First Incompleteness Theorem

Every sound, axiomatizable theory Σ is incomplete.

Sound: $\Sigma \subseteq TA$

Axiomatizable: If there exists $\Gamma \subseteq \Sigma$ such that

① Γ is recursive

② $\Sigma = \{A \in \Phi_0 \mid \Gamma \models A\}$

Incomplete: $\exists A$ such that $A \notin \Sigma$ and $\neg A \notin \Sigma$

Theorem: Σ is axiomatizable iff Σ is re.

- To prove the incompleteness theorem: exhibit a predicate in TA which is not re.

Arithmetical: A predicate is arithmetical if it can be represented by a formula over $\mathcal{L}_A$.

Represented: A formula $A(x_1, \dots, x_n)$ represents a relation $R(x_1, \dots, x_n)$ if $\forall a \in N^n$,

$$R(a_1, \dots, a_n) \iff N \models A(s_{a_1}, \dots, s_{a_n})$$

$$\underbrace{ss \dots s0}_{a_1 \text{ times}}$$

Strategy

We define a predicate $\text{Truth} \subseteq \mathbb{N}$

$$\text{Truth} = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in TA \}$$

We will show that Truth is not r.e.:

1. Every re predicate/language is arithmetical

2. Truth is not arithmetical.

$\therefore$ Truth is not r.e.

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1. Every re predicate/language is arithmetical
   (∃Δ Theorem)

2. Truth is not arithmetical.
   (Tarski's Theorem)

∴ Truth is not r.e.

Since Truth is not r.e.,
there is no r.e. TM that accepts exactly the sentences in TA.

$\therefore$ TA is not axiomatizable

$\therefore$ Any sound and axiomatizable theory Σ is incomplete (there is a sentence $A \in \Phi_0$ such that neither A nor $\neg A$ are in Σ).

$\exists\Delta_0$ Theorem: Every r.e. relation is represented by an $\exists\Delta_0$ formula

unbounded $\exists$ bounded quantifiers.

Main Lemma: Let $f : \mathbb{N}^n \to \mathbb{N}$ be a total computable function. Then $R_f = \{(x,y) \mid f(x) = y\}$ is an $\exists\Delta_0$ relation.

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Proof of $\exists\Delta_0$-Theorem from main Lemma

Let $R(x)$ be an r.e. relation. Then $R(x) = \exists y\, S(x,y)$ where $S(x,y)$ is recursive. Since $S(x,y)$ is recursive,

$$f_s(x,y) = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{o.w.} \end{cases}$$

is total computable. By main lemma R_{f_s} is represented by an $\exists\Delta_0$ relation $\exists z B$.

So $R(x) = \exists y \underbrace{\exists z B}_{R_{f_s}}$ is represented by an $\exists\Delta_0$.

Proof of Main Lemma

Proof idea: Because f is total computable, there exists a TM M_f computing f. To come up with an $\exists\Delta_0$ relation for R_f we will use M_f! Given input (x,y), we will guess M's computation on x using existential quantifiers.

- guess number of steps m of $M_f(x)$
- guess the tableaux $r_1, \ldots, r_m, \ldots, r_{m^2}$ of M_f's computation on x.

Proof of Main Lemma

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- guess number of steps m of $M_f(x)$
- guess the tableaux $r_1, \dots, r_m, \dots, r_{m^2}$ of M_f's computation on x.

issue: In quantifiers this would look like

$$\exists m \underbrace{\exists r_1 \dots \exists r_{m^2}}_{\text{number of quantifiers depends on } m.} R$$

$\therefore$ Need a way to encode the entries of the tableaux $r_1, \dots, r_{m^2}$ an $\exists\Delta_0$-formula.

Proof of Main Lemma

Need a way to encode the entries of the tableaux $r_1, \ldots, r_{m^2}$ as an $\exists\Delta_0$-formula

Usual way: prime power decomposition – encode $(r_1, \ldots, r_n)$ as powers of the first n primes.

e.g. $2^{r_1} 3^{r_2} 5^{r_3} \ldots$

Doesn't work because in $\mathcal{L}_A$ we only have $+, \cdot, S$, no exponentiation.

Proof of Main Lemma

Need a way to encode the entries of the tableaux $r_1, \dots, r_{M^2}$ as an $\exists\Delta_0$-formula

β-function: $\beta(c,d,i) = rm(c, d(i+1)+1)$

where $rm(x,y) = x \bmod y$.

Lemma: $\forall n, r_0, \dots, r_n \; \exists c, d$ such that

$$\beta(c,d,i) = r_i \quad \forall i, \; 0 \leq i \leq n.$$

So the pair (c,d) represents the sequence $r_1, \dots, r_n$.

Proof of Main Lemma

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Chinese Remainder Theorem:

Let $r_0, \ldots, r_n, m_0, \ldots, m_n$ be such that $0 \le r_i \le m_i$ $\forall i$, $0 \le i \le m$ and $gcd(m_i, m_j) = 1$ $\forall i \ne j$.

Then $\exists r$ such that $rm(r, m_i) = r_i$ $\forall i, 0 \le i \le n$.

Proof of Main Lemma

Lemma: $\forall n, r_0, \dots, r_n \; \exists c, d$ such that $\beta(c,d,i) = r_i \quad \forall i,\ 0 \le i \le n$.

$\beta(c,d,i)$
$= rm(c, d(i+1)+1)$

Chinese Remainder Theorem:

Let $r_0, \dots, r_n, m_0, \dots, m_n$ be such that $0 \le r_i \le m_i$ $\forall i,\ 0 \le i \le m$ and $gcd(m_i, m_j) = 1 \quad \forall i \neq j$.

Then $\exists r$ such that $rm(r, m_i) = r_i \quad \forall i,\ 0 \le i \le n$.

Proof: Define d such that $d(i+1)+1$ are coprime.

Use crt to find a value for c.

→ Let $d = (n + r_0 + \dots + r_n)!$ **Claim:** $d(i+1)+1$ are coprime (proof in notes)

By crt $\exists r = c$ so that $\beta(c,d,i) = rm(c, m_i) = r_i \quad \forall i$

Proof of Main Lemma

Lemma: $\forall n, r_0, \dots, r_n \; \exists c, d$ such that $\beta(c,d,i) = r_i \quad \forall i,\; 0 \leq i \leq n.$

Lemma 1: R_β is a Δ_0 relation

Pf: $y = \beta(c,d,i) \iff [\exists q \leq c\,(c = q(d(i+1)+1)+y) \wedge y < d(i+1)+1]$

Lemma 2: If $R(x,y)$ and $R_\beta(x,y)$ are $\exists\Delta_0$ relations then $S(x) = \exists y\,(R_\beta(x,y) \wedge R(x,y))$ is an $\exists\Delta_0$ relation.

Proof of Main Lemma

Main Lemma: Let $f:\mathbb{N}^n \to \mathbb{N}$ be total computable then $R_f = \{(x,y) \mid f(x)=y\}$ is an $\exists\Delta_0$ relation.

Proof: Let M_f be the TM computing f.

- Use M_f to construct an $\exists\Delta_0$-relation $R(x,y)$ saying:

$\exists \underset{\text{\#steps of TM}}{m}, \underbrace{c, d}_{\text{input to } \beta \text{ function.}}$ such that

Proof of Main Lemma

Main Lemma: Let $f: \mathbb{N}^n \to \mathbb{N}$ be total computable then $R_f = \{(x,y) \mid f(x) = y\}$ is an $\exists\Delta_0$ relation.

Proof: Let M_f be the TM computing f.

- use M_f to construct an $\exists\Delta_0$-relation $R(x,y)$ saying:

$\exists m, \underbrace{c, d}$ such that

#steps of TM ↗ ↖ input to β function.

(1) c, d describe the tableaux $r_1, \dots, r_{m^2}$ given by β-function

(2) $r_1, \dots, r_m$ encode the start state of M_f on x.

(3) last m numbers $r_{m(m-1)} \cdots r_{m^2}$ encode the last config, containing y on the first $|y|$ cells then B and state is q_2 (halt state)

(4) all 2x3 local cells are consistent with β's transition function  & other states are not q_2.

Recap: We proved

$\exists\Delta_0$ Theorem: Every re relation is represented by a $\exists\Delta_0$ formula

Which followed by the main lemma:

f total computable $\Rightarrow R_f$ is a $\exists\Delta_0$ relation.

Strategy

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We will show that Truth is not r.e.:

✓ 1. Every re predicate/language is arithmetical
(∃Δ Theorem)

2. Truth is not arithmetical.
(Tarski's Theorem)

∴ Truth is not r.e.

Tarski's Theorem

↳ States that the truth of a sentence in $\mathcal{L}_A$ cannot be expressed by any one formula in $\mathcal{L}_A$.

Define the predicate $\text{Truth} \subseteq \mathbb{N}$

$$\text{Truth} = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in TA \}$$

Tarski's Theorem: Truth is not arithmetical.

- high level - formulate a sentence which says "I am false" which is self contradictory.

Tarski's Theorem

Truth = $\{m \mid m$ encodes a sentence $\langle m \rangle \in TA\}$

$Sub(n,m) = \begin{cases} 0 \text{ if } m \text{ is not a legal encoding of a formula} \\ \text{O.W. Suppose } m \text{ encodes } A(x) \text{ with free variable } x. \text{ Then } sub(m,n) = m' \text{ where } m' \text{ encodes } A(n) \end{cases}$

$d(n) = Sub(n,n)$

$Sub(n,m)$: "decode m, plug in n, re-encode"

Observe: d and $s(n,m)$ are computable.

Tarski's Theorem

Truth $= \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in TA \}$

Proof of Tarski's Theorem: Suppose Truth is arithmetical.

Then $R(x) = \neg \mathrm{Truth}(d(x))$ is as well.

Let $\widetilde{R}(x)$ represent $R(x)$.

Denote by e the encoding of $\widetilde{R(x)}$.

Then $d(e) = R(\widetilde{S}\ )$.

"I am false"

Tarski's Theorem

$\text{Truth} = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in TA \}$

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Denote by e the encoding of $\widetilde{R}(x)$.

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"I am false"

Then

$$\widetilde{R}(s_e) \in TA \iff \neg \text{Truth}(d(e)) \quad \text{Since } \widetilde{R} \text{ represents } R$$

$$\iff \neg \text{Truth}(\widetilde{R}(s_e)) \quad \text{by defn of } d \,\&\, e$$

$$\iff \neg(\widetilde{R}(s_e) \in TA) \quad \text{by defn of Truth}$$

Contradiction $\therefore$ Truth is not arithmetical.

First Incompleteness Theorem

We have proven:

1. Every r.e. predicate/language is arithmetical

2. Truth is not arithmetical.

∴ Truth is not r.e.

Truth not r.e. ⟹ TA not axiomatizable.

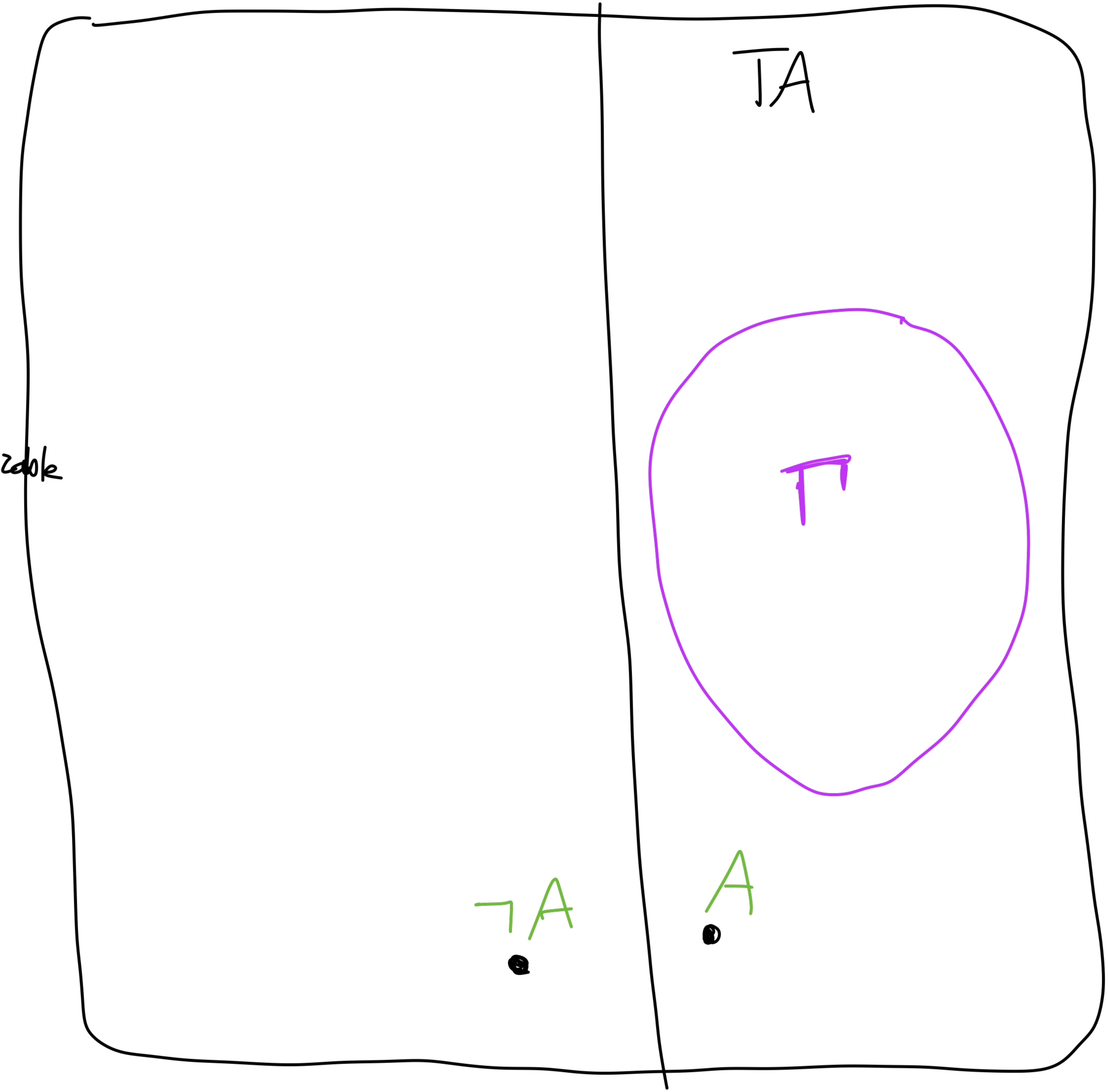
∴ Any sound axiomatizable theory is incomplete.

Φ_ω:

All $\mathcal{L}_A$ sentences

Γ sound & axiomatizable

$\Rightarrow \exists A, \neg A \notin \Gamma$



2nd Incompleteness Theorem

- We define PA (Peano Arithmetic) an axiomatizable theory.
- Most of number theory provable in PA
- We will see that PA cannot prove its own consistency (2nd Incompleteness Theorem)

→ Can be generalized to show that <u>any</u> consistent theory cannot prove its own consistency.

Peano Arithmetic

Peano Postulates (Peano, 1889):

Set of postulates characterizing $\mathbb{N}$

Let $\mathbb{N}$ be a set with $0 \in \mathbb{N}$, and let $S: \mathbb{N} \to \mathbb{N}$ be a function satisfying the following "Peano postulates"

GP1: $S(x) \neq 0 \quad \forall x \in \mathbb{N}$

GP2: If $S(x) = S(y)$ then $x = y$

GP3: Let $A \subseteq \mathbb{N}$ s.t. $0 \in A$ and A is closed under S (i.e. $S(x) \in A \ \forall x \in \mathbb{N}$) Then $A = \mathbb{N}$ (GP3 is a form of induction)

Peano Arithmetic

Peano Postulates (Peano, 1889):

GP1: $S(x) \neq 0 \quad \forall x \in \mathbb{N}$

GP2: If $S(x) = S(y)$ then $x = y$

GP3: Let $A \subseteq \mathbb{N}$ s.t. $0 \in A$ and A is closed under S (i.e. $S(x) \in A \ \forall x \in \mathbb{N}$)
Then $A = \mathbb{N}$

Peano Postulates Characterize $\mathbb{N}$ upto isomorphism
- Any $\langle \mathbb{N}, S, 0 \rangle$, $\langle \mathbb{N}', S', 0' \rangle$ satisfying GP1 – GP3 are isomorphic.

However: Cannot formulate GP3 except with set theory.

Peano Arithmetic

To define PA we will use $\mathcal{L}_A = [0, S, +, \cdot ; =]$.

- GP1 & GP2 are easily formulated
- To try to formulate GP3 by representing Sets by formulas $A(x)$ in $\mathcal{L}_A$.

Axioms of PA: (recursively define $+, \cdot$)

P1 $\forall x (sx \neq 0)$

P2 $\forall x \forall y (Sx = Sy) \supset x = y$ — S is 1-1

P3 $\forall x (x + 0 = x)$

P4 $\forall x \forall y (x + sy = s(x+y))$

$\}$ define +

P5 $\forall x (x \cdot 0 = 0)$

P6 $\forall x \forall y (x \cdot sy = (x \cdot y) + x)$

$\}$ define $\cdot$

Ind(A(x)): $\forall y_1 \cdots \forall y_k [(A(0) \wedge \forall x (A(x) \supset A(sx))) \supset \forall x A(x)]$

Peano Arithmetic

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Axioms of PA: (recursively define $+, \cdot$)

P1 $\forall x (Sx \neq 0)$

P2 $\forall x \forall y (Sx = Sy) \supset x = y)$ S is 1-1

P3 $\forall x (x + 0 = x)$

P4 $\forall x \forall y (x + Sy = S(x+y))$ $\Big\}$ define +

P5 $\forall x (x \cdot 0 = 0)$

P6 $\forall x \forall y (x \cdot Sy = (x \cdot y) + x)$ $\Big\}$ define $\cdot$

Ind(A(x)): $\forall y_1 \ldots \forall y_k [(A(0) \wedge \forall x (A(x) \supset A(Sx))) \supset \forall x A(x)]$

Induction axioms: All sentences Ind(A(x)) for formulas A whose free variables are among $x, y_1, \ldots, y_k$

$T_{PA} = \{P_1, \ldots, P_6\} \cup \{\text{Induction axioms}\}$

Peano Arithmetic

$T_{PA} = \{P_1, \ldots, P_6\} \cup \{\text{Induction axioms}\}$

- T_{PA} is recursive

Peano Arithmetic: $PA = \{A \in \Phi_0 \mid T_{PA} \vdash A\}$

- PA is a sound, axiomatizable theory.