UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2016 EXAMINATIONS

CSC 438H1F/2404H1F

Duration - 3 hours
No Aids Allowed

There are 6 questions worth a total of 100 marks.
Answer all questions on the question paper, using backs of pages for scratch work.
Check that your exam book has 9 pages (including this cover page).

PLEASE COMPLETE THIS SECTION:

Name ______________________________

(Please underline your family name.)

Student Number ______________________________

FOR USE IN MARKING:

1. ____________/10
2. ____________/26
3. ____________/10
4. ____________/4
5. ____________/4
6. ____________/8
7. ____________/8
8. ____________/10
9. ____________/10
10. ____________/10

Total:__________/100

[10] 1. Let f and g be unary function symbols, and let A be the formula $\forall x(fgx = x)$ and let B be the formula $\forall x(gfx = x)$. Prove that $A \not\models B$.

2. Let $\mathcal{L}_s$ (the vocabulary of successor) be the vocabulary $[0, s; =]$. Let $Th(s)$ (theory of successor) be the set of all sentences over this vocabulary which are logical consequences of the following infinite set Ψ_s of axioms:

P1) $\forall x(sx \neq 0)$
P2) $\forall x \forall y(sx = sy \supset x = y)$
Q) $\forall x(x = 0 \vee \exists y(x = sy))$ (every nonzero element has a predecessor)

S1) $\forall x(sx \neq x)$
S2) $\forall x(ssx \neq x)$
S3) $\forall x(sssx \neq x)$
.
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.

[12] (a) Prove that for each $n \geq 1$ the axiom Sn is not a logical consequence of {P1,P2,Q,S1,S2,...S$n-1$}. (Do this by giving a model.)

[8] (b) Prove using (a) that $Th(s)$ is not finitely axiomatizable. That is, show that there is no finite set Γ of sentences in $Th(s)$ such that every sentence in $Th(s)$ is a logical consequence of Γ. (Note that the sentences in Γ are not necessarily among the original set Ψ_s of axioms.)

[6] (c) Use the fact that every sentence true in the standard model $\underline{\mathbb{N}}_s$ for the language $\mathcal{L}_s$ is in $Th(s)$ to show that $Th(s)$ is decidable.

[10] 3. Use results proved in class to prove that the function $f(x) = \mu y T(x, x, y)$ has no total computable extension.

[4] 4. Give an example of an arithmetical relation which is not r.e.

[4] 5. Give an example of a relation which is not arithmetical.

[8] 6. Let $\mathcal{L}$ be a first-order language with finitely many function and predicate symbols. Prove that the set of unsatisfiable $\mathcal{L}$-sentences is r.e., using results proved in class.

[8] 7. Recall that **RA** is a theory with 9 axioms P1, ... P9 over the language $\mathcal{L}_A$.
The **RA Representation Theorem** states that every r.e. relation is representable in **RA** by an $\exists\Delta_0$ formula. Use this theorem to prove that **RA** is undecidable.

[10] 8. Use the **RA Representation Theorem** (see previous question) to prove that every sound theory Σ with vocabulary $\mathcal{L}_A$ is undecidable. (Recall that Σ is *sound* if $\underline{\mathbb{N}}$ is a model of Σ.)

[10] 9. Let f be a unary function (not necessarily total). Recall that $graph(f)$ is the relation $R_f(x, y) = (y = f(x))$. Prove that if $graph(f)$ is r.e. then f is recursive. DO NOT USE CHURCH'S THESIS. (Or use Church's thesis for part credit.)

[10] 10. Let Σ be an axiomatizable theory over the vocabulary $\mathcal{L}_A$ of arithmetic such that every r.e. relation is representable in Σ by some $\exists\Delta_0$ formula. Show that there is a $\forall\Delta_0$ sentence (one of the form $\forall yB$, where B is bounded) such that $\Sigma \nvdash A$ and $\Sigma \nvdash \neg A$.