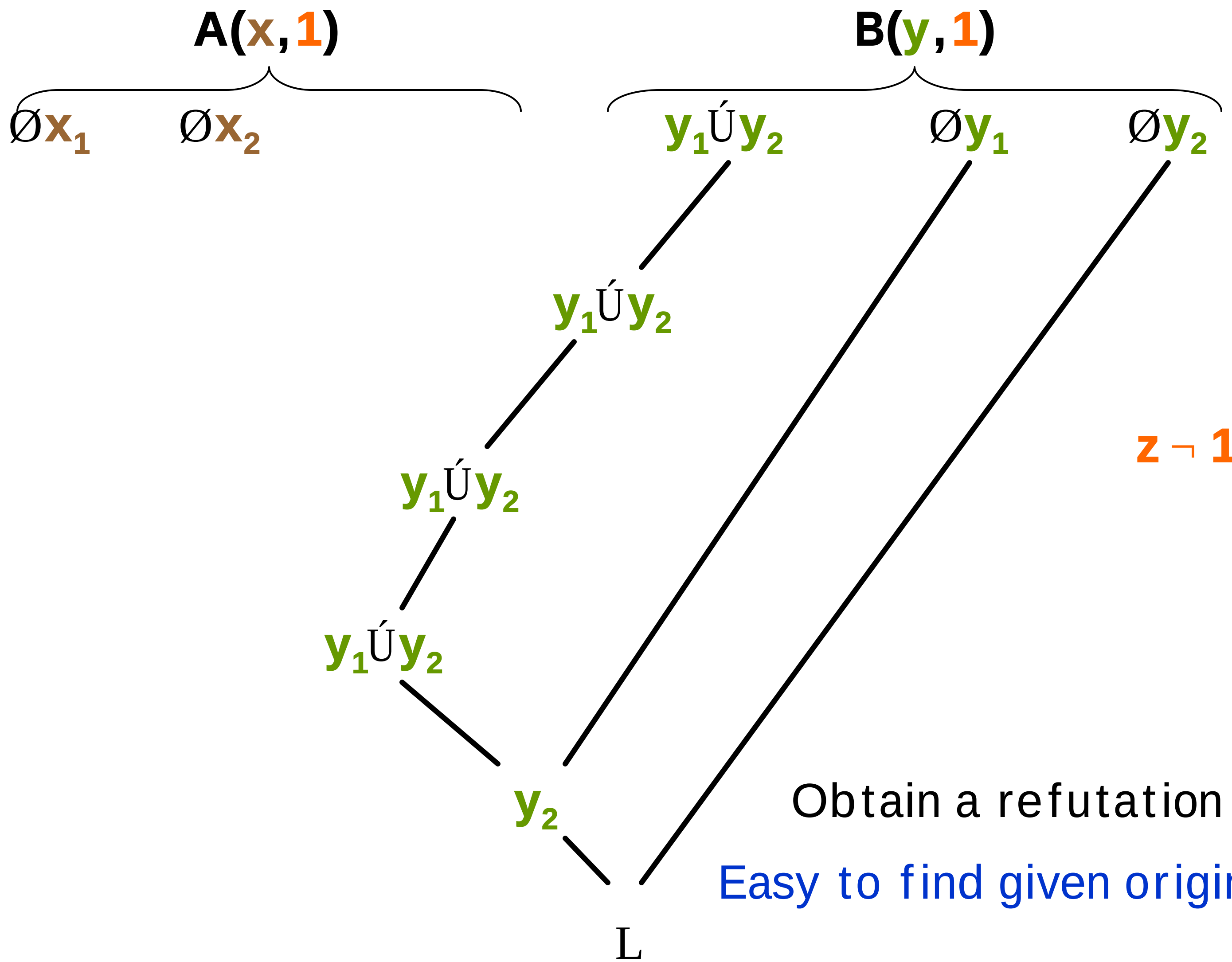
Interpolation for Resolution



Interpolation for Resolution



Interpolation for Resolution



Obtain a refutation of B(y,1)

Easy to find given original proof

Interpolation and Lower Bounds

- General proof strategy:
  - Given
    - a class of circuits for which one has lower bounds
    - a proof system whose interpolants are in the class
  - Build
    - a formula whose interpolant will be a circuit for a hard problem in the circuit class

Interpolation and Lower Bounds

- **Theorem:** If proof system **V** has feasible interpolation and **NPË P/poly** then **V** is not polynomially bounded

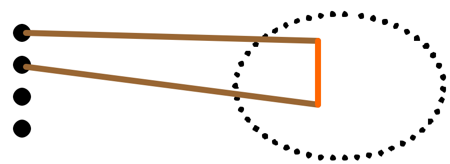
- **Theorem:** [BPR] Any proof system **V** that has monotone feasible interpolation is not polynomially bounded

Interpolation & NP vs P/poly

- **Proof sketch:** Suppose that **V** has feasible interpolation and is polynomially bounded with bound **p**
- Consider formula $\mathbf{A(x,z) \wedge B(y,z)}$ where
  - **z represents a CNF formula**
  - **A(x,z) says that assignment x satisfies z**
  - **B(y,z) says that y - of length p(|x|) - is a proof in V that z is unsatisfiable**
- Feasible interpolation for this formula will give a polysize circuit for deciding satisfiability

Clique-coloring formulas

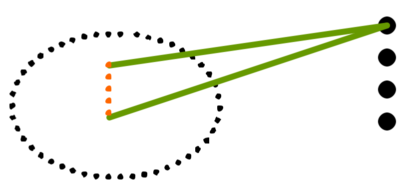
- Formula $A(x,z) \wedge B(y,z)$ where
  - z are the $n(n-1)/2$ variables representing an n node graph $G(z)$
  - $A(x,z)$ is the statement that $G(z)$ has a k-clique

k 

$(\bigvee_v x_{iv})$ $(\neg x_{iv} \vee \neg x_{ju} \vee z_{uv})$

$(\neg x_{iv} \vee \neg x_{jv})$ $(\neg x_{iu} \vee \neg x_{iv})$

  - $B(y,z)$ is the statement that $G(z)$ is $(k-1)$-colorable

 k-1

$(\bigvee_i y_{vi})$ $(\neg z_{uv} \vee \neg y_{ui} \vee \neg y_{vi})$

$(\neg y_{vi} \vee \neg y_{vj})$

Interpolation examples

- **Theorem:** [Krajicek] Resolution has feasible (monotone) interpolation.
- **Theorem:** [Pudlak 95] Cutting Planes has feasible (monotone) interpolation where the interpolants are circuits over the real numbers
  - Also extended monotone lower bounds for clique to real circuits
- **Corollary:** Any Cutting Planes proofs of clique-coloring formulas are exponential
- **Theorem:** Polynomial calculus has feasible interpolation

Limitations of Interpolation

- Theorem: [KP] If one-way functions exist then Frege systems do not have feasible interpolation.

- Theorem: [BPR, Bonet et al] If factoring Blum integers is hard then any proof system that can polynomially simulate TC^0-Frege, or even AC^0-Frege does not have feasible interpolation

Proof idea

- Suppose one has a method of **key agreement**
  - Given two people, one with **x** and another with **y**
  - via exchanging messages, they agree on a secret key **key(x,y)** so that even listening to their conversation without knowing **x** or **y** it is hard to figure out what even a bit of **key(x,y)** is
- The common variables **z** will represent the transcript of their conversation
  - **A(x,z)** will say that the player with **x** correctly computed its side of the conversation and the last bit of **key(x,y)** is **0**
  - **B(y,z)** will say that the player with **y** correctly computed its side of the conversation and the last bit of **key(x,y)** is **1**

Connections with proof systems

- Must encode the computation of each player in such a way that the proof system can prove given **x** and **z** what the value of the bit is
- Can extend **x** by helper extension variables to make the task easier
  - The actual proof uses Diffie-Hellman secret key exchange which is as hard as factoring
  - That requires powering which is not in TC^0 but the extension variables make it easy enough to prove

The Interpolation Line