Extension Complexity vs Communication Complexity

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Linear Programming Relaxation

A discrete optimization problem usually has the formula

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A linear relaxation of (1) is defined by

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We hope P has a 'easy form' to describe. Specifically, P is defined by polynomially many inequalities.

Outline

Main Goal

- Given an n-vertices graph G, we want to solve maximum clique using LP:

$$\max_x c^T x$$
$$s.t. \quad x \in P$$

by giving a poly-size 'description' of P whose vertices correspond to cliques.

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- r is lower bounded by the number of 1-rectangles needed to cover $M = suppmat(S)$.

- The unique disjointness matrix is embedded in M, where $UDISJ$ has nondeterministic communication complextiy $\Omega(n)$.

Polytope

Suppose $P \subset \mathbb{R}^d$ is bounded, we say P is a *polytope* if it can be represented by finite inequalities:

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Equivalently, P is a polytope iff there exists a finite set V such that

$$P = conv(V).$$

Extended Formulation and Extension Complexity

An *extended formulation (EF)* of $P = \{x : Ax \leq b\} \subset \mathbb{R}^d$ is a linear system

$$\pi := \{Ex + Fy \leq g\} \tag{3}$$

in variables $(x, y) \in \mathbb{R}^{d+(k-d)}$, such that $x \in P$ iff (3) holds.

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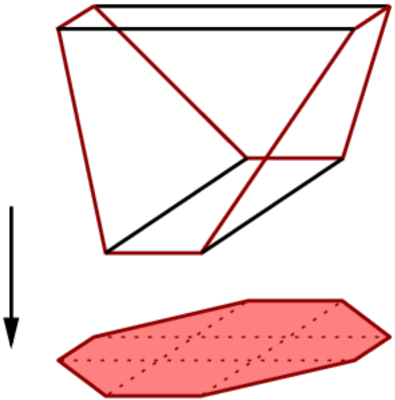
The *extension complexity* of P is defined by

$$xc(P) := \min_{\pi \text{ is an EF}} size(\pi) \tag{4}$$

Example: Regular Polygons



Example: Regular Polygons

Theorem

Let P be a regular n-gon in $\mathbb{R}^2$. Then $xc(P) = O(\log n)$.

Slack Matrix

Suppose P is contained in $Q = \{x \in \mathbb{R}^d : Ax \leq b\}$. Let $V = \{v_1, \ldots, v_k\}$ be a finite set of points in P:

$$V \subset P \subset Q.$$

The *slack matrix* of P with respect to V, Q is the $n \times k$ matrix $S(V, Q)$ with

$$S(V, Q)_{i,j} = b_i - A_i \cdot v_j$$

where A_i is the i-th row of A.

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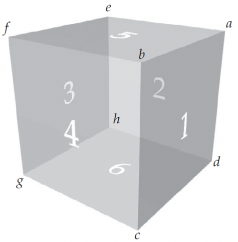
Observation 1: S is a nonnegative matrix.

Observation 2: $S_{i,j}$ is proportional to the distance from v_j to the hyperplane $\{x : A_i x = b_i\}$.

Slack Matrix



	a	b	c	d	e	f	g	h
1	0	0	0	0	1	1	1	1
2	0	1	1	0	0	1	1	0
3	1	1	1	1	0	0	0	0
4	1	0	0	1	1	0	0	1
5	0	0	1	1	0	0	1	1
6	1	1	0	0	1	1	0	0

Nonnegative rank, factorization

The *nonnegative rank* of a matrix $M \in \mathbb{R}_{\geq 0}^{m \times n}$ is defined by the smallest number r such that there exists $T \in \mathbb{R}_{\geq 0}^{m \times r}$ and $U \in \mathbb{R}_{\geq 0}^{r \times n}$ such that $M = TU$, denoted by $rank_+(M)$.

The factorization theorem

Theorem (Yannakakis)

Let $P = conv(V)$ *and* $P \subset Q$*, then*

$$rank_+(S(V, Q)) = xc(P).$$

Clique Polytope (Correlation Polytope)

Let $G_n = (V_n, E_n)$ be a graph of n-vertices. Set $d = n + \binom{n}{2}$, for $A \subset [n]$, define

$$x_i^A = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{otherwise} \end{cases}.$$

$$x_{(i,j)}^A = \begin{cases} 1, & \text{if } i, j \in A \\ 0, & \text{otherwise} \end{cases}.$$

Then $x^A := (x_1^A, \ldots, x_n^A, x_{(1,2)}^A, x_{(1,3)}^A, \ldots, x_{(n-1,n)}^A)^T$.

Define P_n as the convex hull of all these 2^n vectors.

Main Results

Theorem (Fiorini, Massar, Pokutta, Tiwary, De Wolf, 12')

The clique polytope P_n has

$$xc(P_n) = 2^{\Omega(n)}.$$

1st Step: Construct a Slack Matrix

For any $A \subset [n]$, x^A is a clique vector of P_n.

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Define $S(V, Q)$ be the slack matrix with $V = \{x^A : A \subset [n]\}$ and

$$Q = \{x : \ x \text{ satisfies } (5) \text{ for all } B \subset [n]\}.$$

2nd Step: Support of the slack matrix, rectangle covering

Let $M \in \mathbb{R}_{\geq 0}^{m \times n}$ be any nonnegative matrix.

The *support matrix* $suppmat(M)$ is defined by

$$suppmat(M)_{ij} = \begin{cases} 1 & \text{if } M_{ij} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

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Proof.

Let $M = TU$ be the rank-r nonnegative factorization, then M can be written as the sum of r-nonnegative rank-1 matrices:

$$M = \sum_{k=1}^{r} T^k U_k$$

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Taking the support on each side,

$$\begin{aligned}\mathrm{supp}(M) &= \bigcup_{k=1}^{r} \mathrm{supp}(T^k U_k) \\ &= \bigcup_{k=1}^{r} \mathrm{supp}(T^k) \times \mathrm{supp}(U_k).\end{aligned}$$

3rd Step: Reduction to Unique Disjointness

Notice that

$$M_{AB} = \begin{cases} 0, & \text{if } |A \cap B| = 1 \\ 1, & \text{otherwise} \end{cases}.$$

Thus we have

$$M_{AB} = UDISJ(A, B).$$

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Thus, we need $2^{\Omega(n)}$ 1-rectangles to cover M.

suppmat(M) with large 1-monochromatic rectangle cover

Consider the following $2^n \times 2^n$ matrix $M(n)$ with rows and columns indexed by n-bit strings a and b, and nonnegative entries:

$$M(n)_{ab} := (1 - a^T b)^2. \tag{6}$$

Theorem (De Wolf, 03')

Every 1-monochromatic rectangle cover of suppmat$(M(n))$ has size $2^{\Omega(n)}$.

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Lemma (Razborov, 92')

There exists sets $A, B \in \{0,1\}^n \times \{0,1\}^n$ *and probability distribution* μ *on* $\{0,1\}^n \times \{0,1\}^n$ *such that*

1. $a^T b = 0$, *for all* $(a, b) \in A$.
2. $a^T b = 1$, *for all* $(a, b) \in B$.
3. $\mu(A) = \frac{3}{4}$
4. *For all rectangles* R, $\mu(R \cap B) \geq \alpha \cdot \mu(R \cap A) - 2^{-\delta n}$.

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Proof.

Let $R_1, \ldots, R_k$ be a 1-monochromatic rectangle cover of *suppmat*$(M(n))$. Then any R_i cannot contain elements from B. Hence $\mu(R_i \cap B) = 0$ and $\mu(R_l \cap A) \leq 2^{-\delta n}/\alpha$.

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$$\frac{3}{4} = \mu(A) = \mu\left(\bigcup_{i=1}^{k}(R_i \cap A)\right) \leq \sum_{i=1}^{k} \mu(R_i \cap A) \leq k \cdot \frac{2^{-\delta n}}{\alpha}.$$

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Hence, $k \geq 2^{\Omega(n)}$. □

Lower Bounds on Extension Complexity

The following famous polytopes have exponential extension complexity:

- Cut Polytope [Fiorini, Massar, Pokutta, Tiwary, De Wolf, 12']
- Stable Set Polytope [Fiorini, Massar, Pokutta, Tiwary, De Wolf, 12']
- TSP Polytope [Fiorini, Massar, Pokutta, Tiwary, De Wolf, 12']
- Matching Polytope [Rothvoss, 14']

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