NETWORK FLOWS

Supplemental Reading : CLR, pp 579-600

A **FLOW NETWORK** $\mathcal{F} = (\mathcal{G}, c, s, t)$ is a bi-directed graph $\mathcal{G} = (V, E)$
$((u,v) \in E \Rightarrow (v,u) \in E)$
For each edge $(u,v) \in E$, there is a Non-Negative capacity $c(u,v) \geq 0$.
s and t are 2 distinguished vertices.

ASSUME $\forall v \in V$ there is a path $s \rightsquigarrow v \rightsquigarrow t$
<u>and</u> for all $(u,v) \in E$ either $c(u,v) > 0$ or $c(v,u) > 0$
<u>and</u> NO self-loops

A **FLOW** in $\mathcal{G}$ is a function $f : E \to \mathbb{R}$ satisfying:

1) $f(u,v) \leq c(u,v)$ capacity constraint

2) $f(u,v) = -f(v,u)$ symmetry

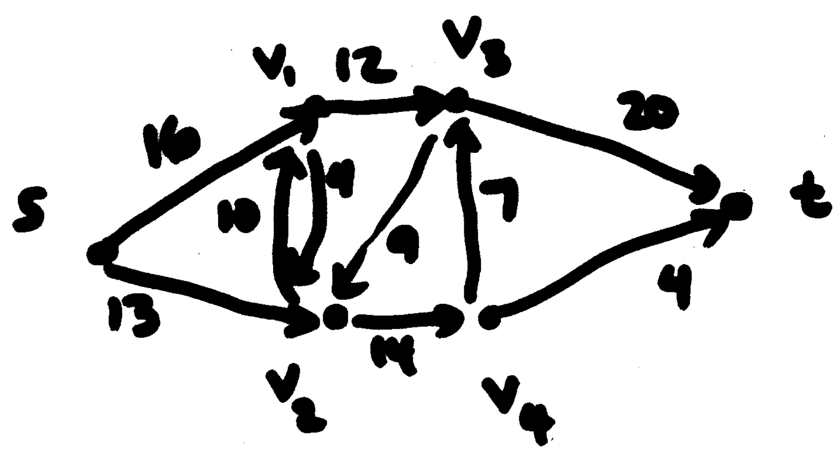
3) $\forall u \neq s,t \quad \sum_{v \in V} f(u,v) = 0$ flow conservation

The **VALUE** of f, $|f| = \sum_{v \in V} f(s,v)$

Maximum Flow Problem

given a flow network g, s, t, c
output a flow of maximum value from s to t.

Example:



Flow Network

RESIDUAL NETWORKS

LET $\mathcal{F}$ BE A FLOW NETWORK, f A FLOW.

FOR ANY $(u,v) \in E$, THE RESIDUAL CAPACITY OF (u,v) IS $c_f(u,v) = c(u,v) - f(u,v)$

THE RESIDUAL GRAPH OF $\mathcal{G}$ INDUCED BY f IS $\mathcal{G}_f = (V, E_f)$

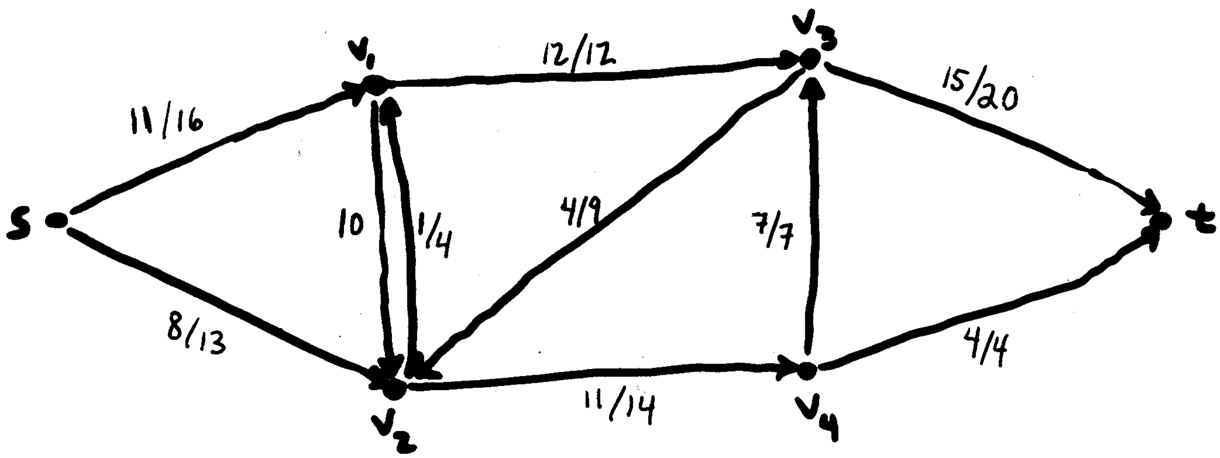
$$E_f = \{ (u,v) \in E \mid c_f(u,v) > 0 \}$$

THE RESIDUAL FLOW NETWORK INDUCED BY f IS $\mathcal{F}_f = (\mathcal{G}, \bar{c}_f, s, t)$

$$\bar{c}_f(u,v) = \begin{cases} c_f(u,v) & \text{for } (u,v) \in E_f \\ 0 \quad \text{otherwise} & (u,v) \in E - E_f \end{cases}$$

EXAMPLE

THE FOLLOWING GRAPH IS A FLOW NETWORK PLUS FLOW



NOTATION x/y on edge (u,v)

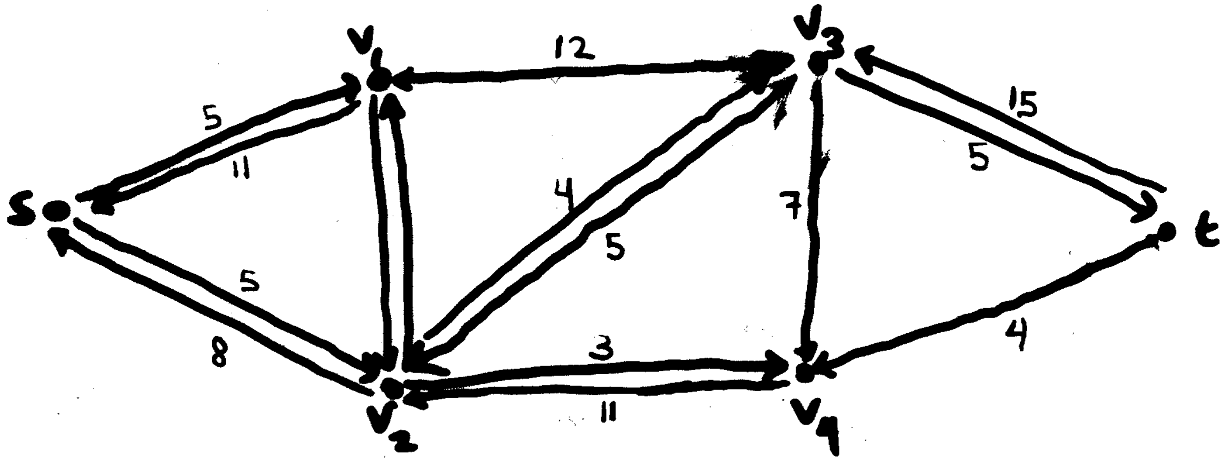
x is flow $x = f(u,v)$

y is capacity $y = c(u,v)$

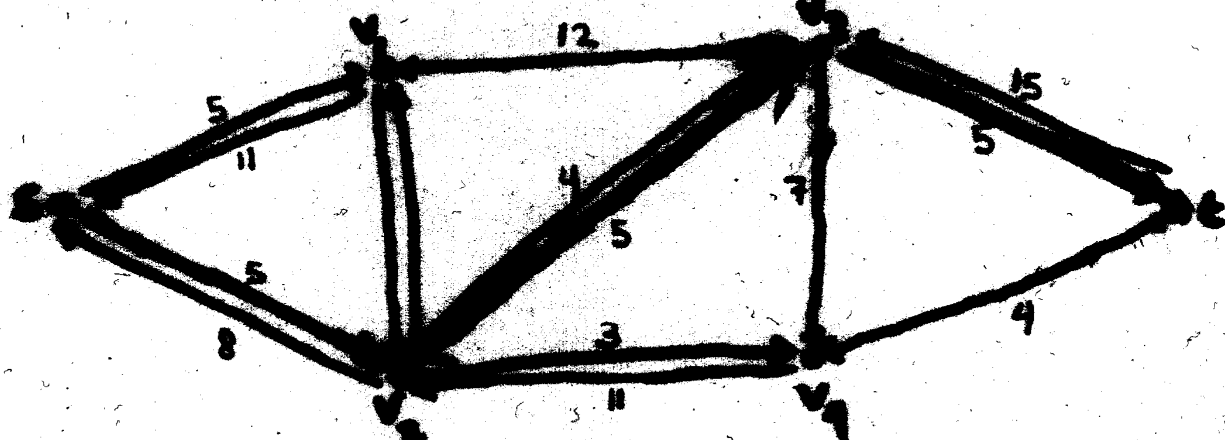
if $f(u,v) \le 0$, then edge labelled only by capacity, y.

reverse flows not indicated are negative.

$|f| = 8 + 11 = 19$



RESIDUAL GRAPH OF G INDUCED BY f



RESIDUAL GRAPH OF G INDUCED BY f

— denotes augmenting path p

$c_f(p) = 4$

$|f_p| = 4$, $|f'| = |f| + |f_p| = 23$

Lemma 27.2

Let G be a flow network with source s, sink t, and let f be a flow in G.

Let G_f be the residual network of G induced by f, and let f' be a flow in $\mathcal{F}_f$.

Then the flow sum

$$(f + f')(u,v) = f(u,v) + f'(u,v)$$

is a flow in G with value

$$|f + f'| = |f| + |f'|.$$

Augmenting Paths

given G, a flow f
an *augmenting path* p
is a simple path from
s to t in the
residual graph G_f.

The maximum amount of
net flow we can ship along p
is called the *residual capacity*
of p.

$$c_f(p) = \min\{c_f(u,v) \mid (u,v) \text{ is on } p\}$$

LEMMA FIX $\mathcal{F} = (G, c, s, t)$, flow f, AUGMENTING PATH p IN G_f AND DEFINE

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \in p \\ -c_f(p) & \text{if } (v,u) \in p \\ 0 & \text{otherwise} \end{cases}$$

THEN f_p IS A FLOW IN $\mathcal{F}_f$,

$|f_p| = c_f(p) > 0.$

Corollary

Fix G, f, p in G_f.
f_p defined as above.

Let $f' = f + f_p$.

Then f' is a flow in G,

$$|f'| = |f| + |f_p| > |f|$$

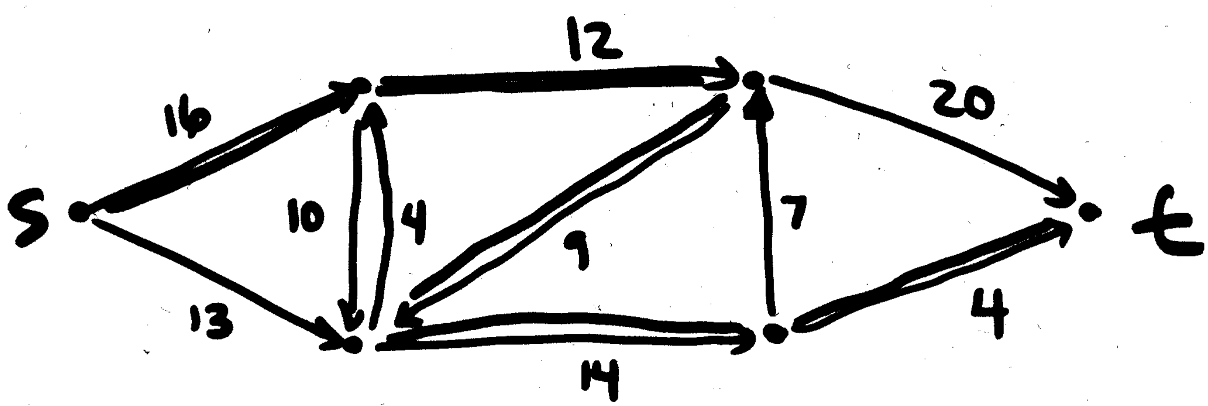
FORD-FULKERSON $(\mathcal{G}, s, t)$

1. Initialize flow f to 0
2. while there exists an augmenting path p in $\mathcal{G}_f$

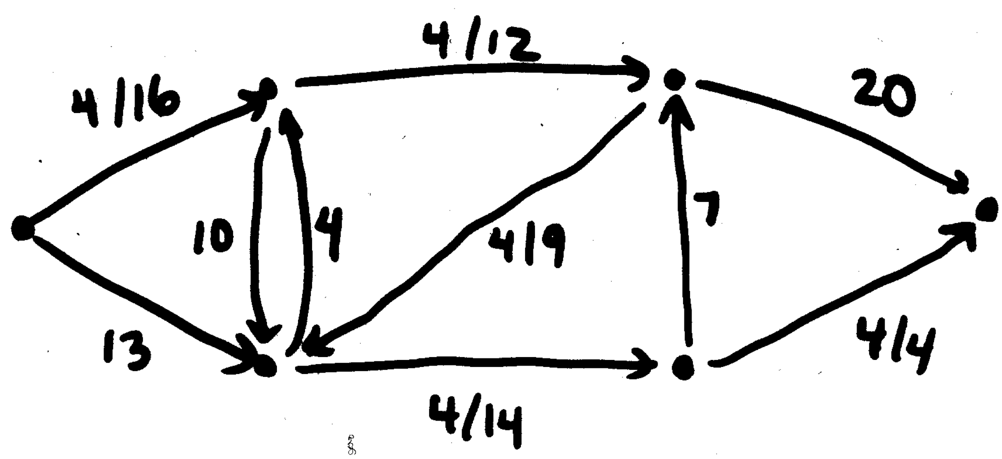
   Do: augment flow f along p
3. Return f.

Example

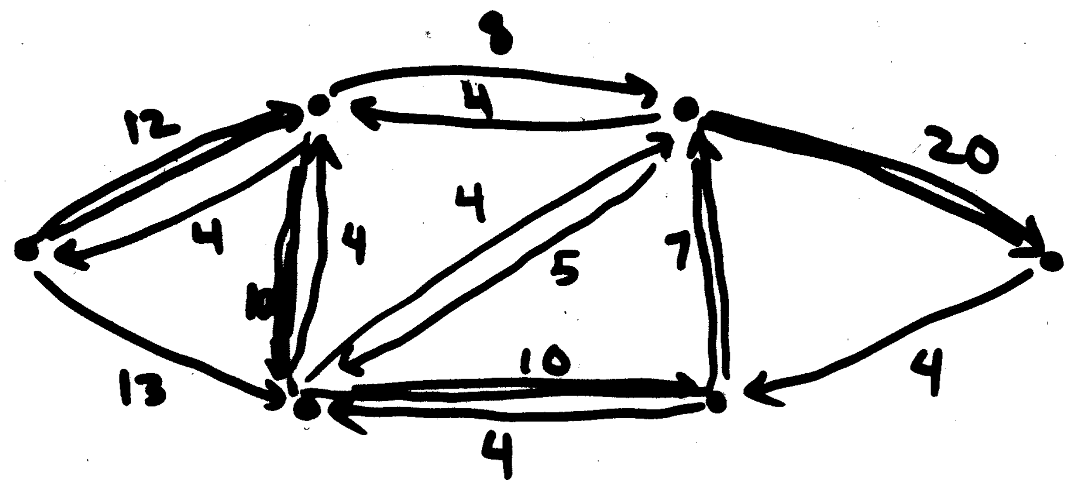
(a) Residual Network (initially = Flow network g)



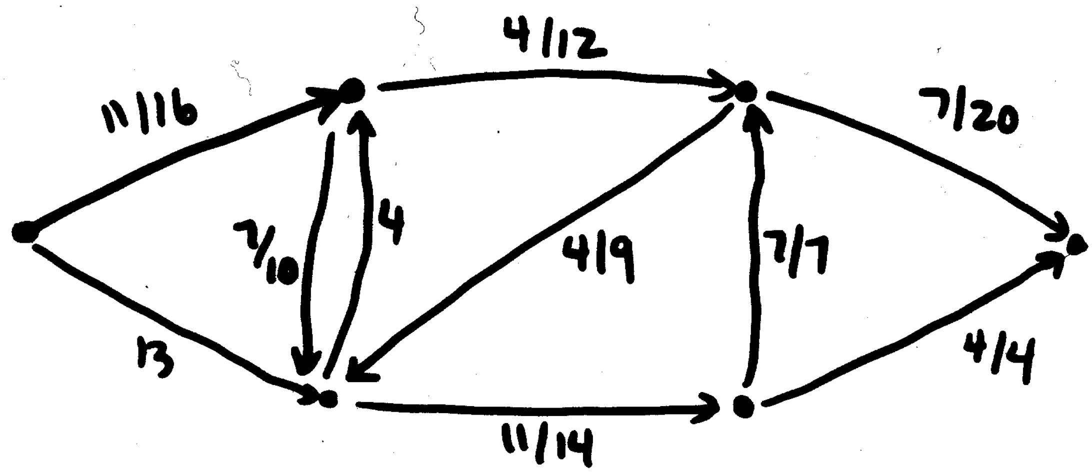
Net flow, adding p



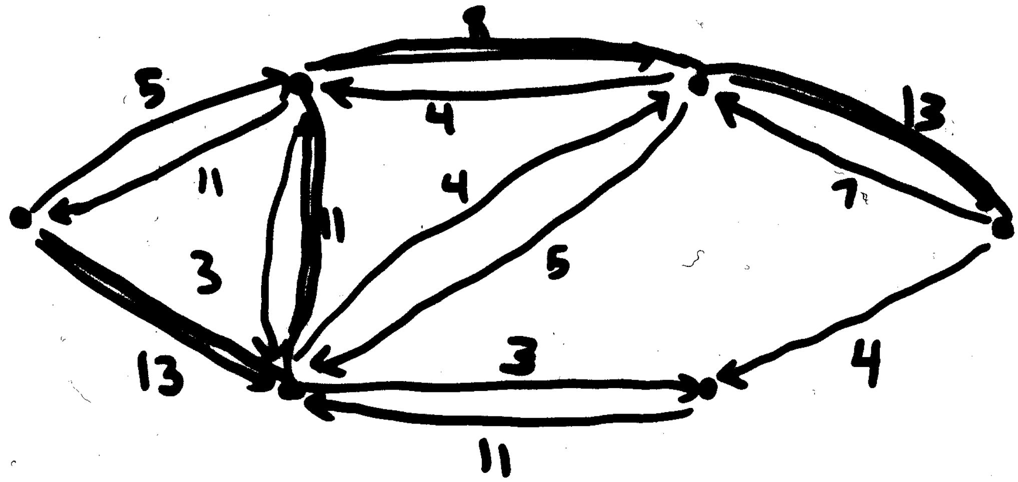
(b) Residual Network



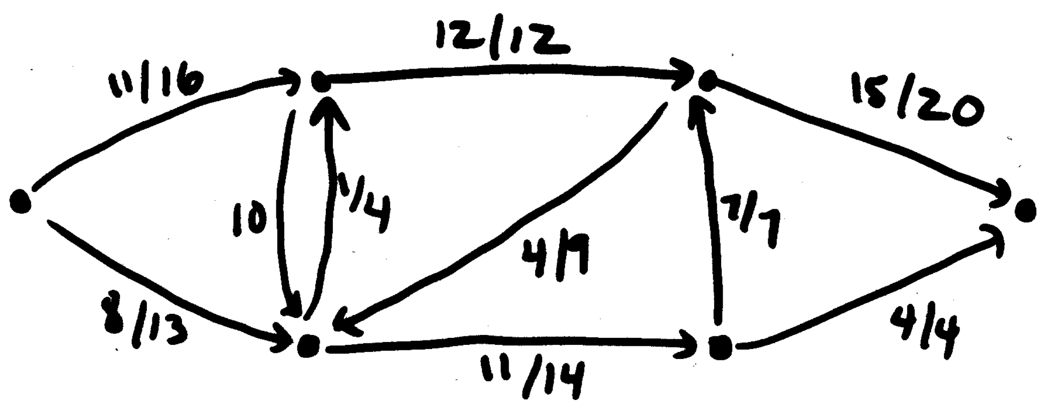
Net flow, adding p



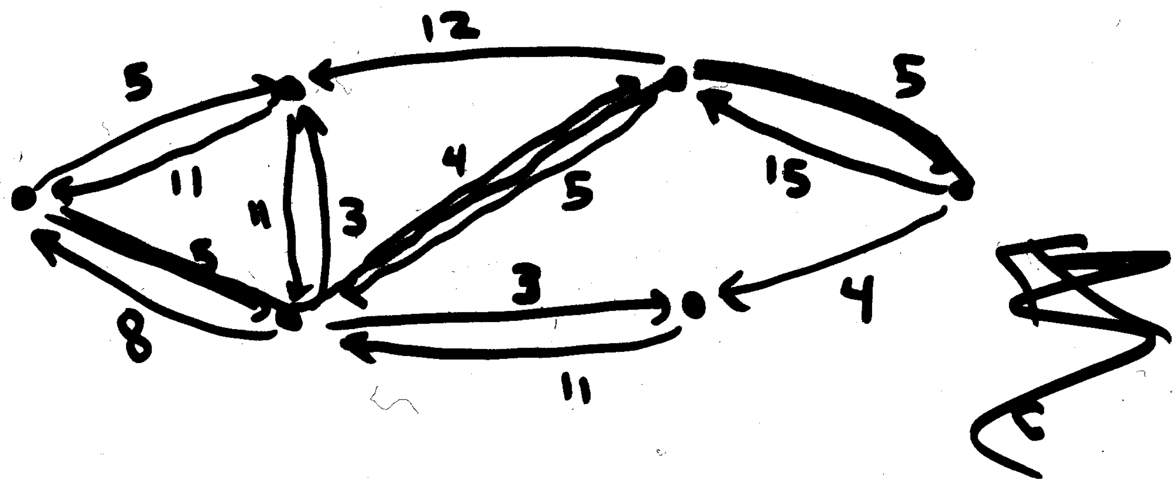
(c) Residual Network



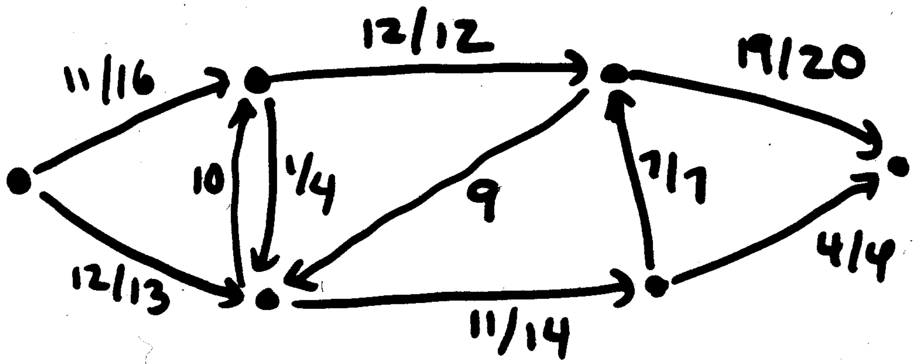
Net flow, adding p



(d) Residual Network

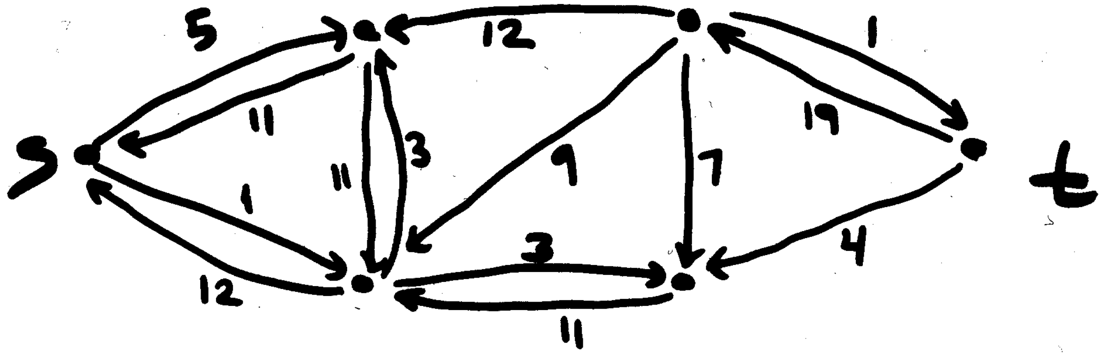


Net flow, adding P



← optimal

(e) Residual Network (No augmenting path)



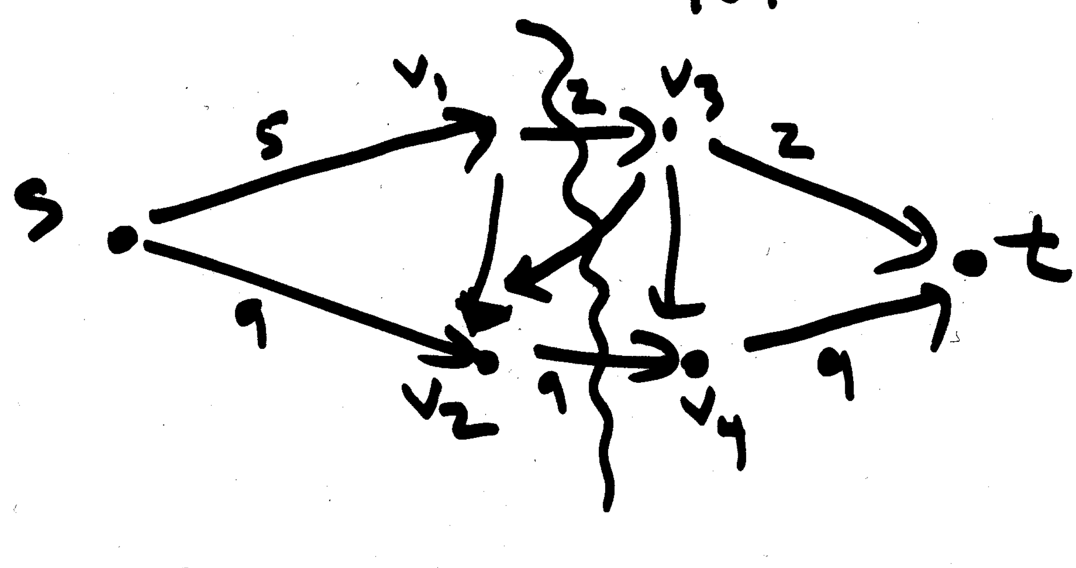
We want to prove that the algorithm finds a maximal flow.

Cuts of flow Networks

A <u>cut</u> (S,T) of $G=(V,E)$ is a partition of V into S and $T=V-S$ such that $s\in S$, $t\in T$

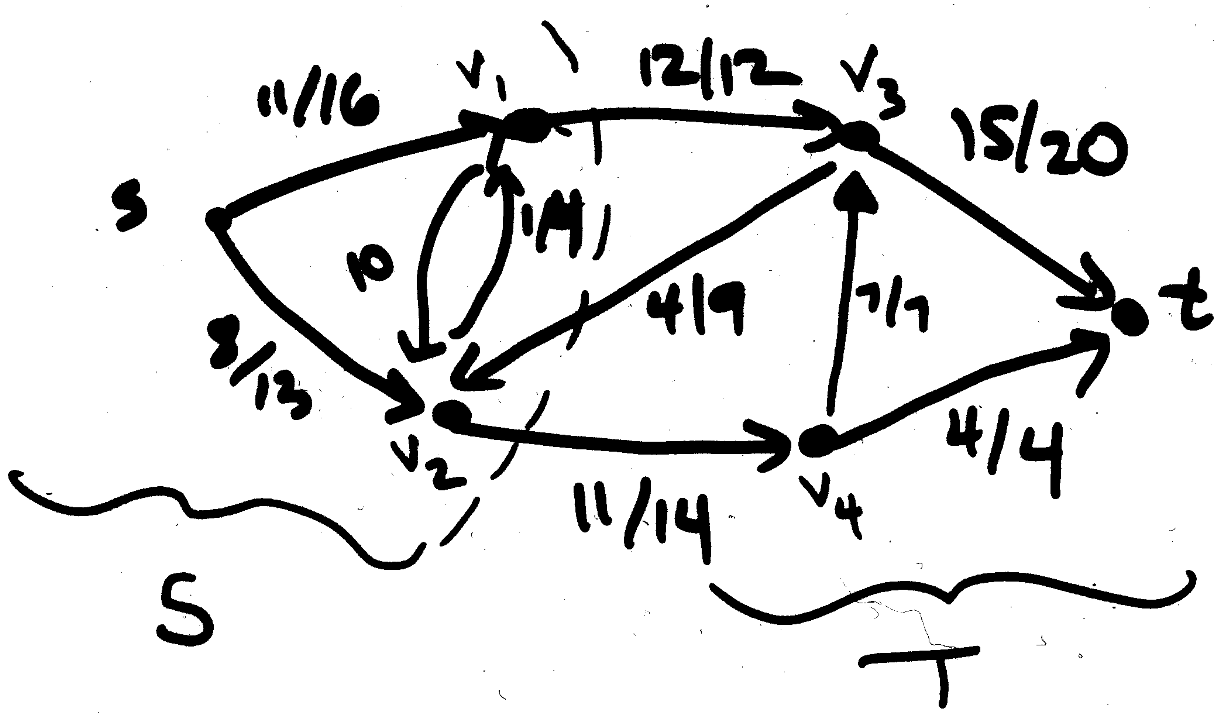
The net flow across (S,T) is

$$f(S,T)=\sum_{x\in S}\sum_{y\in T} f(x,y)$$



The capacity of a cut (S,T) is $c(S,T) = \sum_{x \in S} \sum_{y \in T} c(x,y)$

Lemma 27.5 Fix G, f, s, t and let (S,T) be a cut of G. Then the net flow across (S,T) is $f(S,T) = |f|$.



$f(S,T) = 12 + 11 - 4 = 19$ $\Big)$ $c(S,T) = 26$

Theorem (MAX FLOW MIN CUT THEOREM)

If f is a flow in G, s, t then the following are equivalent

1. f is a max flow in G
2. G_f contains NO augmenting path
3. $|f| = c(S,T)$ for some cut (S,T) of G.

Proof.

(1) $\Rightarrow$ (2). Show $\neg 2 \Rightarrow \neg 1$. If G_f has an augmenting path then $f' = f + f_p$ is a larger flow than f.

(2) $\Rightarrow$ (3) Spose that (2) is true.
Let $S = \{ v \in V \mid \exists$ a path from s to v in $G_f \}$

$T = V - S$.

(S,T) is a cut, and
$\forall (u,v)$ such that $u \in S$, $v \in T$
$f(u,v) = c(u,v)$ since otherwise

$(u,v) \in E_f$ and v is in S.

$\therefore$ by lemma 27.5, $|f| = f(S,T) = c(S,T)$

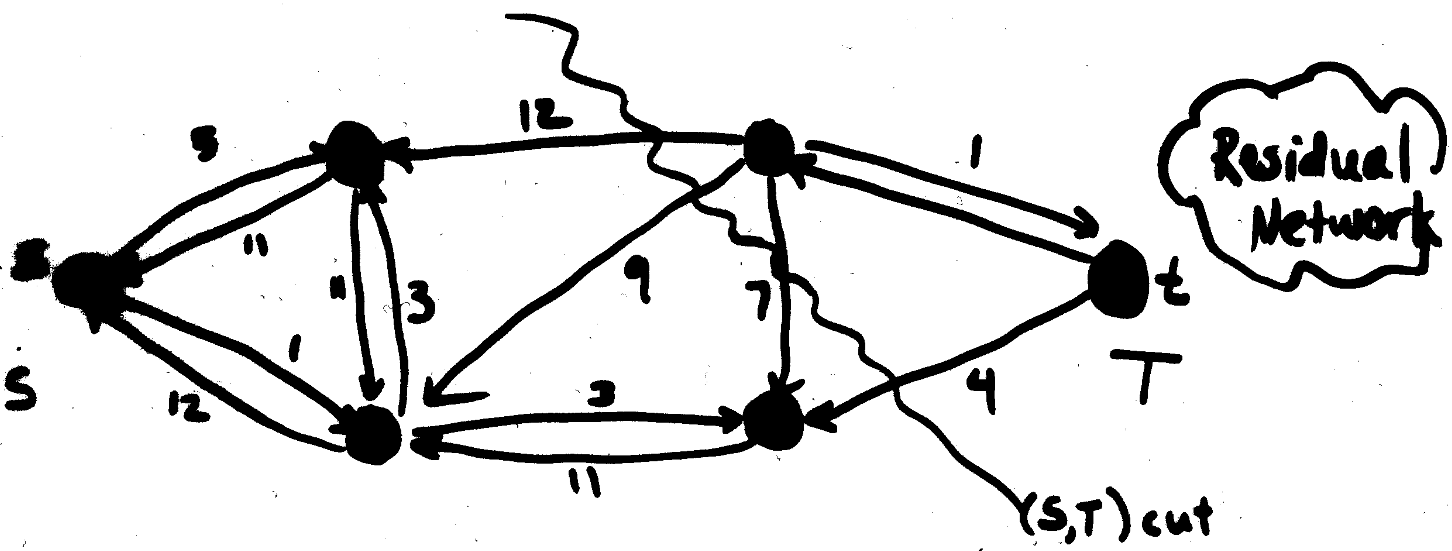
(3) $\Rightarrow$ (1) Show if $|f| = c(S,T)$ then f is max flow in G.

We know $|f| \leq c(S,T)$ $\forall$ cuts (S,T) (by Corollary 27.6)

For all cuts (S,T)
The condition $|f| = c(S,T)$ thus implies that f is a max flow.

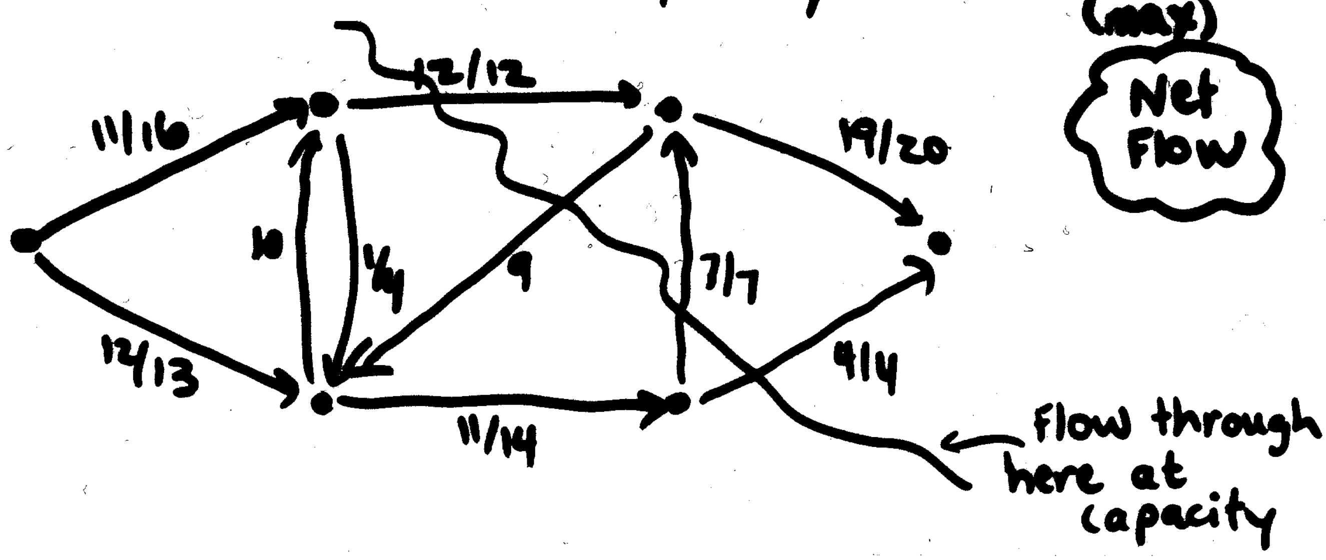
Correctness of Algorithm, an example

In previous example, algorithm ended when we got this residual network:



Since this network has NO augmenting path, the induced cut (S,T) is nontrivial.

★ The flow across (S,T) in the current flow <u>must</u> be maximal (at capacity)



Ford-Fulkerson (g, s, t)

For each $(u,v) \in E[g]$

Do $f[u,v] \leftarrow 0$

$f[v,u] \leftarrow 0$

while there is a path p from s to t in g_f

DO: $c_f(p) \leftarrow \min\{c_f(u,v) \mid (u,v) \in p\}$

For each (u,v) in p

DO $f[u,v] \leftarrow f[u,v] + c_f(p)$

$f[v,u] \leftarrow -f[u,v]$

Choose augmenting path using Breadth-First-Search.

<u>Runtime</u> is

$O(|E| \cdot |f^*|)$,

ignoring finding paths.

Using BFS to find paths,

$O(|E| \cdot |V|)$