Week 10

→ HW 4 Due this Friday Nov 29

→ Extra office hours posted (Today 5-6 as usual; also Wed 2-3)

→ Next Monday: Wrap-Up
Review for test II

→ TEST II : Thursday Dec 5 3-5pm

Review of Definitions

$\mathcal{L}_A = \{0, s, +, \cdot\ ; =\}$ Language of arithmetic

$\Phi_0 =$ all $\mathcal{L}_A$-sentences

$TA = \{A \in \Phi_0 \mid \mathbb{N} \models A\}$ True Arithmetic

A <u>theory</u> Σ is a set of sentences (over $\mathcal{L}_A$) closed under logical consequence

- We can specify a theory by a subset of sentences that logically implies all sentences in Σ

Σ is <u>consistent</u> iff $\Phi_0 \neq \Sigma$ (iff $\forall A \in \Phi_0$, either A or $\neg A$ Not in Σ)

Σ is <u>complete</u> iff Σ is consistent and $\forall A$ either A or $\neg A$ is in Σ

Σ is <u>sound</u> iff $\Sigma \subseteq TA$

Let $\mathfrak{M}$ be a model/structure over $\mathcal{L}_A$

$Th(\mathfrak{M}) = \{ A \in \Phi_0 \mid \mathfrak{M} \models A \}$

$Th(\mathfrak{M})$ is <u>complete</u> (for all structures $\mathfrak{M}$)

<u>Note</u> $TA = Th(\mathbb{N})$ is complete, consistent, & sound

$VALID = \{ A \in \Phi_0 \mid \models A \}$ $\longleftarrow$ smallest theory

Let Σ be a theory

Σ is <u>axiomatizable</u> if there exists a set $\Gamma \subseteq \Sigma$ such that

1. Γ is recursive
2. $\Sigma = \{ A \in \Phi_0 \mid \Gamma \models A \}$

<u>Theorem</u> Σ is axiomatizable iff Σ is r.e.

(P. 76 of Notes)

Recap: First Incompleteness Theorem

1. TA is not r.e. (so by previous theorem, not axiomatizable)

   First Incompleteness Theorem: Every sound axiomatizable theory is incomplete

FIRST INCOMPLETENESS THEOREM

We define a predicate $Truth \subseteq \mathbb{N}$

$$Truth = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in \Phi_0 \text{ that is in } TA \}$$

We will show that $Truth$ is not r.e.

FIRST INCOMPLETENESS THEOREM

SHOW:

1. Every r.e. predicate/language is arithmetical — Exists-Delta Theorem pp. 68-71
2. Truth is not arithmetical — Tarski Theorem 73-74

∴ Truth is not r.e.

Truth not r.e. ⇒ TA Not axiomatizable

∴ Any SOUND, axiomatizable theory is incomplete

① Every R.e. predicate is arithmetical

Definition Let $s_0 = 0$, $s_1 = S0$, $s_2 = SS0$, etc.

Let $R(x_1 .. x_n)$ be an n-ary relation $R \subseteq \mathbb{N}^n$

Let $A(x_1, .., x_n)$ be an $\mathcal{L}_A$ formula, with free variables $x_1, .., x_n$

$A(\vec{x})$ **represents** R iff $\forall \vec{a} \in \mathbb{N}^n \quad R(\vec{a}) \iff \mathbb{N} \models A(s_{a_1}, s_{a_2} .. s_{a_n})$

R is **arithmetical** iff there is a formula $A \in \mathcal{L}_A$ that represents R

Exists-Delta-Theorem every r.e. relation is arithmetical. In fact every r.e. relation is represented by a $\exists\Delta_0$ $\mathcal{L}_A$-formula.

② Truth is not Arithmetical

Define the predicate $\text{Truth} \subseteq \mathbb{N}$

$$\text{Truth} = \{ n \mid n \text{ encodes a sentence } \langle n \rangle \in TA \}$$

Then Truth is not arithmetical

High Level idea:

Formulate a sentence "I am false"
which is self-contradictory

Pf of Tarski's Thm

Let $sub(m,n) = \begin{cases} 0 \text{ if } m \text{ is not a legal encoding of a formula} \\ \text{otherwise say } m \text{ encodes the formula} \\ \quad A(x) \text{ with free variable } x. \\ \quad \text{Then } sub(m,n) = m' \text{ where } m' \\ \quad \text{encodes } A(s_n) \end{cases}$

Let $d(n) = sub(n,n)$

$$\left\{ \begin{array}{l} d(n) = 0 \text{ if } n \text{ not a legal encoding.} \\ \qquad\quad \text{ow say } n \text{ encodes } A(x). \\ \qquad\quad \text{then } d(n) = n' \text{ where } n' \text{ encodes } A(s_n) \end{array} \right\}$$

clearly sub, d are both computable

Proof of Tarski's Thm

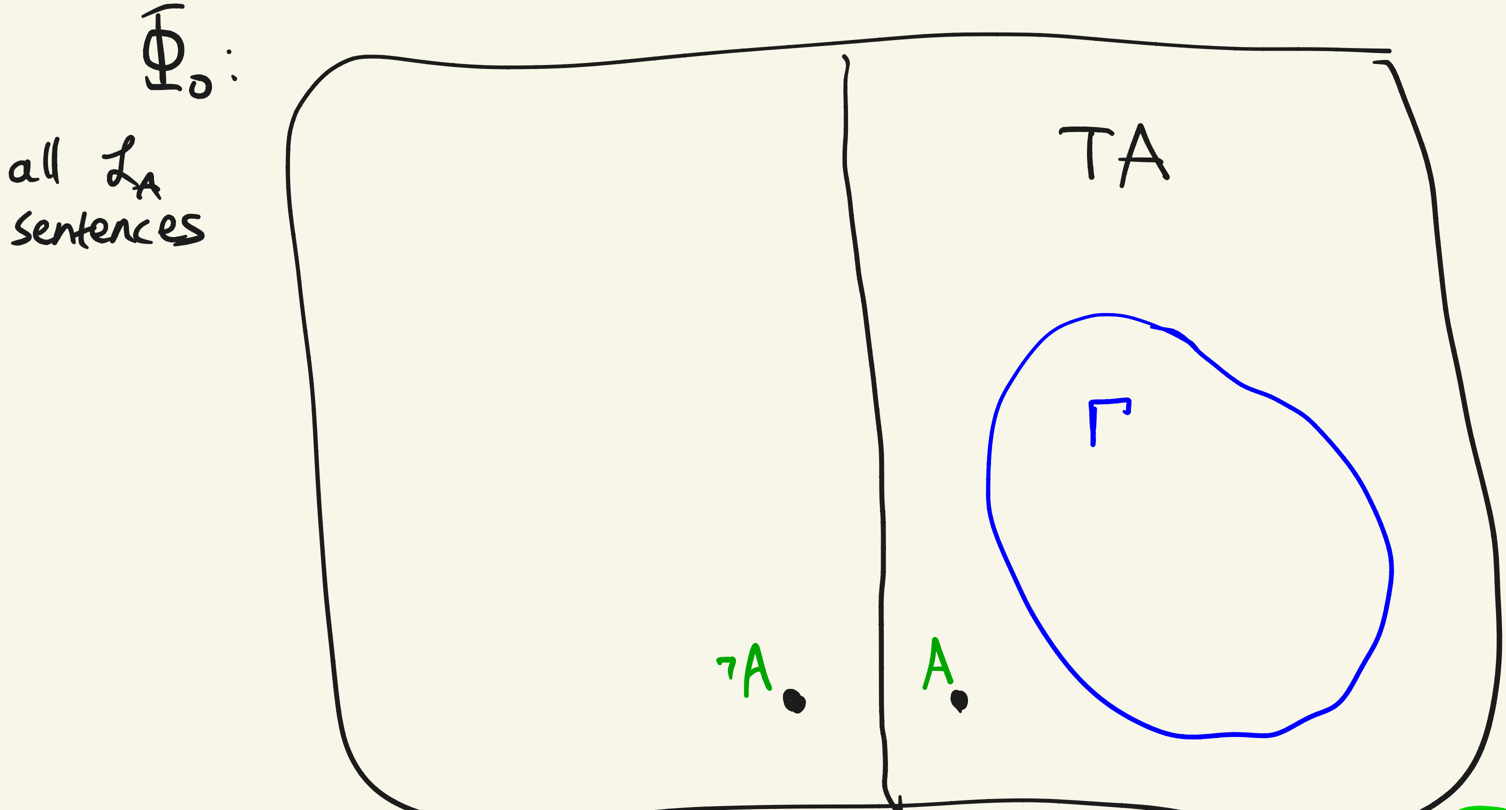
Suppose that Truth is arithmetical.
Then define $R(x) = \neg \mathrm{Truth}(d(x))$

Since d, Truth both arithmetical, so is R
Let $\widetilde{R(x)}$ represent $R(x)$, and let e be the encoding of $\widetilde{R(x)}$
Let $d(e) = e'$ so e' encodes $\widetilde{R(s_e)}$

encodes "I am false"

Then
$$\widetilde{R(s_e)} \in TA \iff \neg \mathrm{Truth}(d(e)) \qquad \text{since } \widetilde{R} \text{ represents } R$$
$$\iff \neg \widetilde{R(s_e)} \in TA \qquad \text{by defn of truth}$$
$$\iff \widetilde{R(s_e)} \notin TA \qquad \text{TA contains exactly one of } A, \neg A$$

↯ this is a contradiction. $\therefore$ Truth is not arithmetical



Γ sound and axiomatizable $\Longrightarrow$ $\exists A, \neg A \not\in \Gamma$

Corollary 3 in Notes

Notes

- Tarski's Theorem holds for any theory that can define $0, s, +, \cdot$ on $\mathbb{N}$

Incompleteness Theorems

① TA is not r.e. (so by previous theorem, not axiomatizable)

First Incompleteness Theorem Every sound axiomatizable theory is incomplete

② Define PA – Peano arithmetic
Sound, axiomatizable (so incomplete by ①)
RA – Finitely axiomatizable subtheory of PA

Strengthened First Incompleteness Theorem Every consistent, axiomatizable extension of RA is incomplete

③ **Second Incompleteness Theorem:**

A specific sentence asserting "PA is consistent" is not a theorem of PA

Incompleteness Theorems

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③ Second Incompleteness Theorem:

A specific sentence asserting "PA is consistent" is not a theorem of PA

Peano Arithmetic

- We introduce a standard set of axioms for $\mathcal{L}_A$
  PA (Peano Arithmetic) is the theory associated with these axioms

- PA is <u>sound</u>, so by (corollary to) Incompleteness
  PA is <u>incomplete</u>

- PA still strong enough to prove all of standard number theory and more

PEANO ARITHMETIC AXIOMS

$$\left.\begin{array}{ll} \text{P1.} & \forall x\,(sx \neq 0) \\ \text{P2.} & \forall x \forall y\,(sx = sy \supset x = y) \end{array}\right\} \text{Define } s$$

$$\left.\begin{array}{ll} \text{P3.} & \forall x\,(x + 0 = x) \\ \text{P4.} & \forall x \forall y\,(x + sy) = s(x+y) \end{array}\right\} \text{Define } +$$

$$\left.\begin{array}{ll} \text{P5.} & \forall x\,(x \cdot 0 = 0) \\ \text{P6.} & \forall x \forall y\,(x \cdot sy = (x \cdot y) + x) \end{array}\right\} \text{Define } \cdot$$

PEANO ARITHMETIC AXIOMS

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Induction Let $A(\underbrace{x, y_1, \ldots, y_k}_{\text{free variables}})$ be a $\mathcal{L}_A$ formula

$$\mathrm{IND}(A(x)) : \forall y_1 \forall y_2 \ldots \forall y_k \left[\left(A(0) \wedge \forall x \left(A(x) \supset A(sx) \right) \right) \supset \forall x\, A(x) \right]$$

$$\Gamma_{PA} = \{P1, \ldots, P6\} \cup \{\mathrm{IND}(A(x))\} \qquad PA = \{A \in \Phi_0 \mid \Gamma_{PA} \models A\}$$

PEANO ARITHMETIC

- PA is recursive, and axiomatizable
- PA is sound
- PA can prove all of elementary number theory even though it is incomplete

PEANO ARITHMETIC

Exercise : Try proving some basic facts about $+, \cdot, S, 0$

Example 1 $\forall x \forall y \forall z \; (x+y)+z = x+(y+z)$

Example 3 $\forall x \forall y \forall z \; ((x \cdot z + y \cdot z) = (z \cdot (x+y))$

RA (Robinson's Arithmetic)

- A weak subtheory of PA
- Axioms of RA: P1, P2, ..., P7, P8, P9

P7. $\forall x (x \le 0 \supset x = 0)$
P8. $\forall x \forall y (x \le sy \supset (x \le y \vee x = sy))$
P9. $\forall x \forall y (x \le y \vee y \le x)$

where $t_1 \le t_2$ stands for $\exists z (t_1 + z = t_2)$

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where $t_1 \leq t_2$ stands for $\exists z (t_1 + z = t_2)$

FACTS

(1) $RA \subseteq PA$ (show $\Gamma_{PA} \models P7$, $\Gamma_{PA} \models P8$, $\Gamma_{PA} \models P9$)

(2) RA is finitely axiomatizable

(3) over $\mathcal{L}_{A,\leq}$ axioms of RA are $\forall$sentences

RA Representation Theorem

Theorem Every r.e. relation is representable in RA by an $\exists\Delta_0$ formula

RA Representation Theorem

Theorem Every r.e. relation is representable in RA by an $\exists\Delta_0$ formula

- Major result that extends the Exists-Delta Theorem (every r.e. relation is represented by an $\exists\Delta_0$ formula)

$R(\vec{x})$ is represented by an $\exists\Delta_0$-formula $A(\vec{x})$:

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow \mathbb{N} \models A(S_{\vec{a}}) \Leftrightarrow TA \models A(S_{\vec{a}})$$

$R(\vec{x})$ is represented in RA by an $\exists\Delta_0$ formula $A(\vec{x})$:

$$\forall \vec{a} \in \mathbb{N} \quad R(\vec{a}) \Leftrightarrow RA \models A(S_{\vec{a}})$$

Corollaries of RA Representation Theorem

Definition A theory is decidable if the associated set of sentences in the theory is recursive

Definition Σ' is an extension of Σ if $\Sigma \subseteq \Sigma'$ (Σ', Σ are theories)

Example $\text{VALID} \subseteq RA \subseteq PA \subseteq TA$

Corollaries of RA Representation Theorem

Corollary 1 Every sound extension of RA is undecidable
(not recursive)

Proof Let Σ be a sound extension of RA,
and consider a language such as K that is
r.e. but not recursive. Since K is r.e., it is represented
in Σ by some $\exists\Delta_0$ formula $A(x)$.

If Σ were recursive then K would be recursive

i.e. $a \in K \iff \underset{\Sigma}{\mathrm{RA}} \vDash A(S_a)$

Corollaries of RA Representation Theorem

<u>Corollary 2</u> (Church's Theorem)

VALID ($= \Phi_0$) is Not recursive

<u>Proof</u> Since RA is finitely axiomatizable

$A \in RA \iff (P1 \wedge P2 \wedge \dots \wedge P9 \supset A)$ is valid

So membership in RA is reducible to membership in VALID!

RA Representation Theorem – Proof

MAIN LEMMA Every bounded (Δ_0) sentence in TA is provable in RA.

Example of a bounded sentence of TA:

$$\forall x \leq 100 \; \exists y \leq 2 \cdot x \quad [x = 0 \vee x < y \wedge \mathrm{Prime}(y)]$$

How to prove $\uparrow$ in RA?

RA Representation Theorem – Proof

MAIN LEMMA Every bounded (Δ_0) sentence in TA is provable in RA.

Technical convenience: $RA_{\leq}$ is RA where $\leq$ is added as a new symbol, and axioms of $RA_{\leq}$ are those of RA (P1 .. P9) plus

P0 $\forall x \forall y (x \leq y \leftrightarrow \exists z (x + z = y))$

We will prove MAIN LEMMA for $RA_{\leq}$

RA Representation Theorem – Proof

MAIN LEMMA Every bounded (Δ_0) sentence in TA is provable in $RA_{\leq}$

Proof by induction on the number of logical operators (other than $\neg$) on A.

RA Representation Theorem – Proof

MAIN LEMMA Every bounded (Δ_0) sentence in TA is provable in $RA_\leq$

Base case A: $t = u$, $\neg(t = u)$, $t \leq u$, $\neg(t \leq u)$

Lemma A1 $RA_\leq \vdash s_m + s_n = s_{m+n}$ $\forall m, n \in \mathbb{N}$

$RA_\leq \vdash s_m \cdot s_n = s_{m \cdot n}$

Lemma A If t is a closed term (no variables in t) and $TA \models t = s_n$ then $RA_\leq \vdash t = s_n$

by induction using A1

Lemma B $\forall m \neq n \in \mathbb{N}$ $RA_\leq \vdash s_n \neq s_m$

Lemma C $RA_\leq \vdash \forall x\,(x \leq s_n \supset (x = 0 \vee x = s_1 \vee \dots \vee x = s_n))$

RA Representation Theorem – Proof

MAIN LEMMA Every bounded (Δ_0) sentence in TA is provable in $RA_{\leq}$

Induction Step (assume all $\neg$'s pushed inside)

(1) Outermost connective of A is $\wedge$ or $\vee$: apply induction hyp

(2) A is $\forall x \leq t\, B(x)$. Then t is closed so

- by Lemma A $RA \vdash t = s_n$ for some n
  Say $n = 23$.
- Show $RA_{\leq} \vdash \forall x \leq 23\, B(x)$
- By Lemma C, $RA_{\leq} \vdash x \leq 23 \supset (x = 0 \vee x = 1 \vee \dots \vee x = 23)$
- By induction hyp, $RA_{\leq} \vdash B(c)$, $c \in \{0, 1, \dots, 23\}$
- Put all together to get $RA_{\leq} \vdash \forall x \leq 23\, B(x)$

RA Representation Theorem — Proof

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Induction Step (assume all $\neg$'s pushed inside)

(1) Outermost connective of A is $\wedge$ or $\vee$: apply induction hyp

(2) A is $\forall x \leq t\ B(x)$. ✓

(3) A is $\exists x \leq t\ B(x)$
easier (don't need Lemma C)

Consequences of MAIN LEMMA

- Every $\exists\Delta_0$ sentence of TA is provable in RA
- The set of $\exists\Delta_0$ sentences of TA is re but not decidable

(the bounded sentences of TA are decidable)

RA Representation Theorem – Proof

RA Representation Theorem Every r.e. relation is represented in RA by an $\exists\Delta_0$ formula

MAIN LEMMA Every bounded (Δ_0) sentence in TA is provable in $RA_{\leq}$

Proof Let $R(\vec{x})$ be an r.e. relation

- by Exists-Delta theorem, $R(\vec{x})$ is represented in TA by some $\exists\Delta_0$ formula $\exists y A(\vec{x}, y)$.

  So $\forall \vec{a} \in \mathbb{N}^n \quad R(\vec{a}) \Leftrightarrow [\exists y\, A(S_{\vec{a}}, y) \in TA]$

- By soundness of RA, and since every $\exists\Delta_0$ sentence of TA is provable in RA

$$R(\vec{a}) \Leftrightarrow [RA_{\leq} \vdash \exists y\, A(S_{\vec{a}}, y)]$$

- So $\exists y A(\vec{x}, y)$ represents $R(\vec{x})$ in $RA_{\leq}$

Results for consistent (but possibly unsound) theories

Theorem Every consistent extension of RA is undecidable (not recursive)

Corollary (Strengthening of First Incompleteness Theorem)
Every consistent, axiomatizable extension of RA is incomplete

↑

Strengthens previous corollary 3 of Tarski's Theorem
Now we don't have to assume soundness.
consistency is syntactic notion (no proof of $0=1$ from axioms)
soundness is semantic

Results for consistent (but possibly unsound) theories

<u>Theorem</u> Every consistent extension of RA is undecidable

<u>Definition</u> (strongly represents)

$A(\vec{x})$ strongly represents $R(\vec{x})$ in Σ if $\forall \vec{a} \in \mathbb{N}^n$

$$R(\vec{a}) \Rightarrow A(s_{\vec{a}}) \in \Sigma$$

$$\neg R(\vec{a}) \Rightarrow \neg A(s_{\vec{a}}) \in \Sigma$$

(Before we had: $R(\vec{a}) \Leftrightarrow A(s_{\vec{a}}) \in \Sigma$)

as long as Σ is consistent, strongly represents $\Rightarrow$ represents

Results for consistent (but possibly unsound) theories

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Strong RA Representation Theorem Every recursive relation is strongly representable in RA by an $\exists\Delta_0$ formula

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Strong RA Representation Theorem Every recursive relation is strongly representable in RA by an $\exists \Delta_0$ formula

Undecidability theorem If every recursive relation is representable in Σ, then Σ is undecidable (not recursive)

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Like Proof of RA Representation Theorem

Undecidability theorem If every recursive relation is representable in Σ, then Σ is undecidable

Like Proof of Tarski's Theorem

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Like Proof of RA Represent'n Theorem

Undecidability theorem If every recursive relation is representable in Σ, then Σ is undecidable

Like Proof of Tarski's Theorem

Proof (of Theorem) $R(\vec{x})$ recursive

$\Rightarrow R(\vec{x})$ strongly rep in RA (strong RA Rep Thm)

$\Rightarrow R(\vec{x})$ strongly rep. in every extension of RA

$\Rightarrow R(\vec{x})$ rep. in every consistent extension of RA

$\Rightarrow$ If Σ a consistent extension of RA then Σ is undecidable (Undec. Thm)