EQUIVALENCE WITH OTHER MODELS

SO FAR WE HAVE PRESENTED SOME VARIANTS OF THE BASIC TM MODEL AND HAVE SHOWN THEM TO BE EQUIVALENT IN POWER.

GENERAL PURPOSE COMPUTATIONAL MODELS ALL SHARE ESSENTIAL FEATURES OF TMS: UNLIMITED MEMORY

IT TURNS OUT THAT ALL MODELS WITH THAT FEATURE ARE EQUIVALENT IN POWER, ASSUMING A FEW REASONABLE REQUIREMENTS.

THIS LEADS US TO THE FAMOUS CHURCH-TURING THESIS

$$\boxed{\text{INTUITIVE NOTION OF ALGORITHMS} \equiv \text{TM ALGORITHMS}}$$

BECAUSE "INTUITIVE NOTION OF ALGORITHM" IS NOT RIGOROUSLY DEFINED, IT IS A THESIS NOT A THEOREM. BUT IT IS TRUE FOR ALL CURRENT MODELS OF COMPUTATION.

FROM NOW ON, WE ACCEPT THE ABOVE THESIS.
ALGORITHM $\equiv$ TM ALGORITHM.

ENCODING TURING MACHINES BY BINARY STRINGS

WE WANT TO ASSOCIATE A BINARY NUMBER WITH EACH TURING MACHINE ACCEPTING A LANGUAGE OVER $\Sigma = \{0,1\}$. HERE IS ONE WAY TO DO THIS.

$$M = \Big(Q = \{q_1, q_2, \ldots, q_n\},\ \Sigma = \{\overset{X_1}{0}, \overset{X_2}{1}\},\ \Gamma = \{X_1, X_2 \ldots X_k\},$$
$$\delta,\ \underset{\text{START}}{q_1},\ \underset{\text{ACCEPT}}{q_2},\ \underset{\text{REJECT}}{q_3} \Big)$$

ASSUME AN ORDERING OF THE UNDERLYING STATES AND TAPE SYMBOLS. LET "L" BE D_1, "R" BE D_2. WE REPRESENT $\delta(q_i, X_j) \rightarrow (q_k, X_l, D_m)$ BY:

$$0^i 1 0^j 1 0^k 1 0^l 1 0^m$$

THE BINARY CODE FOR M IS:

$$111\,\text{code}_1\,11\,\text{code}_2\,11\,\text{code}_3\,11 \ldots\ 11\,\text{code}_r\,111$$

WHERE CODE_i IS THE CODE FOR ONE OF THE TRANSITIONS.

<u>EXAMPLE</u> $M = \Big(Q = \{q_1 q_2 q_3\},\ \Sigma = \{0, 1\},\ \Gamma = \{\underset{X_1}{0}, \underset{X_2}{1}, \underset{X_3}{b}\},\ \delta,\ q_1, q_2, q_3 \Big)$

$\delta(q_1, 1) \rightarrow (q_3, 0, R)$	$0^1 1 0^2 1 0^3 1 0 1 0^2$	$= \text{code}_1$
$\delta(q_2, 0) \rightarrow (q_1, 1, R)$	$0^2 1 0 1 0\ 1 0^2 1 0^2$	$= \text{code}_2$
$\delta(q_2, 1) \rightarrow (q_2, 0, R)$	$0^2 1 0^2 1 0^2 1 0 1 0^2$	$= \text{code}_3$
		$\vdots$

Code FOR M:

$$\langle M \rangle = 111\,\text{code}_1\,11\,\text{code}_2\,11 \ldots 111$$

A UNIVERSAL TURING MACHINE

A UNIVERSAL TURING MACHINE $\mathcal{U}$ TAKES AS INPUT A STRING $\underbrace{\langle M, w\rangle}_{111\,code_1\,11..11\,code_r\,111\,w}$

WHERE $\langle M, w\rangle$ ENCODES A TURING MACHINE M OVER $\Sigma = \{0,1\}$, FOLLOWED BY A STRING $w \in \{0,1\}^*$.

$\mathcal{U}$ ACCEPTS $\langle M, w\rangle$ IF AND ONLY IF M ON w ACCEPTS.

* NOTE THAT IF M DOES NOT ACCEPT w, THEN U MAY NOT HALT ON $\langle M, w\rangle$

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ IS A TURING MACHINE AND } M \text{ ACCEPTS } w\}$$

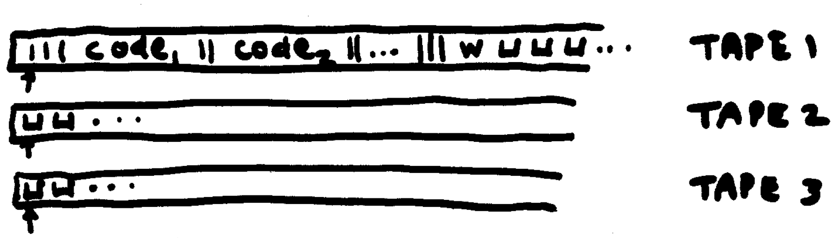
THEOREM A_{TM} IS RECURSIVELY ENUMERABLE.

PROOF SKETCH WE WILL CREATE A 3 TAPE TM, $\mathcal{U}$. ON INPUT $\langle M, w\rangle$, $\mathcal{U}$ WILL SIMULATE M ON w.

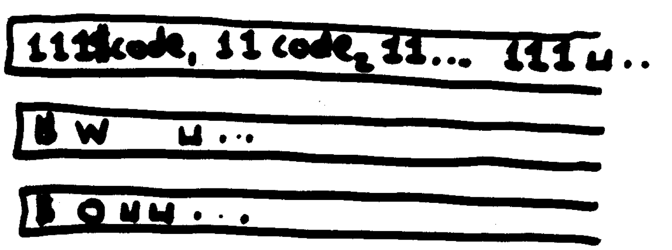
- IF M HALTS AND REJECTS w THEN $\mathcal{U}$ HALTS AND REJECTS $\langle M, w\rangle$
- IF M HALTS AND ACCEPTS w THEN $\mathcal{U}$ ACCEPTS $\langle M, w\rangle$
- IF M DOES NOT HALT ON w THEN $\mathcal{U}$ DOES NOT HALT ON $\langle M, w\rangle$

HIGH LEVEL DESCRIPTION OF $\mathcal{U}$ ON $\langle M, w\rangle$

0.) INITIALLY $\langle M, w\rangle$ ON TAPE 1



1. CHECK IF <M, w> IS A VALID ENCODING. IF NOT, REJECT.
    - VALID CODES BEGIN WITH "111" AND END WITH "111" FOLLOWED BY w
    - BETWEEN "111" IT HAS THE FORM $x_1 11 x_2 11 .. 11 x_q$ WHERE x_n LOOKS LIKE $0^i 1 0^j 1 0^k 1 0^l 1 0^m$, $m = 1$ or 2
    - TRANSITION FUNCTION SPECIFIED BY x_n's IS COMPLETE
2. INITIALIZE TAPE 2 TO CONTAIN #w
3. INITIALIZE TAPE 3 TO CONTAIN \$0 (q_1 IN BINARY)
4. INITIALIZE TAPE 1 TO HOLD <M>, with \$ AFTER 1st '111'



5. REPEAT:
    - IF TAPE 3 HOLDS \$000, HALT AND REJECT
    - IF TAPE 3 HOLDS \$00, HALT AND ACCEPT
    - LET x_j BE SYMBOL SCANNED BY HEAD 2, AND LET 0^i BE CONTENTS OF TAPE 3
    - SCAN TAPE 1 FROM \$ TO '111', LOOKING FOR STRING BEGINNING WITH $0^i 1 0^j 1$. SAY IT LOOKS LIKE: $0^i 1 0^j 1 0^k 1 0^l 1 0^m$
    - PUT 0^k ON TAPE 3; WRITE x_l ON TAPE CELL SCANNED BY HEAD 2; AND MOVE HEAD 2 ACCORDING TO $m = 1$ or 2 (L OR R).

UNDECIDABILITY

NOW WE ARE READY TO SHOW THAT THERE IS A LANGUAGE L OVER $\{0,1\}^*$ THAT IS NOT R.E. (RECURSIVELY ENUMERABLE.)

WE HAVE ALREADY SEEN THAT EACH TM CAN BE REPRESENTED BY A UNIQUE BINARY NUMBER.
THEREFORE IT IS POSSIBLE TO ORDER ALL TM'S OVER $\Sigma = \{0,1\}$:

$$T_{M1}, T_{M2}, T_{M3}, \ldots$$

WHERE $T_{Mi} < T_{Mj}$ IF THE CODE FOR T_{Mi} (Mi) IS LESS THAN THE CODE FOR T_{Mj} (Mj).

WE CAN ALSO ORDER ALL BINARY INPUTS

$w_1, w_2, w_3, \ldots$

$\varepsilon \quad 0 \quad 1 \quad 00 \quad 01 \quad 10 \quad 11 \quad \ldots$

DEFINE $L_D = \{ w_i \mid w_i$ IS THE i^{th} BINARY INPUT AND T_{Mi}, THE i^{th} TURING MACHINE DOES NOT ACCEPT $w_i \}$

<u>THEOREM</u> L_D IS NOT RECURSIVELY ENUMERABLE

PROOF SKETCH

CONSTRUCT THE TABLE BELOW. HORIZONTAL AXIS IS LABELLED WITH ALL BINARY INPUT STRINGS (IN ORDER). VERTICAL AXIS IS LABELLED WITH ALL TM's (IN ORDER). ENTRY (i,j) HAS VALUE 1 IF T_{Mi} ACCEPTS W_j AND HAS VALUE 0 OTHERWISE

* L_D WILL NOT BE ACCEPTED BY ANY OF THE TURING MACHINES IN THE TABLE BECAUSE FOR EVERY i, T_{Mi} GIVES THE WRONG ANSWER (DIFFERS FROM L_D) ON W_i.

	$\varepsilon = W_1$	$0 = W_2$	$1 = W_3$	$00 = W_4$	$01 = W_5$	...
T_{M1}	0	1	0	1	1	...
T_{M2}	1	0	1	0	0	..
T_{M3}	0	1	1	1	...	
T_{M4}	0	$\boxed{0}$	0	1	...	
⋮		...				

THE ONE'S OF THIS ROW CORRESPOND TO THE LANGUAGE ACCEPTED BY T_{M4}.

ENTRY (4,2) IS 0, SO T_{M4} DOES NOT ACCEPT $W_2 = 0$

DIAGONALIZATION

THE PREVIOUS PROOF IS AN EXAMPLE OF CANTOR'S DIAGONALIZATION METHOD, ORIGINALLY USED TO PROVE THAT THERE IS NO 1-1, ONTO FUNCTION FROM $\mathbb{N}$ TO $\mathbb{R}$ IN 1873!

__Def'n__ A SET S IS COUNTABLE IF S IS FINITE OR THERE IS A 1-1 ONTO FUNCTION FROM $\mathbb{N}$ TO S.

__THEOREM__ $\mathbb{R}$ (THE REAL NUMBERS) ARE NOT COUNTABLE

__PROOF__ ASSUME FOR SAKE OF CONTRADICTION THAT $\mathbb{R}$ IS COUNTABLE. LET f BE A 1-1, ONTO FUNCTION FROM $\mathbb{N}$ TO $\mathbb{R}$. EACH $x \in \mathbb{R}$ IS UNIQUELY REPRESENTED BY A DECIMAL

ie. $x = 0.425200\cdots$, $\pi = 3.141526\cdots$ ARE BOTH IN $\mathbb{R}$.

CONSTRUCT THE FOLLOWING TABLE BASED ON f:

n	f(n) truncated to digits after decimal	
1	.1 4 1 5 9	$f(1) = 3.14159\cdots$
2	.5 5 5 5 5	$f(2) = 55.555\cdots$
3	.1 $\boxed{2}$ 3 4	$f(3) = 555.1234$
4	.5 0 0 0 0	etc.
5	.4 5 · ·	
⋮		

entry (i,j) = value of j^{th} digit after decimal in $f(i)$

Let $X_d = 0.X_d^1 X_d^2 X_d^3 X_d^4 \cdots$ BE A REAL NUMBER WHERE

$$X_d^j = \begin{cases} 2 & \text{IF ENTRY } (j,j) \neq 2 \\ 3 & \text{IF ENTRY } (j,j) = 2 \end{cases}$$

X_d IS A REAL NUMBER, AND $\forall j$, $f(j) \neq X_d$.
THIS IS BECAUSE FOR EACH j $f(j)$ DIFFERS FROM X_d ON THE DIGIT X_d^j. CONTRADICTION.
THUS $\mathbb{R}$ IS NOT COUNTABLE (SINCE $\mathbb{R}$ ALSO NOT FINITE)

∎

THEOREM $\bar{L}_D = \{w_i \mid w_i$ IS THE i^{th} BINARY STRING AND T_{Mi} ACCEPTS $w_i\}$

IS NOT RECURSIVE (BUT IS R.E.)

PROOF

SUPPOSE $\bar{L}_D$ IS RECURSIVE FOR SAKE OF CONTRADICTION.
THEN SOME TM M ACCEPTS $\bar{L}_D$ AND ALWAYS HALTS.
CONSTRUCT M' FROM M: M' ACCEPTS AND HALTS ON x IF AND ONLY IF M HALTS AND REJECTS x
M' ALWAYS HALTS AND ACCEPTS L_D.
BUT THIS CONTRADICTS THE FACT THAT L_D IS NOT R.E.
$\therefore$ $\bar{L}_D$ IS NOT RECURSIVE.

GENERAL LEMMA L NOT R.E. $\Rightarrow$ $\bar{L}$ NOT RECURSIVE.

THEOREM $A_{TM} = \{\langle M, w\rangle \mid$ THE TM ENCODED BY M ACCEPTS $w\}$

IS NOT RECURSIVE BUT IS R.E.

PROOF WE WILL SHOW THAT A_{TM} IS NOT RECURSIVE BY REDUCING $\bar{L}_D$ TO A_{TM}.

WE WILL DEFINE A COMPUTABLE FUNCTION f FROM Σ^* TO Σ^* SUCH THAT $x \in \bar{L}_D$ IFF $f(x) \in A_{TM}$:

GIVEN $w_i = x$, COMPUTE $\langle T_{Mi}\rangle$, THE ENCODING OF THE i^{th} TM.
$f(w_i) = \langle T_{Mi}, w_i\rangle$

NOW SUPPOSE THAT A_{TM} IS RECURSIVE FOR SAKE OF CONTRADICTION
LET M ALWAYS HALT AND ACCEPT A_{TM}. CONSTRUCT M' FROM M:
M' ACCEPTS AND HALTS ON x IFF M ACCEPTS $f(x)$. OTHERWISE M' HALTS AND REJECTS x. NOW M' ALWAYS HALTS AND ACCEPTS $\bar{L}_D$ -- CONTRADICTION.