CS 2401
Complexity Theory
ASSIGNMENT # 1
DUE DATE: October 27, 2015

(1.) A language L is *unary* if the underlying alphabet is $\{1\}$. (That is, every string in L is of the form 1^i for some $i \geq 0$.) Prove that if every unary NP language is in P, then EXP=NEXP.

(2.) (a) Show that 2SAT is in NL.

(b) Prove that 2SAT is NL-hard with respect to logspace reductions.

(3.) Let $S = \{S_1, S_2, \ldots, S_m\}$ be a collection of subsets of a finite set U. Let $|U| = n$. Then each S_i will be represented by a bit string of length $n = |U|$, where the j^{th} position will indicate whether or not the j^{th} element of U is in S_i. The VC-dimension of S, denoted by $VC(S)$, is the size of the largest set $X \subset U$ such that for every $X' \subseteq X$, there is an i such that $S_i \cap X = X'$. (That is, X is *shattered* by S.) Let $VCdim$ be the set of pairs (S, k) such that the VC-dimension of S is at least k.

(a) Prove that $VCdim$ is in NP.

(b) Explain why it is unlikely that $VCdim$ is NP-complete.

HINT: There is an algorithm for $VCdim$ that runs in *quasi-polynomial* time. That is, time $n^{O(\log n)}$, where n is the total input size. The algorithm is based on a simple lemma which upper bounds the maximal size of the VC dimension of a set S, as a function of the size of S. State and prove this lemma, and show how it implies both an NP-algorithm, as well as a quasi-polynomial time algorithm for $VCdim$. Then explain why the existence of such an algorithm makes it unlikely that $VCdim$ is NP-complete.

(4.) This problem also concerns the VC-dimension of a set S, only now the set S will be represented more succinctly. A boolean circuit C succinctly represents collection S if S_i consists of exactly those elements $x \in U$ for which $C(i, x) = 1$. Let $|U| = n$. Then C will have $\log m + \log n$ inputs, where the first $\log m$ inputs will be i in binary notation, and the last $\log n$ inputs will be x in binary notation. C itself will be encoded by some string of length polynomial in the size of C. Define $VCdimSuccinct$ to be the set of all strings $< C, k >$ such that C represents a collection S such that the VC-dimension of S is at least k.

(a) Show that $VCdimSuccinct$ is in Σ_3^p.

(b) Prove that $VCdimSuccinct$ is Σ_3^p complete. (Hint: Reduce from Σ_3-3SAT.)

(5.) Show that $SPACE(n) \neq NP$.