Week 11

- Test II : Thursday 3-5 pm
- Extra office hrs posted
- See course webpage for practice problems

TODAY:

1. A specific sentence "I am not provable" $\equiv g$ such that neither g nor $\neg g$ are provable in PA (assuming PA is consistent)

2. Consistency of PA, Con(PA) is not provable in PA (assuming PA is consistent)

- Let Γ_{PA} be the set of axioms of PA
- Let $\mathrm{Proof}(x, y)$: true if and only if y codes a LK-Γ_{PA} proof of the sentence coded by x
- Recall $d(n) = \# A(s_n)$ where $\# A(x) = n$

  (so n codes the formula $A(x)$, and $d(n)$ codes $A(s_n)$)

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- Let $S(x)$ be the r.e. relation: $\exists y \, \text{Proof}(d(x), y)$
- By RA representation Theorem, let $A(x)$ be a $\exists\Delta_0$ formula that represents $S(x)$ in RA (& hence in PA)
- Then $\forall n \in \mathbb{N}$ $\exists y \, \text{Proof}(d(n), y) \Longleftrightarrow PA \vdash A(s_n)$ $(*)$

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- Let $e = \#\neg A(x)$, so $d(e) = \#\neg A(s_e)$

- Let $g \stackrel{d}{=} \neg A(s_e)$

  says that "I am not provable" since $\neg A(s_e)$ says the formula encoded by $d(e)$ -- which is g -- is not provable in PA

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- Let $e = \#\neg A(x)$, so $d(e) = \#\neg A(S_e)$
- Let $g \overset{d}{=} \neg A(S_e)$

<u>Theorem</u> PA consistent $\Longrightarrow$ PA $\nvdash g$

<u>Pf</u> Suppose PA $\vdash g$

Then sentence number $d(e)$ is provable, so $\exists y\, Proof(d(e), y)$ holds

Thus PA $\vdash A(S_e)$ by <u>left-to-rt</u> direction of (*)

<u>Thus PA $\vdash$</u> $\neg g$ and PA $\vdash g$ so PA not consistent

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- By RA representation Theorem, let $A(x)$ be a $\exists\Delta_0$ formula that represents $S(x)$ in RA (& hence in PA)
- Let $e = \# \neg A(x)$, so $d(e) = \# \neg A(S_e)$
- Let $g \stackrel{d}{=} \neg A(S_e)$

Theorem PA consistent $\Rightarrow$ PA $\nvdash \neg g$

Pf Suppose $PA \vdash \neg g$. ie PA proves $A(S_e)$

Then $\underline{\exists y \, Proof(d(e), y)}$ by rt-to-left direction of $(*)$

So PA proves $\neg A(S_e)$

So $PA \vdash g$ and $PA \vdash \neg g$, so PA not consistent

Formulating consistency in PA

Let $B(x,y)$ be a $\exists\Delta_0$ formula that represents
$\text{Proof}(x,y)$ in RA (and thus also in PA)

Then for every sentence C

$$PA \vdash C \iff PA \vdash \exists y\, \underbrace{B(\#C, y)}_{\text{stands for } B(S_{\#C}, y)}$$

Then $PA \vdash A(S_n) \supset \exists y\, B(S_{d(n)}, y)$

[recall $A(x)$ represents $\exists y\, B(d(x), y)$]

Formulating consistency in PA

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[recall $A(x)$ represents $\exists y\, B(d(x), y)$]

Define $\mathrm{con}(PA) \stackrel{d}{=} \neg \exists y B(\#0 \neq 0, y)$

Theorem If PA is consistent, then $PA \nvdash con(PA)$

Proof :

Main Lemma : $PA \vdash (con(PA) \supset g)$

[recall $g \stackrel{d}{=} \neg A(s_e)$, $e = \# \neg A(x)$ says "I am not provable"]

If $PA \vdash con(PA)$ by main Lemma $PA \vdash g$

But by previous theorem
PA consistent $\Rightarrow PA \nvdash g$

$\therefore$ PA consistent $\Rightarrow PA \nvdash con(PA)$

It is left to prove:

Main Lemma: $PA \vdash con(PA) \supset g$

[recall $g \stackrel{d}{=} \neg A(S_e)$, $e = \# \neg A(x)$ says "I am not provable"]

Need to formalize Proof of Gödel's Incompleteness Thm in PA. Main step is to formalize in PA that every true $\exists \Delta_0$ sentence is provable in RA.

Review for Test II

1. completeness of LK
   derivational completeness

2. computability : diagonalization D
   recursive / r.e.

   recursive, re not rec, not re

K, D, Halt
$\leftarrow x, y$

3. Incompleteness.

- Defns: theory consistent, sound, axiomatizable
- a relation $R(\vec{x})$ is represented by a ($\exists\Delta_0$) formula $A(\vec{x})$

$$\forall \vec{a} \in \mathbb{N}^n \quad R(\vec{a}) \text{ iff } \underset{(\underline{\mathbb{N}})}{TA} \models A(S_{\vec{a}})$$

- A relation $R(\vec{x})$ is represented in Σ by $A(\vec{x})$

$$\forall \vec{a} \in \mathbb{N}^n \quad R(\vec{a}) \text{ iff } \Sigma \vdash A(S_{\vec{a}})$$

* every $\exists$ r.e. relation is represented by a $\exists\Delta_0$ formula } $\exists\Delta_0$ Theorem

every $\exists\Delta_0$ formula is r.e.

Corollaries

Cor 2: TA is not axiomatizable

(Tarski's thm: TA is not arithmetical so not r.e.)

Cor 3 every sound axiom theory is incomplete

PA, RA

__RA represent. Thm__

every r.e. relation is represented in RA by $\exists\Delta_0$ formula.

__Strong Rep Thm__

every recursive reln is strongly represented in RA by $\exists\Delta_0$ formula

strongly: $\left.\begin{array}{l} R(\vec{a}) \\ \neg R(\vec{a}) \end{array}\right\} \begin{array}{l} \Longrightarrow RA \vdash A(S_{\vec{a}}) \\ \Longrightarrow RA \vdash \neg A(S_{\vec{a}}) \end{array}$

Corollary every ~~sound~~ consistent extension of RA is undecidable

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Today

1. $PA \nvdash g$, $PA \nvdash \neg g$

So PA is incomplete (assuming PA is consistent)

$\Rightarrow$ 2. $PA \nvdash cons(PA)$

5 questions (~65 pts)

~~3~~ → 20 - 25 computability

→ ~~rest~~ one q on PA proof & something

rest (other half)
incompleteness / defns

→ P1 → Ind(A)

?

PA axioms, equal axioms