CS 2429
PCP and Hardness of Approximation
ASSIGNMENT # 2
DUE DATE: December 5, 2007

1. In class we discussed the Fourier representation of a boolean function $f : \{1, -1\}^n \to \{1, -1\}$.

   a. Prove that for all n sufficiently large, there exists a function f (a random function) such that all Fourier coefficients of f are small. How small are they?

   b. For each $S \subset [n]$, compute the Fourier coefficient $\hat{f}_S$ where f is the mod 3 function over n input variables.

   c. For each $S \subset [n]$, compute the Fourier coefficient $\hat{f}_S$ where f is the majority function over n input variables.

   d Give another example of a boolean function where all Fourier coefficients are small.

2. (Due to Eli Ben Sasson) let $L \subseteq \{0,1\}^*$. As in class, define

$$PCP[t(n),\ l(n),\ r(n),\ q,\ c,\ s > 0],$$

   where $t(n)$ is the runtime, $l(n)$ is the length of the proof, $r(n)$ is the number of random bits, q is the number of queries to the proof, c is the completeness, and s is the soundness, and such that q, s are constants independent of n. Prove the following statements and replace the "??" with the best possible missing parameters.

   a. If $t(n) = n^{O(1)}$, $l(n) = n^{O(1)}$, $r(n) = O(\log n)$, $q = 2$, then L is decidable in polynomial time.

   b. There exists $s' > 0$ such that

$$L \in PCP[??,\ q = 2,\ c = 1,\ s = s'].$$

   What is s'?

   c. There exists $s' > 0$ and finite alphabet Σ such that

$$L \in PCP[??,\ q = 2, c = 1, s = s'].$$

   (Each query is answered by a symbol of Σ.) What is $|\Sigma|$? What is s'?

   d. For all $s' < 1$, there exists q' such that

$$L \in PCP[??,\ q = q',\ c = 1,\ s = s'].$$

   What is q'?

3. (A Long Code Test) Let $\mathcal{C}$ be a set of boolean functions from $\{-1,1\}^n$ to $\{-1,1\}$. A *local test* for $\mathcal{C}$ works as follows. Given an unknown $f : \{-1,1\} \to \{-1,1\}$ as a table of values, a local test makes q queries to f. If $f \in \mathcal{C}$ then the test should accept with probability 1, and if f is δ-far from every function in $\mathcal{C}$ then the test should reject with probability at least $\Omega(\delta)$. In class, we saw the BLR test, which is a 3-query local test for the class of linear functions, $\mathcal{L} = \{\chi_S \mid S \subseteq [n]\}$. In this problem we will develop a 6-query local test for the set of *dictator functions*, $\mathcal{D} = \{\chi_{\{i\}} \mid i \in [n]\}$; i.e., the set of n functions of the form $f(x) = x_i$.

   a. Explain why a local test for a class $\mathcal{C}$ is not necessarily also a local test for a subclass $\mathcal{C}' \subset \mathcal{C}$. Given an example of a function that demonstrates that BLR is not a proper local test for $\mathcal{D}$.

   b. Let $a, b, c \in \{-1,1\}$. Write an expression that is 1 if they are "not all equal" (NAE) and that is 0 if they are all equal.

   c. Consider the following 3-query test, called the "NAE" test, on an unknown function f: Pick 3 strings $x, y, z \in \{-1,1\}^n$ at random by choosing each triple (x_i, y_i, z_i) independently and uniformly at random from the set of strings in $\{-1,1\}^3 - \{(-1,-1,-1),(1,1,1)\}$; then test that $f(x)$, $f(y)$, and $f(z)$ are NAE. Show that the probability that the NAE test accepts is equal to $3/4 - 3/4 \sum_{S \subseteq [n]} \hat{f}(S)^2 (-1/3)^{|S|}$. Clearly if $f \in \mathcal{D}$, the NAE test accepts with probability 1.

4. Let $f : \{1,-1\}^n \to \{1,-1\}$ and $x \in \{1,-1\}^n$. Let $N_f(x)$ be the number of indices i such that if we define string y to be just like x except that the i^{th} coordinate is flipped, then $f(x) \neq f(y)$. Let $as(f)$, the average sensitivity of f, be the expectation of $N_f(x)$ over $x \in \{1,-1\}^n$.

   a. Prove that for every balanced function $f : \{1,-1\}^n \to \{1,-1\}$, $as(f) \geq 1$. (A function is balanced if it is 1 half the time and -1 half the time.

   b. Let $f : \{1,-1\}^n \to \{1,-1\}$ be a balanced function with $as(f) = 1$. Prove that f is either a dictator function, or the negation of a dictator function. (I.e. $f(x) = x_i$ or $f(x) = -x_i$ for some $i \in [n]$.)