Definition (Optimization problem associated with $\mathcal{S}$)

$\boxed{MLP_{\mathcal{S}}}$ Instance: A propositional tautology f

Solution: An $\mathcal{S}$-proof of f

Objective: Minimize the size of proof

* Runtime should be polynomial in the size of the shortest $\mathcal{S}$-proof of f.

A proof system $\mathcal{S}$ is automatizable if there is a polynomial-time algorithm that approximates $MLP_{\mathcal{S}}$ to within a polynomial factor.

Hardness Results for MLP

1. Buss 1995 : For a particular Frege system F_1, MLP_{F_1} is NP-hard

2. Iwama-Miyano 1995
   Iwama 1997 :

   $MLP_{\text{Resolution}}$ is NP-hard

3. Alekhnovich, Buss, Moran, Pitassi 1998:

   $P \neq NP \Rightarrow$ there is no polytime approximation scheme for $MLP_{\mathcal{S}}$

   $NP \not\subseteq QP \Rightarrow$ there is no polytime algorithm to approximate $MLP_{\mathcal{S}}$ to within a factor of $2^{\log^{(1-\epsilon)} n}$ for any ϵ

$\mathcal{S}$ can be almost any proof system:
Frege, Extended Frege, Sequent Calculus, Cut-Free Sequent Calculus, Resolution, Polynomial Calculus, ...
in tree-like or dag-like form.

Monotone Minimum Satisfying Assignment

$\boxed{\text{MMSA}}$

Instance: A monotone formula $\phi(x_1 \ldots x_n)$ over $\vee, \wedge$

Solution: A truth assignment τ s.t. $\phi(\tau)=1$

Objective: Minimize the number of 1's in τ

a. We give polynomial-time, approximation-preserving reductions from MMSA to MLP

b. We show MMSA cannot be ε-approximated (unless $P=NP$).

Proof sketch of a.

Let $\varphi(x_1 \ldots x_k)$ be an instance of MMSA, $|\varphi| = n$

Enumerate subformulas of φ as

$$\varphi_1, \varphi_2 \ldots \varphi_\ell \quad \text{(bottom-up)}$$

Γ_φ has the following clauses:

(a) $\{\bar{y}_\ell\}$ ← output value $= 0$

(b) $\forall i \leq \ell$ if $\varphi_i = (\varphi_j \wedge \varphi_k)$ then $\{\bar{y}_j \bar{y}_k y_i\} \in \Gamma_\varphi$

(c) $\forall i \leq \ell$ if $\varphi_i = (\varphi_j \vee \varphi_k)$ then $\{\bar{y}_j y_i\}$, $\{\bar{y}_k y_i\} \in \Gamma_\varphi$

$\wedge, \vee$ gates preserve truth values

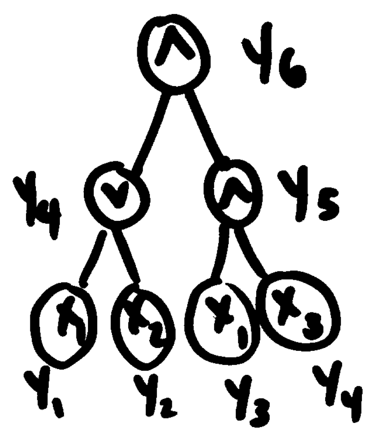
(d) $\forall i \leq k$

$$\{x_{i_1}\}\{\bar{x}_{i_1} x_{i_2}\}\{\bar{x}_{i_2} x_{i_3}\} \ldots \{\bar{x}_{i_m} y_i\} \in \Gamma_\varphi$$

input variable $y_i = 1$

Example

φ:



Γ_φ: $(\bar{y}_6)$

$(\bar{y}_4 \bar{y}_5 y_6)$

$\rightarrow (\bar{y}_1 y_4)$ $(\bar{y}_2 y_4)$

$\rightarrow (\bar{y}_1 \bar{y}_3 y_5)$

$(x_{11})(\bar{x}_{11} x_{12})(\bar{x}_{12} x_{13}) \ldots (\bar{x}_{1m} y_1)$

$(x_{21})(\bar{x}_{21} x_{22}) \quad \ldots \quad (\bar{x}_{2m} y_2)$

$(x_{31}) \quad \ldots \quad (\bar{x}_{3m} y_3)$

$y=1$, $y_3=1$ $y_4=1$

Refutation:

a. $(\bar{y}_4 \bar{y}_5)$, $(\bar{y}_1 \bar{y}_5)$, $(\bar{y}_1 \bar{y}_3 \bar{y}_4)$

b. (y_1) (y_3) (y_4)

c. ϕ

Lemma Let p = cardinality of the min. satisfying assignment for ϕ. Γ_ϕ has a tree-like Resolution refutation with $O(pm+n)$ clauses

(a) work top-down from $\{\bar{y}_e\}$ to derive $\{\bar{y}_{i_1}, \ldots, \bar{y}_{i_p}\}$, $i_1, \ldots, i_p \in P$

(b) Derive $\{y_i\}$ $\forall i \in P$

(c) Resolve to get ϕ

Lemma Let φ, p be as above.
Then any Resolution refutation has $\geq pm$ clauses.

Proof sketch

x_i is R-analyzed if every one of the $(m+1)$ clauses associated with x_i is used in R.

Show at least p variables are R-analyzed.

- Let I = set of R-analyzed variables
- From I, obtain τ, an assignment to all variables of Γ_φ

  τ: $x_{i,k} = 1$ iff $(x_{i1})(\bar{x}_{i1}\, x_{i2}) \ldots (\bar{x}_{ik-1}\, x_k)$ used in R

- If I does not satisfy φ, then τ would satisfy all clauses in R -- a contradiction

$\therefore$ I satisfies φ

$\therefore$ $|I| \geq p$

$\therefore$ $|R| \geq pm$.