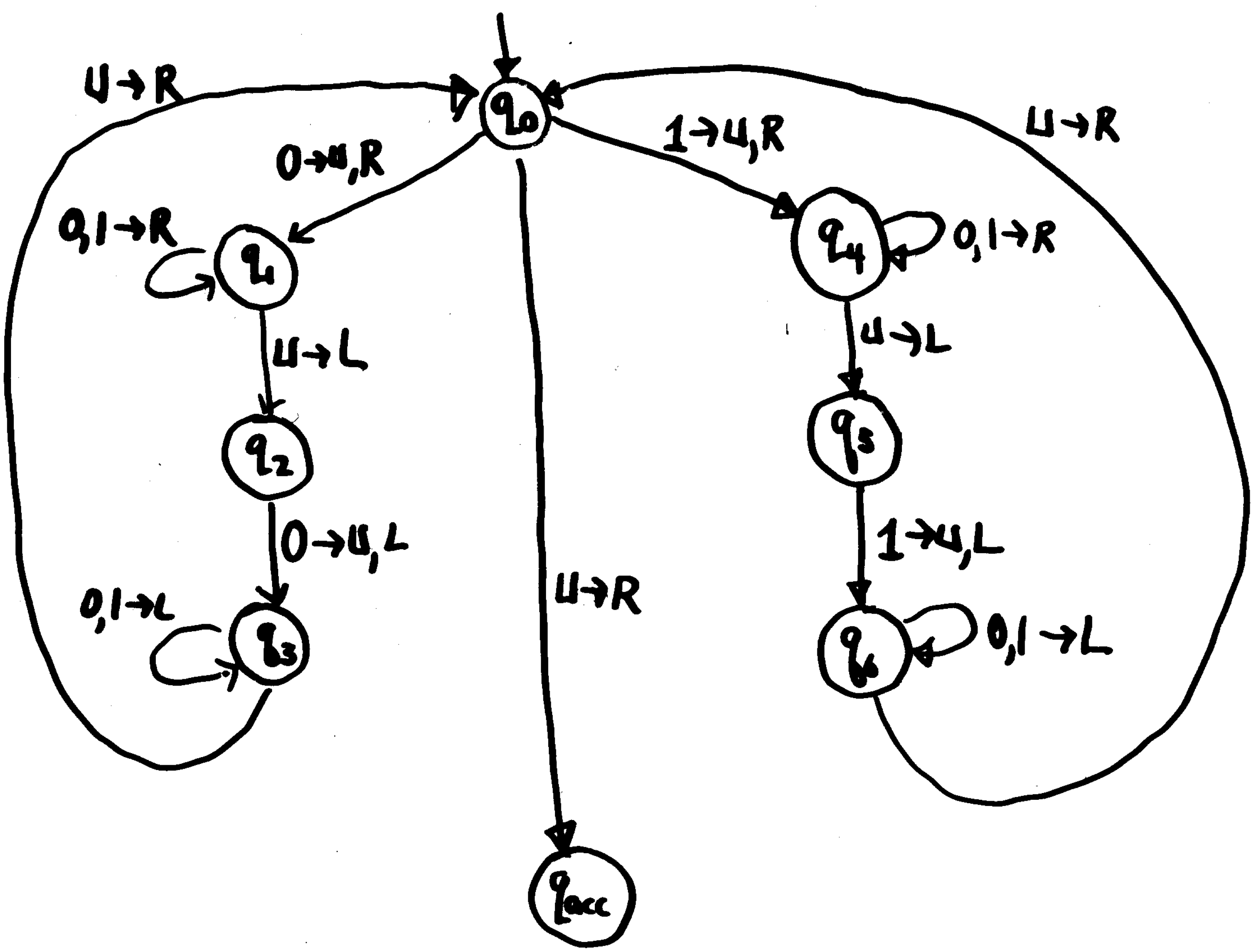
(3) $PAL = \{ y y^r \mid y \in \{0,1\}^*$ where y^r is y spelled backwards$\}$

$0110 \in PAL$

$0111 \notin PAL$

$$M = (Q, \Sigma = \{0,1\}, \Gamma = \{0,1,\sqcup\}, \delta, q_0, q_{acc}, q_{rej})$$

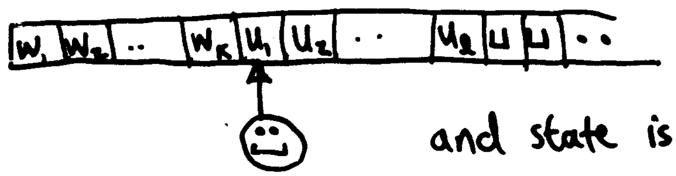
$$Q = \{q_0, q_1, \ldots, q_6\}$$



FORMAL DEF'N OF ACCEPTANCE

DEF'N A CONFIGURATION OF A TM IS A COMPLETE DESCRIPTION OF THE CURRENT STATE OF COMPUTATION

FORMALLY A CONFIGURATION OF M IS A TRIPLE (q,w,u) (ALSO WRITTEN wqu), MEANING



and state is q

$w = w_1 w_2 \cdots w_k$
$u = u_1 u_2 \cdots u_l$

$(q,w,u) \xrightarrow[\text{yields}]{M} (q',w',u')$ IF IN A SINGLE STEP OF M WE TRANSITION FROM (q,w,u) TO (q',w',u').

$(q,w,u) \xrightarrow{M^k} (q',w',u')$ IF IN k STEPS OF M WE TRANSITION FROM (q,w,u) TO (q',w',u').

$(q,w,u) \xrightarrow{M^*} (q',w',u')$ IF $\exists k$ SUCH THAT WE TRANSITION FROM (q,w,u) TO (q',w',u') IN k STEPS.

DEF'N (ACCEPTANCE)
M ACCEPTS w IF THERE EXISTS A SEQUENCE OF CONFIGURATIONS $C_1 C_2 .. C_k$ S.T.

(1) C_1 IS THE START CONFIG OF M ON w $(C_1 = q_0 w)$
(2) $\forall i < k$ C_i YIELDS C_{i+1}
(3) C_k IS AN ACCEPTING CONFIG
$(C_k = w\, q_{accept}\, u)$
(4) $\forall i < k$ C_i IS NOT A REJECTING CONFIG.

L(M) IS THE SET OF ALL STRINGS THAT M ACCEPTS.

TM'S FOR COMPUTING A FUNCTION

WE CAN EASILY MODIFY OUR DEF'N OF A TM SO THAT IT COMPUTES A FUNCTION AS FOLLOWS

A TM (COMPUTING A FUNCTION) IS A 6-TUPLE

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{value})$$

NOW, WHENEVER M ON x ENTERS q_{value}, THE COMPUTATION STOPS AND $M(x)=y$ WHERE y IS THE LARGEST CONSECUTIVE STRING ON THE TAPE BEGINNING AT THE LEFTMOST CELL, AND NOT CONTAINING ANY BLANK ($\sqcup$) SYMBOLS.

THE FUNCTION COMPUTED BY M IS
$f: \Sigma^* \to \Sigma^*$ SUCH THAT $f(x)=y$ IFF $M(x)=y$

IF M DOES NOT HALT ON ALL INPUTS
THEN $f: \Sigma^* \to \Sigma^* \cup \perp$ and

$f(x)=y$ IFF $M(x)=y$

If $M(x) = \nearrow$ (never halts), then $f(x) = \perp$

CLASSIFICATION OF LANGUAGES / FUNCTIONS

TYPICALLY $\Sigma = \{0,1\}$ $\Gamma = \{0,1,\sqcup\}$

DEF'N LET $L \subseteq \Sigma^*$ BE A LANGUAGE.
M ACCEPTS OR RECOGNIZES L IF $\forall x \in L$
$M(x) =$ yes AND $\forall x \notin L$ $M(x) =$ no or $M(x) = \nearrow$.
L IS TURING-RECOGNIZABLE OR RECURSIVELY ENUMERABLE (RE) IF SOME TM ACCEPTS L.

DEF'N A DECIDER IS A TM THAT HALTS ON ALL INPUTS. ($\forall x$ $M(x) =$ yes or $M(x) =$ no)
L IS DECIDABLE OR RECURSIVE IF SOME DECIDER ACCEPTS L.

EASY CLAIM L RECURSIVE $\Rightarrow$ L IS RE

DEF'N LET $f: \Sigma^* \to \Sigma^*$ AND LET M BE A TM WITH ALPHABET Σ. THEN M COMPUTES f IF $\forall x \in \Sigma^*$ $M(x) = f(x)$.
IF SUCH AN M EXISTS THEN f IS A RECURSIVE FUNCTION.

Example 1 THE SET OF ALL PALINDROMES OVER 0,1 IS A RECURSIVE LANGUAGE.

Example 2 THE FUNCTION $f: \{0,1\}^* \to \{0,1\}^*$ THAT TAKES A NUMBER n WRITTEN IN BINARY AND RETURNS $2n$ (IN BINARY) IS A RECURSIVE FUNCTION.

VARIANTS OF TURING MACHINES

OUR NEXT GOAL IS TO SHOW THAT THE TM MODEL CAPTURES ARBITRARY COMPUTATION. MORE SPECIFICALLY WE WANT TO ARGUE THAT ANY FUNCTION/LANGUAGE COMPUTED BY SOME COMPUTER CAN ALSO BE COMPUTED BY A TM.

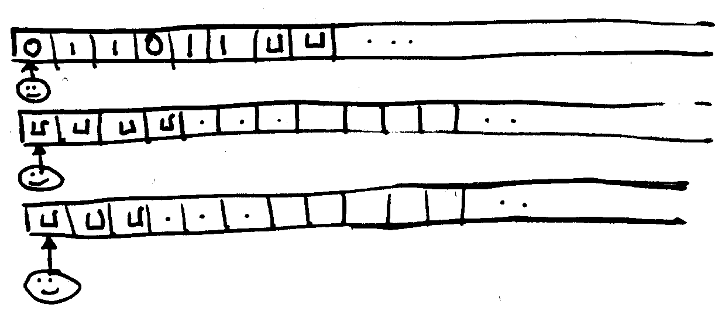
TO WORK TOWARD THIS GOAL, WE WILL ADD SEEMINGLY MORE AND MORE POWERFUL FEATURES TO OUR BASIC TM MODEL, AND ARGUE THAT THE BASIC MODEL WAS ALREADY CAPABLE OF SIMULATING THESE FEATURES.

WE WILL CONSIDER TWO SUCH FEATURES:

(1) MULTITAPE TURING MACHINES
(2) NONDETERMINISTIC TURING MACHINES

MULTITAPE TM'S

THERE ARE K TAPES AND K-HEADS



K=3

- INITIALLY INPUT IS ON TAPE 1, ALL HEADS AT LEFTMOST POSITIONS

- M TAKES A STEP ACCORDING TO δ; READ CONTENTS UNDER ALL HEADS SIMULTANEOUSLY, EACH HEAD PRINTS A SYMBOL AND HEADS MOVE

ONE CELL LEFT OR RIGHT INDEPENDENTLY,
CHANGE STATE

$$\delta: Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$$

i.e. $\delta(q_i\ a_1 .. a_k) = (q_j\ b_1 .. b_k\ L\ R\ ..\ L)$ MEANS

IF M IN STATE q_i AND HEAD i READING a_i
THEN WRITE SYMBOL b_i (TO REPLACE a_i)
MOVE TO STATE q_j AND MOVE HEAD L/R
AS SPECIFIED.

<u>Exercise</u> Show how to recognize $\{w\#w \mid w \in \{0,1\}^*\}$ with a simple multitape TM.

<u>THEOREM 3.8</u>

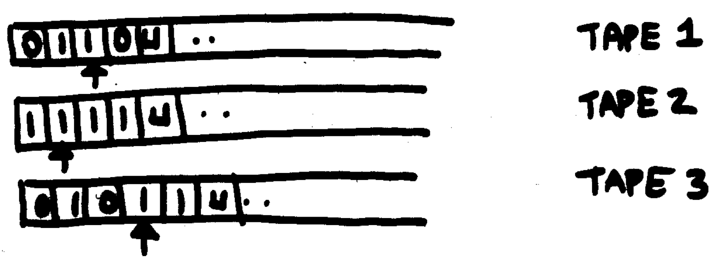
EVERY MULTITAPE TM HAS AN EQUIVALENT SINGLE TAPE TM.

<u>PROOF SKETCH</u> ($k=3$)

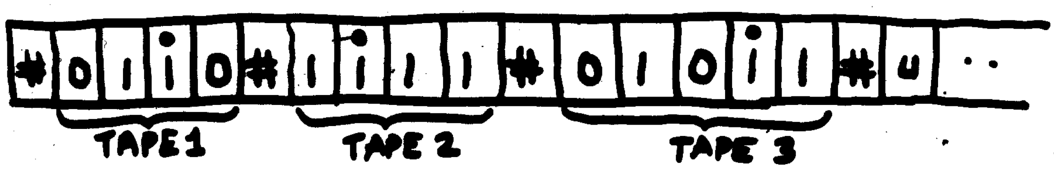
LET $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ BE A 3-TAPE TM

WE CONSTRUCT A BASIC TM, M', S.T. $L(M) = L(M')$

M:



M' :



M' ON INPUT w : (INFORMAL DESCRIPTION)

1. PUT M' IN START CONFIGURATION:

$$\#\underbrace{\dot{w}_1 w_2 \cdots w_n}_{\text{TAPE 1}}\#\underbrace{\dot{\sqcup}}_{\text{TAPE 2}}\#\underbrace{\dot{\sqcup}}_{\text{TAPE 3}}\#$$

2. TO SIMULATE ONE STEP OF M, M' SCANS TAPE FROM FIRST '#' TO FOURTH '#', KEEPING TRACK OF SYMBOLS UNDER THE 3 HEADS (VIA STATE). THEN M' MAKES A SECOND PASS TO UPDATE TAPES ACCORDING TO M.

2a. IF IN 2, WE TRY TO MOVE A VIRTUAL HEAD RIGHT TO # OVERWRITE WITH ⊔ AND SHIFT EVERYTHING TO RIGHT BY ONE CELL.

HOW TO CONVERT OUR INFORMAL DESCRIPTION OF M' TO A FORMAL TM?

$$M' = (Q', \Sigma', \Gamma', \delta', q_0', q_{acc}', q_{rej}')$$

$$\Sigma' = \Sigma$$
$$\Gamma' = \{c \mid c \in \Gamma\} \cup \{\dot{c} \mid c \in \Gamma\} \cup \{\#\}$$

Q' AND δ' ARE COMPLICATED. HERE IS A HIGH LEVEL VIEW OF HOW TO DEFINE Q' AND δ'.

THINK OF A STATE OF M' AS A FINITE TUPLE OF INFORMATION:
[EACH ENTRY FINITE]

$$(2, 1, 3r, 0, 1, q) \in Q'$$

STEP 2 — PASS 1 — LOOKING FOR HEAD 3 GOING R — VALUE UNDER HEAD 1 — VALUE UNDER 2 — CURRENT STATE OF M

EXAMPLE TRANSITIONS

$$\delta'((2, 1, 3r, c_1\ c_2\ q)\ \ 0) \to ((2, 1, 3r, c_1, c_2, q), 0, R)$$

STEP 2 PASS 1 LOOKING FOR HEAD 3, GOING R
c_1 c_2 UNDER HEADS 1 + 2 RESPECTIVELY. CURRENTLY LOOKING AT 0.
CONTINUE TO LOOK FOR HEAD 3, GOING R

$$\delta'((2, 1, 3r, c_1\ c_2\ q), \dot{0}) \to ((2, 1, 1L, c_1\ c_2\ 0, q), \dot{0}, L)$$

FOUND HEAD 3. REMEMBER 0 UNDER HEAD 3
GO LEFT LOOKING FOR 1st '#'

$$\delta'((2,1,1L,c_1c_2c_3\,q),\#) \to ((2,1,2L,c_1c_2c_3\,q),\#,L)$$

step 2 pass 1 looking for 1st '#' values under heads state of M under M''s head look for second '#'

$$\delta'((2,1,3L,c_1c_2c_3\,q),\#) \to ((2,2,1r,c_1c_2c_3\,q),\#,R)$$

looking for third '#' value under M''s head begin pass 2 looking for 1st head, going R

$$\delta'((2,2,1r,c_1c_2c_3\,q),\dot{c}_1) \to ((2,2,1u,c_1c_2c_3\,q),d_1,L)$$

step 2 pass 2 FOUND 1st head update head 1 overwrite with d_1 go L

$$\delta'((2,2,1u,c_1c_2c_3\,q),c') \to ((2,2,2r,c_1c_2c_3\,q),\dot{c}',R)$$

look for 2nd head put head symbol over this cell

$$\delta'((2,2,2r,c_1c_2c_3\,q),\dot{c}_3) \to ((2,2,1L,c_1c_2c_3\,q')\,d_3,L)$$

FOUND 3rd head REMEMBER NEW STATE overwrite with d_3 go LEFT, LOOKING FOR 1st '#'

ASSUME

$$\delta(q,c_1,c_2,c_3) \to (q',d_1,d_2,d_3,L,L,R)$$

NONDETERMINISTIC TURING MACHINES

$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$

$\delta: Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$

powerset

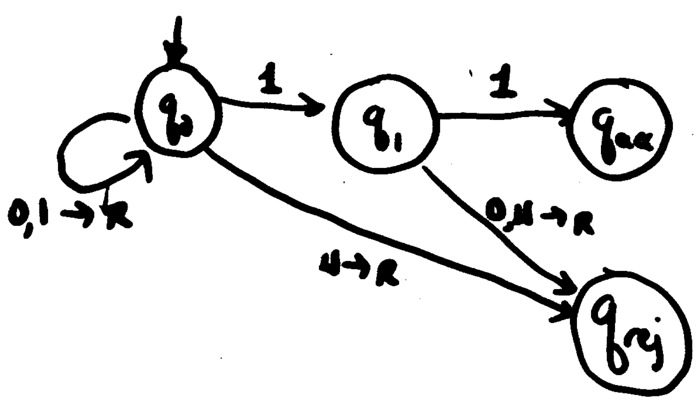
M <u>accepts</u> a string x if there is some sequence of transitions of M on x leading to q_{acc}.

$\mathcal{L}(M) = \{x \mid M \text{ accepts } x\}$

<u>Example</u> $L = \{y\,1\,1\,y' \mid y, y' \in \{0,1\}^*\}$

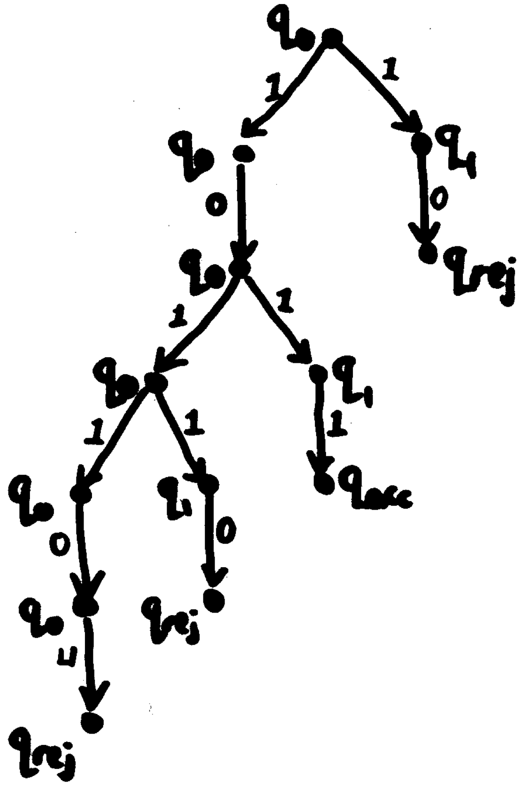
$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ $\quad Q = \{q_0, q_1, q_{acc}, q_{rej}\}$

$\Sigma = \Gamma = \{0,1\}$



A NONDETERMINISTIC COMPUTATION (OF M ON x) IS A COMPUTATION TREE:

$x = 10110$



M ACCEPTS x IFF SOME PATH OF COMPUTATION TREE (OF M ON x) LEADS TO q_{acc}

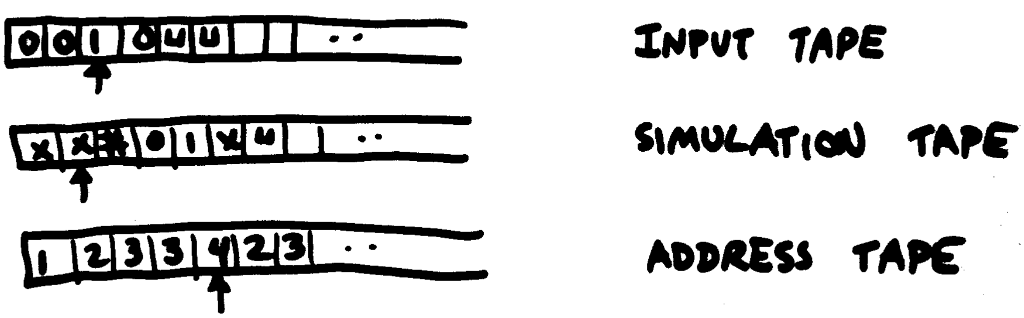
<u>THEOREM</u> IF L IS ACCEPTED BY SOME NONDETERMINISTIC TM N, THEN L IS ACCEPTED BY A BASIC TM D.
FURTHER, D ALWAYS HALTS IF N ALWAYS HALTS (ON ALL COMPUTATION PATHS).

PROOF SKETCH

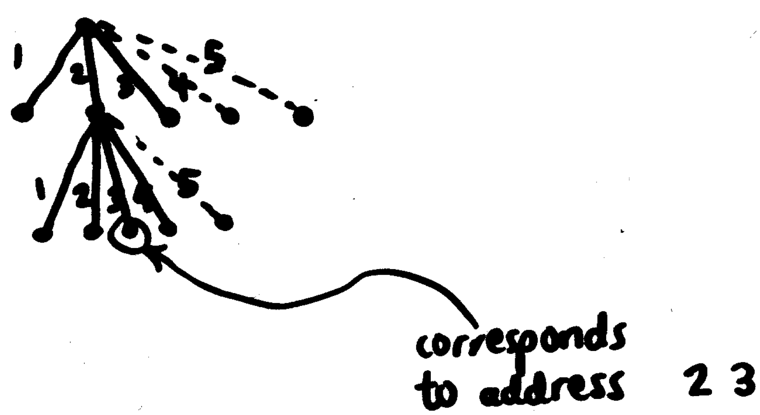
LET $N = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$

SINCE $|Q|$, $|\Gamma|$ ARE FINITE, $|\mathcal{P}(Q \times \Gamma \times \{L, R\})| = b$ IS ALSO FINITE. THEREFORE ON ANY INPUT x, COMPUTATION TREE HAS FANOUT AT MOST b.

D WILL HAVE 3 TAPES. (THEN USE SIMULATION OF A MULTITAPE TM BY A BASIC TM.)



ADDRESS TAPE SPECIFIES A PARTICULAR NODE IN COMPUTATION TREE:



HIGH LEVEL DESCRIPTION OF D

1. INITIALLY TAPE 1 CONTAINS W, TAPE 2, 3 EMPTY

2. COPY TAPE 1 TO TAPE 2
   INITIALIZE TAPE 3 TO ϵ

3. USE TAPE 2 TO SIMULATE N (ON W) ON ONE BRANCH OF COMPUTATION. SEE WHICH CHOICE TO MAKE BY CONSULTING NEXT SYMBOL OF TAPE 3.
   IF NO SYMBOLS LEFT ON TAPE 3, OR NONDETERMINISTIC COMPUTATION IS INVALID, OR q_{rej} ENCOUNTERED, GO TO STEP 4. IF q_{acc} ENCOUNTERED, HALT AND ACCEPT.

4. REPLACE STRING ON TAPE 3 BY LEXICOGRAPHICALLY NEXT STRING.
   (ORDERING: 1, 2, ..., b, 11, 12, ..., 1b, 21, ..., 2b, ..., bb,
   111, 11b, ... bbb, .. etc)

Q HOW TO MODIFY SO THAT D HALTS ON W IF N HALTS ON W?