Log Rank Conjecture: Upper bounds

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Log Rank Conjecture (LRC)

Theorem (Mehlhorn-Schmidt '82)

Let $F: X \times Y \to \{0,1\}$. Then

$$P^{cc}(F) \geq \log(\mathrm{rank}(M_F)).$$

Conjecture (Lovasz-Saks '88)

Let $F: X \times Y \to \{0,1\}$. Then

$$P^{cc}(F) \leq \log^c(\mathrm{rank}(M_F)).$$

State-of-the-art

$$\Omega(\log^2(\mathrm{rank}(M_F))) \leq P^{cc}(F) \leq O(\sqrt{\mathrm{rank}(M_F)}\log(\mathrm{rank}(M_F))).$$

↑ Göös, Pitassi, Watson '15 ↑ Lovett '14

Outline

- Log Rank Conjecture for Parities
  1. a weaker version of LRC.
  2. well-studied.
  3. proving it could potentially shed new light on LRC.
- Proof of the square root upper bound for LRC.

Log Rank Conjecture for Parities

Lifting functions with Parities:

Suppose $f: \{0,1\}^n \to \{0,1\}$ is a Boolean function.

Consider the communication function $f^{\oplus}: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,

where $f^{\oplus}(x,y) := f(x \oplus y)$.

Conjecture:

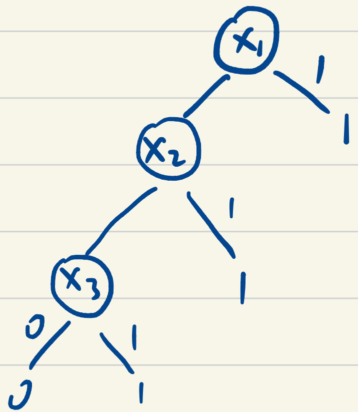
Let $f: \{0,1\}^n \to \{0,1\}$ be a Boolean function. Then

$$\underbrace{P^{cc}(f^{\oplus})}_{\text{Parity decision tree}} \le \log^c(\underbrace{\text{rank}(Mf^{\oplus})}_{\text{Fourier sparsity of } f}).$$

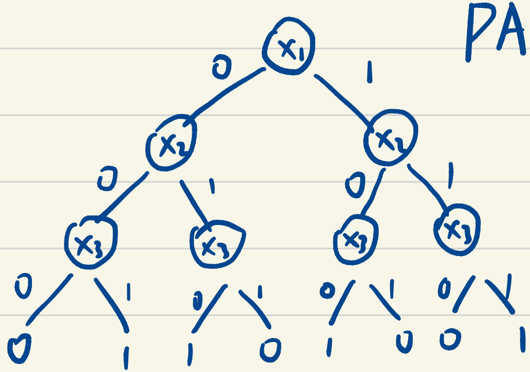
The method is Fourier analysis.

Decision Tree

OR

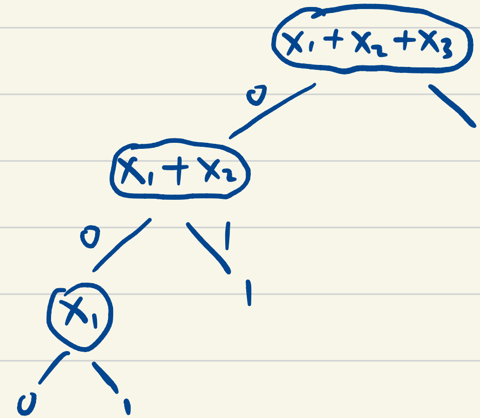


PARITY

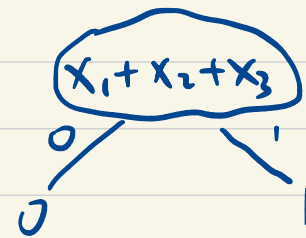


Parity Decision Tree

OR



PARITY



$P^{cc}(f^{\oplus})$ versus $PDT(f)$

Observation:

$$P^{cc}(f^{\oplus}) \le 2 \cdot PDT(f).$$

Theorem (Hatami, Hosseini, Lovett '16)

$$PDT(f) \le \left(P^{cc}(f^{\oplus})\right)^6.$$

So the corollary is:

Log Rank Conjecture for Parities is equivalent to

$$PDT(f) \le \log^c(\mathrm{rank}(M_{f^{\oplus}})).$$

Fourier expansion

- Fourier expansion of a Boolean function f

$$f \overset{\text{uniquely}}{=\!=\!=} \sum_{\gamma \subseteq [n]} \hat{f}(\gamma) \cdot \chi_\gamma \qquad \chi_\gamma(x) := (-1)^{\gamma \cdot x}$$

- Fourier sparsity

view Fourier spectrum $\hat{f}$ as a vector with length 2^n.

$$\text{sparsity}(\hat{f}) := \|\hat{f}\|_0$$

- Spectral norm

is defined as $\|\hat{f}\|_1$.

Observation:

$\|\hat{f}\|_1 \leq \sqrt{\|\hat{f}\|_0}$ by Cauchy-Schwarz inequality.

But for AND function:

$\|\hat{f}\|_1 \leq 3$, $\|\hat{f}\|_0 = 2^n$.

rank$(M_{f^{\oplus}})$ versus Fourier sparsity of f

Observation:

$$\mathrm{rank}(M_{f^{\oplus}}) = \mathrm{sparsity}(\hat{f}). \qquad \text{LRC for XOR} \iff \boxed{\mathrm{PDT}(f) \le \log^c(\|\hat{f}\|_0).}$$

Proof:

$$f(x \oplus y) = \sum_{S \subseteq [n]} \hat{f}_S \cdot \chi_S(x \oplus y) = \sum_{S \subseteq [n]} \hat{f}_S \cdot \chi_S(x) \cdot \chi_S(y).$$

$$M_{f^{\oplus}} = \begin{array}{cc} & \begin{matrix} S=\phi & \cdots & S=[n] \end{matrix} \\ \begin{matrix} x=0^n \\ \vdots \\ x=1^n \end{matrix} & \begin{bmatrix} & & \\ & \chi_S(x) & \\ & & \end{bmatrix} \end{array} \times \begin{bmatrix} \hat{f}(\phi) & & 0 \\ & \ddots & \\ 0 & & \hat{f}([n]) \end{bmatrix} \times \begin{array}{cc} & \begin{matrix} y=0^n & \cdots & y=1^n \end{matrix} \\ \begin{matrix} S=\phi \\ \vdots \\ S=[n] \end{matrix} & \begin{bmatrix} & & \\ & \chi_S(y) & \\ & & \end{bmatrix} \end{array}$$

Note that it is also sufficient (but not necessary) to prove:

$$\mathrm{PDT}(f) \le \log^c(\|\hat{f}\|_1) \text{ because } \|\hat{f}\|_1 \le \sqrt{\|\hat{f}\|_0}.$$

State-of-the-art

Suppose the $\mathbb{F}_2$-degree of f is d.

Note that $d \le \log \|\hat{f}\|_0$.

Theorem (Tsang, Wong, Xie, Zhang '13.04)

$$PDT(f) \le O(\|\hat{f}\|_1) \cdot d$$

$$\Rightarrow P^{cc}(f^{\oplus}) \le O\left(\sqrt{\|\hat{f}\|_0} \cdot \log \|\hat{f}\|_0\right).$$

Theorem (Tsang, Wong, Xie, Zhang '13.04)

$$PDT(f) \le O\left(2^{d^2/2} \cdot \log^{d-2} \|\hat{f}\|_1\right) \cdot d .$$

Polynomial Rank

Definition

Suppose $f \in \mathbb{F}_2[x_1, \cdots, x_n]$ and $\deg_2(f) = d$.

Then $\text{rank}(f)$ is defined as the minimum integer r s.t. f can be expressed as

$$f = \underbrace{\ell_1}_{\deg_2(\ell_1)=1} \underbrace{f_1}_{\deg_2(f_1)=d-1} + \cdots + \ell_r f_r + \underbrace{f_{r+1}}_{\deg_2(f_{r+1}) \le d-1}$$

Then we will have:

$$\begin{aligned} PDT(f) &\le \text{rank}(f) \cdot \deg_2(f) \quad (\text{by querying } \ell_1 \cdots \ell_r) \\ &\le \text{rank}(f) \cdot \log \|\hat{f}\|_0 . \end{aligned}$$

The notion of rank

Definition 1.5 (Rank)**.** Let $d \geqslant 0$, and let $P : \mathbb{F}^n \to \mathbb{F}$ be a function. We define the *degree d rank* $\mathrm{rank}_d(P)$ of P to be the least integer $k \geqslant 0$ for which there exist polynomials $Q_1, \ldots, Q_k \in \mathcal{P}_d(\mathbb{F}^n)$ and a function $B : \mathbb{F}^k \to \mathbb{F}$ such that we have the representation $P = B(Q_1, \ldots, Q_k)$. If no such k exists, we declare $\mathrm{rank}_d(P)$ to be infinite (since $\mathbb{F}^n$ is finite-dimensional, this only occurs when $d = 0$ and P is non-constant).

This is from a paper by
Ben Green and Terence Tao.

A simple but powerful Lemma

Lemma (Tsang, Wong, Xie, Zhang '13.04)

For all non-constant $f:\{0,1\}^n \to \{0,1\}$, if there is an affine subspace H s.t. $\deg_2(f|_H) < \deg_2(f)$, then

$$\text{rank}(f) \le \text{co-dim}(H)$$

Parity kill Number

Definition (Parity kill Number) [O'Donnell, Sun, Tan, Wright and Zhao'14]

$$C^{\oplus}_{\min}(f) = \min\{\text{co-dim}(H) \mid H \text{ is an affine subspace, on which } f \text{ is a constant}\}.$$

Observation

$\text{rank}(f) \le C^{\oplus}_{\min}(f)$ b.c. $f|_H$ is degree 0.

Comment:

We will not lose too many things using rank(f) and parity kill number.

$$PDT(f)/\log\|\hat{f}\|_0 \le \text{rank}(f) \le C_{\oplus,\min}(f) \le PDT(f).$$

Recall that

$$P^{cc}(f^{\oplus}) \le O(PDT(f)) \le \mathrm{rank}(f) \cdot \deg_2(f)$$
$$\le C_{\oplus,\min}(f) \cdot \log \|\hat{f}\|_0$$

It remains for us to upper bound $C_{\oplus,\min}(f)$.

In the world that LRC is true:

$$C_{\oplus,\min}(f) \le \log^c \|\hat{f}\|_0 .$$

What is known (Tsang, Wong, Xie, Zhang '13.04)

$$C_{\oplus,\min}(f) \underset{\triangle}{\le} \|\hat{f}\|_1 \le \sqrt{\|\hat{f}\|_0}$$

Proof sketch of $C_{\oplus,\min}(f) \le \|\hat{f}\|_1$

1) sort the $|\hat{f}(\gamma)|$ in the decreasing order $\gamma^1 \cdots \gamma^s$.

$b=0 \Rightarrow \hat{f}(\gamma_1)+\hat{f}(\gamma_2)$

$b=1 \Rightarrow \hat{f}(\gamma_1)-\hat{f}(\gamma_2)$

2) greedy folding process:

ⓐ Fold $\beta=\gamma^1+\gamma^2$ and selete $b\in\{0,1\}$ s.t. the subfunction has its largest Fourier coefficience to be $|\hat{f}(\gamma_1)|+|\hat{f}(\gamma_2)|$.

ⓑ Use $\sum_{i\in\beta} x_i = b$ as a linear restriction (query β and get answer b).

Claim:

1) after at most $\|\hat{f}\|_1$ queries, $|\hat{f}(\gamma_1)| \geq 1/2$.

2) then for each query, $\|\hat{f}\|_1$ will decrease $2\cdot|\hat{f}(\gamma_1)| \geq 1$.

3) when $\|f\|_1=1$, it will be a constant function after one more query.

State-of-the-art

Theorem (Tsang, Wong, Xie, Zhang '13.04)

$$P^{cc}(f^{\oplus}) \le O(\|\hat{f}\|_1)\cdot d = O\left(\sqrt{\|\hat{f}\|_0}\cdot \log \|\hat{f}\|_0\right).$$

Theorem (Tsang, Wong, Xie, Zhang '13.04)

$$P^{cc}(f^{\oplus}) \le O\left(2^{d^2/2}\cdot \log^{d-2}\|\hat{f}\|_1\right)\cdot d$$

Conjecture (Tsang, Wong, Xie, Zhang '13.04)

$$C_{\oplus,\min}(f) \underset{\triangle}{\le} \log^c(\|\hat{f}\|_1) \le \log^c(\|\hat{f}\|_0).$$

State-of-the-art

Theorem (Tsang, Wong, Xie, Zhang '13.04)

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$$C_{\oplus,\min}(f) \underset{\triangle}{\le} \log^c(\|\hat{f}\|_1) \le \log^c(\|\hat{f}\|_0).$$

This has been disproved…

Theorem (Chattopadhyay, Mande, Sherif '19)

There is a Boolean function f with $O(m)$ spectral norm, but $C_{\oplus,\min}(f) = \lceil 2m/3 \rceil$.

Communication is bounded by root of rank

State of the art

Theorem (Lovett '14)
Let $f: |X| \times |Y| \to \{-1,1\}$ be a Boolean function with rank$(M_f) = r$, then

$$P^{cc}(f) \le O(\sqrt{r}\log r).$$

The method is to analize large monochromatic rectangle using (taking advantage of) discrepancy.

Theorem from Nisan and Wigderson

Theorem (Nisan and Wigderson '94)

Assume that for any function $f: X \times Y \to \{-1, 1\}$ of $\mathrm{rank}(M_f) = r$ there exists a monochromatic rectangle of size $|R| \geq 2^{-c(r)} |X \times Y|$.

Then we have

$$P^{cc}(f) \leq O\Big(\log^2 r + \sum_{i=0}^{\log r} c(r/2^i)\Big).$$

In particular, by master theorem

$$c(r) \leq p(r) \Rightarrow P^{cc}(f) \leq p(r).$$

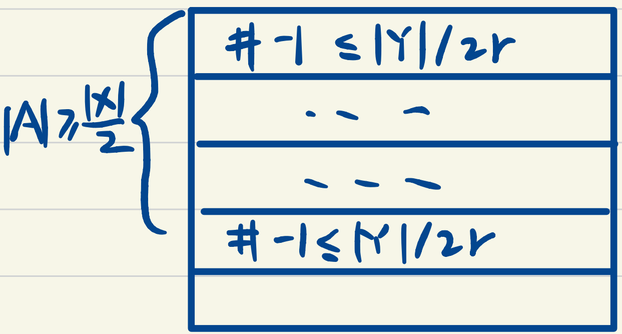
$$c(r) \leq \log^{\alpha} r \Rightarrow P^{cc}(f) \leq \log^{\alpha+1} r.$$

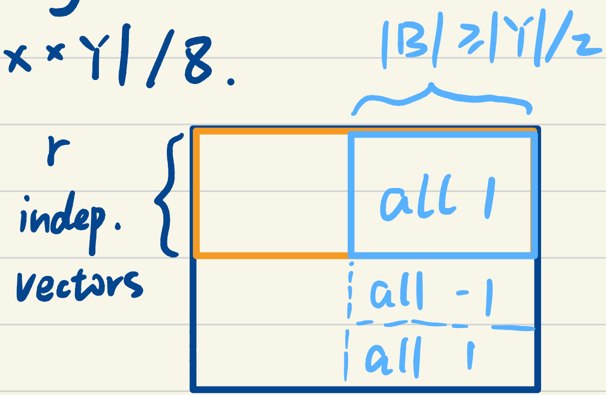
Lovett proved $c(r) \leq \sqrt{r} \log r$.

Closed Monochromatic Rectangle

Lemma (Gavinsky and Lovett '13)
Let $f: |X| \times |Y| \to \{-1, 1\}$ be a Boolean function with $\mathrm{rank}(M_f) = r$ and $\mathbb{E}[f] \geq 1 - \frac{1}{2r}$, then there exists a monorectangle R with

$$|R| \geq |X \times Y| / 8.$$





Discrepancy

$f: X \times Y \to \{-1, 1\}$ is a total function
μ is a distribution over $X \times Y$.

$$\mathrm{disc}_\mu(f) := \max_R \left| \sum_{(x,y)\in R} \mu(x,y) f(x,y) \right|.$$

$$\mathrm{disc}(f) := \min_\mu \max_R \left| \sum_{(x,y)\in R} \mu(x,y) f(x,y) \right|.$$

What we know: $\mathrm{disc}(f) \geq 1/8\sqrt{\mathrm{rank}(M_f)}$ $\leftarrow$ sharp when f is IP

We are going to prove

$\exists$ mono $|R| \geq 2^{-O(\log r/\delta)} |X \times Y|$, δ is $\mathrm{disc}(f)$.

High-Level idea: purifying matrix

Lemma 1

Let $f: X \times Y \to \{-1, 1\}$ and $\mathbb{E}[f] = 1 - \beta \geq 0$ and $\mathrm{disc}(f) = 3\delta$.

Then there exists R s.t.

$$|\mathbb{E}_R[f]| \geq 1 - \beta/2$$

$$|R| \geq 2^{-2/\delta} |X \times Y|.$$

Repeat this Lemma $O(\log r)$ times, we will get a R^* s.t.

1. $|\mathbb{E}_R[f]| \geq 1 - 1/2r$

2. $|R| \geq 2^{-O(\log r/\delta)} |X \times Y|$

How to use discrepancy?

Lemma: Let $f: X \times Y \to \{-1, 1\}$ with $E[f] = \alpha \geq 0$ and $\mathrm{disc}(f) = 3\delta$.

Then there exists R s.t.

$$E_R[f] \geq \alpha + \delta(1-\alpha^2)\frac{|X \times Y|}{|R|}.$$

Proof:

Design a distribution μ.

$$\mu(x,y) = \begin{cases} \dfrac{1}{(1+\alpha)|X \times Y|} & f(x,y) = 1 \\[2ex] \dfrac{1}{(1-\alpha)|X \times Y|} & f(x,y) = -1 \end{cases}$$

Then we have $\sum \mu(x,y) f(x,y) = 0$.

R_1	R_2
R_3	R_4

$$\left|\sum_{(x,y)\in R_1} \mu(x,y) f(x,y)\right| \geq 3\delta$$

$$\exists R \text{ s.t. } \sum_{(x,y)\in R} \mu(x,y) f(x,y) \geq \delta$$

By some calculation:

$$\sum_{(x,y)\in R} f(x,y) \geq \alpha|R| + |X \times Y|(1-\alpha^2)\delta.$$

Final step

Suppose $\mathbb{E}[f] = \alpha = 1 - \beta \geq 0$. Lemma 2: $E_R[f] \geq \alpha + \delta(1-\alpha^2)\frac{|X \times Y|}{|R|}$.

From Lemma 2 to Lemma 1:

Suppose we iteratively apply Lemma 2 t times and get

$$R_0 = X \times Y,\ R_1,\ R_2,\ \cdots,\ R_t.$$

We have $\alpha_t \geq \alpha_0 + \delta \cdot \beta/2 \cdot \sum |R_{i-1}|/|R_i|$.

So $\beta \geq \alpha_t - \alpha_0 \geq \delta \cdot \beta/2 \cdot \sum |R_{i-1}|/|R_i|$.

$\Rightarrow \sum |R_{i-1}|/|R_i| \leq 2/\delta$.

$\Rightarrow \prod |R_{i-1}|/|R_i| \leq 2^{2/\delta}$.

$\Rightarrow \frac{|X \times Y|}{R_t} \leq 2^{2/\delta}$.

QED