CS 2429
PCP and Hardness of Approximation
ASSIGNMENT # 1
DUE DATE: October 17, 2007

1. (Due to Andrej Bogdanov) Recall in our analysis of Gap Amplification (and the Powering step in particular), we proved that our procedure doubled the satisfiability gap, *provided the gap was less than some constant*. That is, we saw that $gap' \geq 2\ min(gap, 10^{-6})$. But what if the gap is already quite large? Could repeating the powering step push gap' all the way towards 1? This problem gives a negative answer to this question.

   a. Assume that for infinitely many constants d, there exist infinitely many (n, d, λ)-expanders G for infinitely many n with the following two properties: First, $\lambda(G) \leq 2\sqrt{d}$ and secondly, G has girth at least $2/3 \log_d n$, where the girth of a graph is the length of the smallest cycle in it. (Such graphs exist.) Suppose we make G into a constraint graph over the alphabet $\{0, 1\}$ by putting an inequality constraint on every edge. Prove that the satisfiability gap of G is at least $1/2 - O(\sqrt{1/d})$. [Hint: Look up the "expander mixing lemma", e.g. in Wikipedia.]

   b. On the other hand, show that for every fixed constant t, if n is large enough, then the output G' of the Powering step has satisfiability gap at most $1/2$.

2. As mentioned in the first lecture, Hastad proved a version of the PCP theorem in which: completeness is $1-\epsilon$; soundness is $1/2+\epsilon$; the number of queries is 3; and the predicates that the verifier uses are of the form $x_i + x_j + x_k = b\ mod\ 2$ where b is 0 or 1. (Here, ϵ can be any constant.) Assuming $P \neq NP$, prove that the following statement is false: "There is a (nonadaptive) probabilistically checkable proof for an NP-complete problem in which (a) three bits of the proof are queried; (b) all predicates are of the form $x_i + x_j + x_k = b\ mod\ 2$; (c) completeness is 1; and (d) soundness is 2/3."

3. Assuming the PCP theorem, prove that the Vertex Cover problem is NP-hard to approximate to within some constant, even in bounded-degree graphs. How small can you make the degree bound?

4. Assuming the PCP theorem, prove that the Steiner Tree problem is NP-hard to approximate to within some constant. Can you prove this even for instances that are complete graphs in which all edge costs are either 1 or 2? [Hint: Reduce from Vertex Cover in bounded-degree graphs. Given a VC instance $G = (V, E)$, construct a Steiner tree instance on the vertex set $V \cup E$.] The Steiner Tree problem is defined as follows. The input is a weighted graph $G = (V, E)$ together with a subset $T \subset V$ of its vertices. The output should be a minimum weight

connected subgraph containing all of the vertices of T. (Note that the subgraph can include vertices outside of T, and this is what distinguishes it from the easier minimum spanning tree problem.)