Contact Representations of Planar Graphs: Extending a Partial Representation is Hard

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Abstract. Planar graphs are known to have geometric representations of various types, e.g. as contacts of disks, triangles or - in the bipartite case - vertical and horizontal segments. It is known that such representations can be drawn in linear time, we here wonder whether it is as easy to decide whether a partial representation can be completed to a representation of the whole graph. We show that in each of the cases above, this problem becomes NP-hard. These are the first classes of geometric graphs where extending partial representations is provably harder than recognition, as opposed to e.g. interval graphs, circle graphs, permutation graphs or even standard representations of plane graphs.
On the positive side we give two polynomial time algorithms for the grid contact case. The first one is for the case when all vertical segments are pre-represented (note: the problem remains NP-complete when a subset of the vertical segments is specified, even if none of the horizontals are). Secondly, we show that the case when the vertical segments have only their x-coordinates specified (i.e., they are ordered horizontally) is polynomially equivalent to level planarity, which is known to be solvable in polynomial time.

1 Introduction

An intersection representation of a graph $G = (V, E)$ is a set family $\{S_v : v \in V\}$ such that uv is an edge of G iff $S_u \cap S_v \neq \emptyset$. Geometric representations (i.e., intersection representations where each set is a geometric object) of graphs have been intensively studied both for their practical motivations and interesting algorithmic properties. The motivations stem from VLSI designs, graphic layouts

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including the rectangular windows overlays, bioinformatics applications (including DNA sequencing), cellular description of reachability and interference in mobile networks, and many others. Geometric representations often also allow problems which are NP-hard in general to be solved in polynomial time.

The oldest and probably best understood class of intersection graphs are *interval graphs*, i.e., intersection graphs of intervals on a line [13]. They can be recognized in linear time and all basic optimization problems like independent set, clique or coloring can be solved on them in linear time as well. Generalizations of interval graphs include *circular arc graphs* [25, 12], the intersection graphs of arcs on a circle. *Circle graphs* [4] are intersection graphs of chords of a circle and as such include *permutation graphs* [2, 14], the intersection graphs of straight-line segments connecting points on two parallel lines. Intersection graphs of curves connecting points on two parallel lines, sometimes called the *function graphs* [14], are exactly the complements of comparability graphs (graphs admitting a transitive orientation). All these classes can be recognized in polynomial time and the independent set and clique problems can be solved in polynomial time on them as well. An overview of these and many other intersection-defined graph classes is given in many textbooks [22, 5].

Geometric representations of graphs also help in visualizing the information grasped by the graph structure. Thus, the question of recognizing these classes and constructing a representation of a given type is rather important. Additionally, in some cases the polynomial time algorithms mentioned above exploit geometric representations. For the vast majority of the interesting classes of graphs the complexity of their recognition is well understood on the level of Polynomial versus NP-hard, with some cases where NP-membership is not known (for intersection graphs of straight-line segments or intersection graphs of convex sets in the plane, which are both known to be NP-hard to recognize, only PSPACE membership of the recognition problem is known, as their recognition is complete for the existential theory of the reals [24]). Recently, more attention has been paid to the question of extending partial representations of graphs. This setting corresponds to a situation where a part of the graph comes already represented from the applied instance or when the visualization task comes from a customer who does not want to see some part of the picture changed. Formally, we discuss the following decision problem, parameterized by an intersection-defined class $\mathcal{C}$:

RepExt($\mathcal{C}$)
Instance: A graph G and a $\mathcal{C}$-representation R' of an induced subgraph of G.
Question: Does there exist a $\mathcal{C}$-representation R of G such that $R' \subseteq R$?

This question falls into a natural paradigm of extending a partial solution of a problem rather than building a solution from scratch, the latter approach being often easier. This common knowledge of architects and engineers can be observed in graph coloring problems where it is well known that every cubic bipartite graph is 3-edge-colorable, but extending a partial edge-coloring is NP-complete [11], even for planar bipartite graphs [21]. Therefore it feels somewhat unexpected that for the resolved cases of geometric intersection graphs, extending partial solutions has not been harder than recognizing the particular classes[6]. In particular, RepExt($\mathcal{C}$) is decidable in polynomial time when $\mathcal{C}$ is: interval graphs [16], proper interval graphs [16], unit interval graphs [19], circle graphs [6], permutation graphs [17], and function graphs [17]. These algorithms tend to extend the plain recognition ones in nontrivial ways through the use of special data structures which capture all representations. Interestingly, even though the classes of unit and proper interval graphs coincide, they are separated by the partial representation extension problem; i.e., there are instances of partial representations consisting of unit intervals that are extendible to a proper interval representation, but not to a unit one [16].

In this paper we consider RepExt($\mathcal{C}$) when $\mathcal{C}$ is a *contact* graph class. This work is motivated by several elegant theorems that show that all planar graphs have geometric representations by contacts of various geometric objects. In general, an intersection representation is a contact one if the interiors of any two objects of the representation are disjoint. The classical example is Koebe's theorem, often referred to as the *kissing lemma* or the *coin representation*, which was rediscovered several times by several authors. It states that every planar graph is the contact graph of a collection of disks in the plane [20]. The proof of this theorem is nonconstructive but later Mohar [23] gave a polynomial time algorithm for producing an approximate representation (there are planar graphs that require irrational coordinates for some disk centers in any coin representation, and so approximate constructions are the best one can hope for, at least if we want to describe the coordinates and radii by rational numbers). De Fraysseix et al. [8] constructively proved that every planar graph is a contact graph of triangles in the plane. In 1991, Hartman et al. [3] showed that every bipartite planar graph has a *grid contact* representation, i.e., a contact representation in which vertices of one class of bipartition are represented by vertical segments and vertices of the other class by horizontal ones (this was also independently shown by de Fraysseix et al. [7]).

We prove that in all of these cases, deciding whether a partial contact representation of a planar (bipartite) graph can be extended to a contact representation of the entire graph is NP-hard. For geometric intersection graphs (i.e., for intersection graphs of planar objects defined by their shape or geometrical properties), this collection of results provides the first examples where extending partial representations is harder than deciding or constructing representations with no initial constraints. Note that for extending partial representations by triangles, convex sets, or disks, we only claim NP-hardness and not NP-completeness. This is because the membership in the class NP is not known (similarly to recognizing intersection graphs of disks, convex sets or straight line segments, where only PSPACE membership is known).

In the last section of the paper we refocus on grid contact representations of planar graphs and show that the partial representation extension problem

[6] The only exception is formed by partial subtree-in-tree or path-in-tree representations of chordal graphs, but there the NP-hardness follows from limited space issues, not any geometrical ones [18].

remains NP-hard if only some of the vertical segments are prerepresented. On the contrary, the problem becomes polynomially decidable if all of the vertical segments are given in the input, and also if only their x-coordinates are given (i.e., all horizontal segments and the vertical position of the vertical ones are unspecified). The last mentioned case is shown to be polynomially equivalent to testing level-planarity, a problem known to be decidable in linear time.

2 Grid Contact Graphs

For this section, let $G = (V \cup H, E)$ be a planar bipartite graph and let $n_1 = |V|, n_2 = |H|$. As already mentioned, de Fraysseix et al. [7] proved that G has a contact representation in which vertices of V are represented by vertical line-segments, vertices of H by horizontal line-segments, no parallel segments intersect, no two segments cross and any two segments u, v share a point (i.e., a point of contact) if and only if $uv \in E$ (for simplicity we use the same symbol for a vertex and the segment representing it). In particular, both V and H are independent sets of vertices in G. The proof [7] is based on bipolar orientations of planar graphs and their visibility representations. Such a representation can be constructed in polynomial time. We show that the task becomes harder if some of the vertices are pre-represented. The proof of the following theorem plays an important role in the rest of the paper. The NP-hardness reductions are all based on modifications of the gadgets constructed in it.

Theorem 1. *Given a planar bipartite graph G and some of its vertices represented by vertical or horizontal line-segments, it is NP-complete to decide if the partial representation can be extended to a grid contact representation of G.*

Proof. The NP-membership is straightforward, since a grid intersection representation can be described by the linear quasi-orders of n_1 coordinates for the vertical segments and n_2 coordinates for the horizontal ones.

For the NP-hardness proof we reduce from Planar-3-SAT. Given a Boolean formula Φ with a set C of clauses over a set X of variables such that the graph $G_\Phi = (C \cup X, \{xc : (x \in c \in C) \vee (\neg x \in c \in C)\})$ is planar, it is NP-complete to decide if Φ is satisfiable. This problem remains NP-complete even if every variable occurs in 3 clauses, once negated and twice positive, and every clause contains 2 or 3 literals [10] (in fact, Fellows et al. show NP-completeness even in a stronger way, for planar clause-linked formulas, i.e., for formulas whose incidence graphs remain planar after adding a cycle through all clause vertices, but we do not need this assumption). Given such a formula, we first draw the graph G_Φ in a rectilinear way so that edges are piece-wise linear curves with all segments either vertical or horizontal. We may further assume that the edges leaving each variable are positioned so that the edges corresponding to the two positive occurrences start with horizontal segments while the edge corresponding to the negative occurrence starts with a vertical one. The planarity of G_Φ can be tested in linear time, and a rectilinear drawing can be also constructed in linear time, even with a bounded number of bends per edge.

From this drawing we construct a graph G by a sequence of local replacements. Every variable is replaced by a copy of a variable gadget, every clause

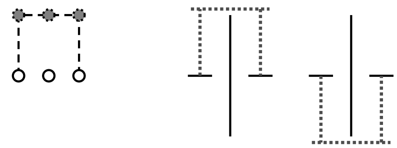

Fig. 1. The brick and its two possible representations.

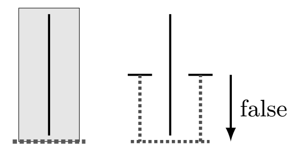

Fig. 2. The schematic encoding of a brick.

by a copy of a clause gadget, and the edges are replaced by chains of gadgets whose length depends on the number of bends on the edge. All gadgets are constructed from two building blocks. The basic one, the so called *brick*, is depicted in Fig. 1. The left part of the figure shows the subgraph, the right one a representation by contacts of segments. In all figures the black vertices and segments are those whose position is prescribed, and white vertices (dotted segments) are the flexible ones. The middle vertical black segment is an isolated vertex and thus cannot be crossed by any of the dotted ones. The dotted path connecting the other two black vertices can be represented either above or below the middle black segment. We use the schematic light grey rectangle depicted in Fig. 2 for the brick, and the side which bears the dotted segment encodes the value false. The bricks can also be rotated into a horizontal position, thus sending the false value to the left or to the right.

The variable gadget consists of three bricks whose vertices are pairwise nonadjacent. It is depicted in Fig. 3. From the overlapping corners of the bricks, it either sends the value false along the vertical edge, in which case both horizontal edges may transfer the value true, or it sends the value true along the vertical edge, in which case both horizontal ones must send the value false. The former case corresponds to the variable being evaluate as true, in the latter to false.

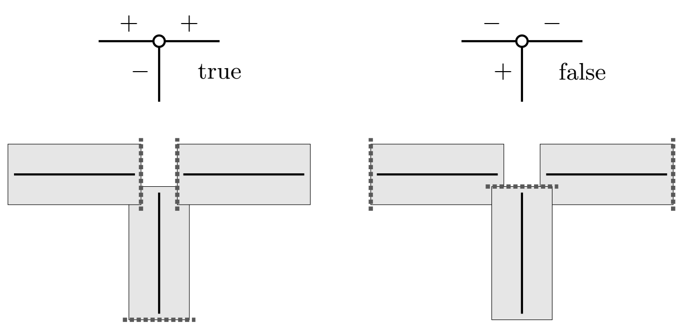

Fig. 3. The variable gadget.

Each rectilinear edge of G_Φ is replaced by a sequence of bricks, one for each linear segment, where these segments are linked again by overlapping corners. In every feasible representation, the value false is transferred along the edge, see Fig. 4. Note that it is possible for the edge gadget to transmit the value false even when its first brick is set to true, but this does not change the satisfiability of Φ.

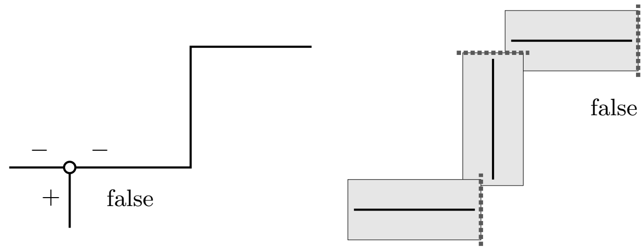

Fig. 4. The edge gadget.

For the clause gadget we use a *modified brick* depicted in Fig. 5. In any representation, at least one of the corners of the bounding rectangle must be used by a dotted segment. The clause gadget consists of two normal bricks and a modified one linked as depicted in Fig. 6. (For clauses containing 2 literals, we use the same gadget with one dummy variable represented by a single brick whose dotted path is pre-represented in the false position.) It is straightforward to see that if all three literals in the clause evaluate to false, all of the corners of the modified brick inside the clause gadget are blocked and the modified brick itself cannot be represented. Thus if the graph constructed as above has a representation, each clause must have at least one true literal and Φ is satisfiable. On the other hand, if Φ is satisfiable, we construct a representation following the lines and pictures above. Feasible representations of the clause gadget for the cases when the bottom or a side incoming literal is true are depicted in Fig. 7. ■

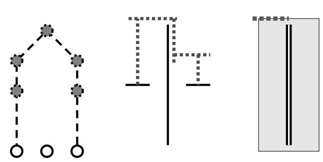

Fig. 5. The modified brick.

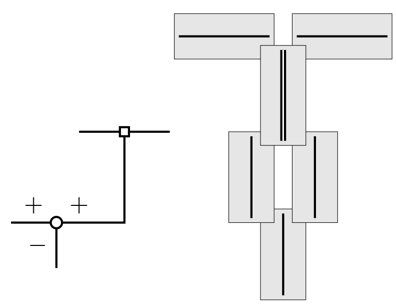

Fig. 6. The clause gadget.

3 Triangles, Disks, and Convex Sets

De Fraysseix et al. [8] proved that every planar graph is a contact graph of isosceles triangles with parallel bases. The construction is based on canonical ordering of the vertices in planar triangulations and can be performed in polynomial time. We show that again, given some vertices pre-represented, it is NP-hard to decide if the representation can be extended to a contact representation of the entire graph. We prove this in a stronger form, noting that triangles are convex sets.

Theorem 2. *Given a planar graph G and a partial representation R' by contacts of isosceles triangles, the following questions are NP-hard to decide*

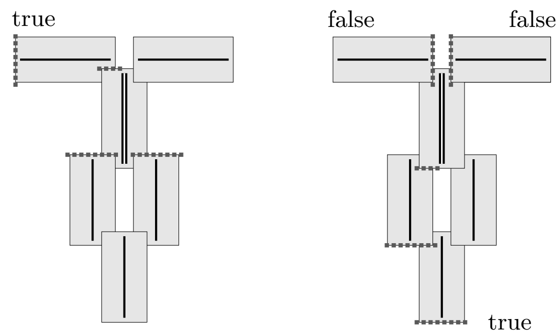

Fig. 7. The representations of the clause gadget.

1. *if R' can be extended to an intersection representation of G by convex sets,*
2. *if R' can be extended to a contact representation of G by convex sets,*
3. *if R' can be extended to a contact representation of G by isosceles triangles.*

Proof. We modify the proof of Theorem 1. First, note that the graph G constructed in the proof is a disjoint union of isolated vertices and paths and all *flexible* vertices (i.e., those whose segments are not prescribed) are of degree 2. Thus, any intersection representation by closed convex sets can be reduced to a contact representation by segments (take the sets representing flexible vertices one by one and replace each of them by the segment connecting the closest intersection points with its two neighbors). These segments, however, do not need to follow the vertical and horizontal directions. To force them to be "almost" bi-directional, adjust the bricks by predrawing auxiliary guiding segments, very close to each other, that leave a very narrow angle for the flexible (dotted) segments, as depicted in Fig. 8 (the guiding segments represent isolated vertices and so must not be crossed or touched by any other segments of the representation). If the width of the corridor between the guiding segments is small enough with respect to the length of the central black segment, the corners of the bounding rectangle are blocked by the dotted flexible segments as in the proof of Theorem 1. A similar modification is applied to the modified brick. To keep all flexible segments under control, we add extra blocking predrawn segments – all loose corners will be blocked by an extra black segment and every corner of the modified brick will be filled with three predrawn segments in an H-position as shown in Fig. 9.

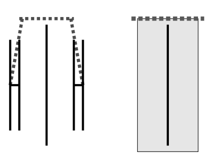

Fig. 8. The brick for the convex set reduction.

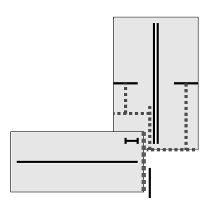

Fig. 9. Extra blockage for the convex set reduction.

So if G has a representation by intersections of convex sets (extending the given partial representation), then G has a contact representation by segments with similar properties propagating the false assignment of variables as in the proof of Theorem 1, and Φ is satisfied by the corresponding valuation. Moreover, if Φ is satisfiable, a contact representation by vertical and horizontal segments is a contact (i.e., also intersection) representation by convex sets. To achieve a contact representation by isosceles triangles, it suffices to replace vertical segments by very thin triangles and the horizontal segments by very fat ones (with very small height and whose base corresponds to a horizontal segment). ■

A similar modification of the gadgets by leaving a controlled space for the flexible vertices shows the next result on disk contact (intersection) graphs (the proof will be given in the full version of the paper).

Theorem 3. *Given a planar graph G and a partial representation R' by contacts of disks, the following questions are NP-hard to decide*

1. *if R' can be extended to an intersection representation of G by disks,*
2. *if R' can be extended to a contact representation of G by disks.*

4 Contacts of Regions

One can further relax the conditions on the representation by geometrical objects. From a non-crossing drawing of a planar graph one can easily construct a contact graph of closed regions bounded by simple Jordan curves. (Disks are of course such regions, but the proof for contacts of simple regions is much easier.) For partial contact representations by regions we encounter a polynomially solvable case. To maintain planarity, we insist that no three regions share a point.

Theorem 4. *Given a graph G and a partial representation by contacts of simple regions, one can decide in linear time if the representation can be extended to a contact region representation of the entire graph G.*

Proof. Add a master point M_u inside every region representing a vertex u and connect it by non-crossing curves to contact points on the boundary of its region. Consider this as a non-crossing drawing of a graph H and add vertices for the unrepresented vertices of G connected to the master points of their neighbors. Call the graph obtained in this way H'. Then G has a representation by contacts of regions if and only if H' has a planar drawing that extends the fixed drawing of H. This can be decided in linear time [1]. See an illustration in Fig. 10. ■

5 Grid Contact Graphs Revisited

In this section we modify the construction from the proof of Theorem 1 once more. We note that we may require that the pre-represented vertices belong to the same bipartition class.

Theorem 5. *Given a planar bipartite graph $G = (V \cup H, E)$ and some of the V vertices represented by vertical line-segments, it is NP-complete to decide if the partial representation can be extended to a grid contact representation of the entire graph.*

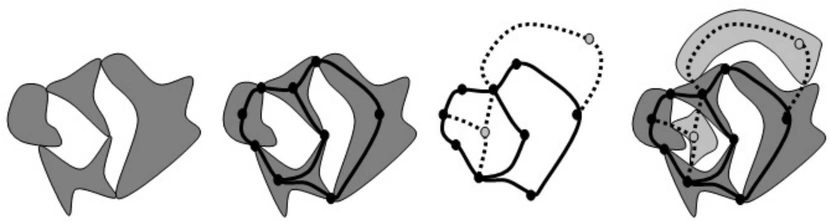

Fig. 10. Connecting region contact graphs and partially embedded planarity.

Proof. From the construction in the proof of Theorem 1, we replace each pre-represented horizontal segment as shown in Fig. 11. Specifically, we replace it with a flexible horizontal segment with three prescribed vertical neighbors. The new flexible horizontal segment has the same adjacency as the original prescribed segment and is locked in the same place by its new black vertical neighbors. ■

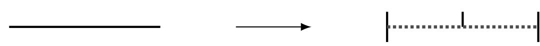

Fig. 11. Lifting the pre-representation of horizontal segments.

We conclude by observing that for the NP-hardness, it is important that some vertical vertices remain flexible:

Theorem 6. *Given a planar bipartite graph $G = (V \cup H, E)$ with all of the V vertices represented by vertical line-segments, it can be decided in polynomial time if the partial representation can be extended to a grid contact representation of the entire graph.*

Proof. (Sketch) For every horizontal vertex $u \in H$, the x-coordinates of its (vertical) neighbors determine the x-coordinates of the left and right endpoints of the segment representing u. What remains to be determined is its height, i.e., the y-coordinate. The positions of its (vertical) neighbors determine a range $I(u)$ of possible y-coordinates of this segment. It can be shown that $I(u)$ is a union of at most $|V|$ real intervals, and hence can be described in polynomial time. Finally, one has to resolve conflicts among the horizontal segments as segments with equal y-coordinates must not intersect (i.e., segments with overlapping projections to the x axis must not have the same y-coordinate). The vertices u with infinite $I(u)$ can be disregarded for the moment, since their y-coordinates can always be chosen different than y-coordinates of all other vertices (we are processing a finite graph). It can be seen that if $I(u)$ is finite, it contains at most 2 values. The choice of the y-coordinates of such vertices can then be modeled by 2-Satisfiability.

Pseudocode for this algorithm, a detailed proof of its correctness, and its running time analysis will be given in the full version of the paper. ■

A further relaxation is when the vertical segments do not come with specified endpoints, but only their x-coordinates (or, equivalently, their left-to-right order) are given:

Theorem 7. *Given a planar bipartite graph $G = (V \cup H, E)$ and the order of x-coordinates of the vertical segments V, there is a polynomial-time algorithm to decide if there is a grid contact representation of G respecting this order.*

Proof. Here we sketch a polynomial reduction from this problem to *level-planarity testing* (a detailed proof will be given in the full version of the paper where we also show that level-planarity testing can be reduced to this problem). In level-planarity testing we are given a *leveled* graph G, i.e., a graph whose vertex set is partitioned into independent sets (*levels*) $S_1, \ldots, S_k$. The goal is to determine if there is a planar drawing of G where the vertices of each S_i are represented by points on the line $x = i$. (For convenience we represent levels vertically, rather than horizontally.)

Level-planarity testing is known to be solvable in linear time [9, 15]; the algorithm proceeds level-by-level and uses PQ-trees to record possible orderings on previous levels. Now to our problem.

Let $G = (V \cup H, E)$ be a given planar bipartite graph and let $v_1, v_2, \ldots, v_{n_1}$ be a given ordering of V by prescribed x-coordinates. For simplicity we use the same symbol for a vertex and the segment representing it.

We may assume that G is connected, otherwise we simply solve the corresponding problem on each connected component of G and put the representations one above the other. We may also assume that the degree of each vertex in H is at least two; all vertices of degree one in H can be safely removed and reattached later at arbitrary contact points.

The intuition regarding the connection between these problems comes from the special case when every horizontal segment has degree two. In this case we see the reduction immediately by respectively mapping vertical segments and horizontal segments to vertices and edges in the level-planarity instance. In particular, by giving each vertical segment its own level and ordering the levels by the x-coordinates of the verticals we are done.

We now turn to the more interesting case, when some horizontal segments have high (≥ 3) degree. In this case we replace each high degree horizontal segment h by a gadget which involves O(degree of h) new segments where the horizontal segments have degree two and the vertical segments have degree three. This replacement is depicted in Fig. 12 and is formally described as Rule 1 below.
Rule 1. If H contains a vertex h of degree at least 3, then let $v_{i_0}, v_{i_1}, \ldots, v_{i_{k+1}}$ denote neighbors of h, where $i_0 < i_1 < \ldots < i_{k+1}$, and do the following:

(a) remove h from H and add a path
$v_{i_0}, z_0, x_1^-, y_1^-, v_{i_1}, y_1^+, x_1^+, z_1, x_2^-, y_2^-, v_{i_2}, \ldots, v_{i_k}, y_k^+, x_k^+, z_k, v_{i_{k+1}}$
where $x_j^-, x_j^+, y_j^-, y_j^+, z_j$ are new vertices such that:
x_j^-, x_j^+ are put in V and the rest in H.

(b) add new vertices $h_1, h_2, \ldots, h_k$ where each h_i is adjacent to x_i^- and x_i^+,

(c) modify the ordering of V by
– inserting x_j^- right before and x_j^+ right after v_{i_j}, for all $j = 1, \ldots, k$.

Moreover, from Fig. 12, it is easy to see that any solution to the original problem is preserved by applying Rule 1. So, we need to argue that any solution

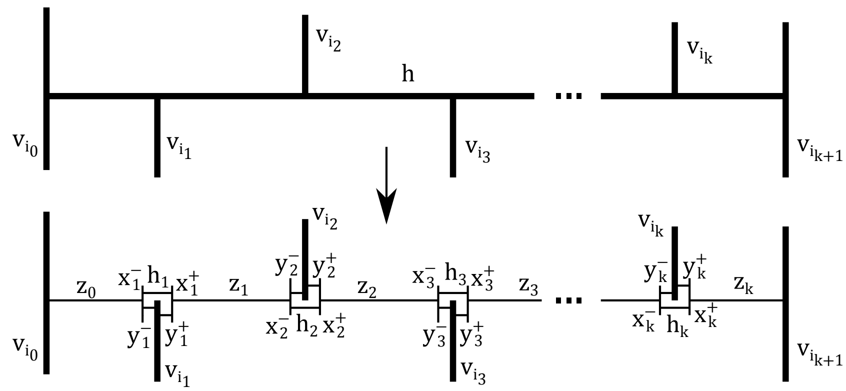

Fig. 12. Replacing a horizontal segment.

to the instance post-replacement must correspond to a solution pre-replacement. This amounts to two properties that we will need. The first is that the faces x_j^-, h_j, x_j^+, y_j^+, v_{i_j}, y_j^- are empty in any solution and the second is that the path

$$z_0,\, x_1^-,\, h_1,\, x_1^+,\, z_1,\, x_2^-,\, h_2,\, x_2^+,\, z_2,\, \ldots,\, x_k^-,\, h_k,\, x_k^+,\, z_k$$

can be "straightened". Both of these conditions are easily observed, but require a bit of care to prove formally. Moreover, once they are attained we then simply reverse the replacement as in Fig. 12 to obtain a solution to our original instance.

Notice that the size of the instance of level-planarity we produced is linear with respect to our input graph. Thus, via Rule 1 and our argument regarding the degree two case, the reduction is complete. ■

6 Conclusion

In most of the cases we have encountered NP-hardness. One certainly wonders if additional assumptions may make the partial representation extension problems polynomially solvable. One possible direction is requiring the input graph to be highly connected (since the graph used in the proof of Theorem 1 is very sparse):

Problem 1 Is extendability of partial grid contact representations of planar quadrangulations decidable in polynomial time?

In view of Theorem 7 one may wonder what happens if only a part of the vertical segments is partially described:

Problem 2 Given a planar bipartite graph and a linear order of the x-coordinates of some of the vertical segments, can one decide in polynomial time if there is a grid contact representation respecting this order?

References

1. Angelini, P., Battista, G.D., Frati, F., Jelínek, V., Kratochvíl, J., Patrignani, M., Rutter, I.: Testing planarity of partially embedded graphs. In: SODA 2010: Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms (2010)
2. Baker, K. A., Fishburn, P., Roberts, F. S.: Partial orders of dimension 2, Networks 2 (1): 11–28 (1972)

3. Ben-Arroyo Hartman, I., Newman, I., Ziv, R.: On grid intersection graphs. Discrete Mathematics 87(1): 41–52 (1991)
4. Bouchet, A.: Reducing prime graphs and recognizing circle graphs. Combinatorica, 7: 243–254 (1987)
5. Brandstaedt, A., Le, V.B., Spinrad, J.P.: Graph classes: a survey. Society for Industrial and Applied Mathematics, Philadelphia (1999)
6. Chaplick, S., Fulek, R., Klavík, P.: Extending Partial Representations of Circle Graphs, In: Proceedings Graph Drawing 2013, LNCS 8242: 131–142 (2013)
7. de Fraysseix, H., Ossona de Mendez, P., Pach, J.: Representation of planar graphs by segments. in: Intuitive Geometry (K. Böröczky, G. Fejes Tóth, eds), Coll. Math. Soc. J. Bolyai 63:109–117 (1994)
8. de Fraysseix, H., Ossona de Mendez, P., Rosenstiehl, P.: On Triangle Contact Graphs. Combinatorics, Probability & Computing 3: 233–246 (1994)
9. di Battista, G., Nardelli, E.: Hierarchies and planarity theory, IEEE Transactions on Systems, Man, and Cybernetics, 18: 1035-1046 (1988)
10. Fellows, M. R., Kratochvíl, J., Middendorf, M., Pfeiffer, F.: The Complexity of Induced Minors and Related Problems. Algorithmica 13(3): 266–282 (1995)
11. Fiala, J.: NP completeness of the edge precoloring extension problem on bipartite graphs. J. Graph Theory 43(2): 156–160 (2003)
12. Gavril, F.: Algorithms on circular-arc graphs, Networks 4 (4): 357–369 (1974)
13. Gilmore, P. C., Hoffman, A. J.: A characterization of comparability graphs and of interval graphs, Canadian Journal of Mathematics 16: 539–548 (1964)
14. Golumbic, M. C.: Algorithmic Graph Theory and Perfect Graphs, Computer Science and Applied Mathematics, Academic Press (1980)
15. Jünger, M., Leipert, S., Mutzel, P.: Level Planarity Testing in Linear Time, In: Proceedings Graph Drawing 1998, LNCS 1547: 224–237 (1998)
16. Klavík, P., J. Kratochvíl, T. Vyskočil: Extending Partial Representations of Interval Graphs. In: Proceedings TAMC 2011, LNCS 6648: 276–285 (2011)
17. Klavík, P., Kratochvíl, J., Krawczyk, T., Walczak, B.: Extending Partial Representations of Function Graphs and Permutation Graphs. In: Proceedings ESA 2012, LNCS 7501: 671–682 (2012)
18. Klavík, P., Kratochvíl, J., Otachi, Y., Saitoh, T.: Extending Partial Representations of Subclasses of Chordal Graphs. In: Proceedings ISAAC 2012, LNCS 7676: 444–454 (2012)
19. Klavík, P., Kratochvíl, J., Otachi, Y., Rutter, I., Saitoh, T., Saumell, M., Vyskočil, T.: Extending Partial Representations of Proper and Unit Interval Graphs. In: Proceedings SWAT 2014, LNCS 8503: 253–264 (2014) `http://arxiv.org/abs/1207.6960`
20. Koebe, P.: Kontaktprobleme auf der konformen Abbildung, Ber. Verh. Saechs. Akad. Wiss. Leipzig Math.-Phys. Kl., 88: 141–164 (1936)
21. Marx, D.: NP-completeness of list coloring and precoloring extension on the edges of planar graphs. J. Graph Theory 49(4): 313–324 (2005)
22. McKee, T.A., McMorris, F.R.: Topics in Intersection Graph Theory. SIAM Monographs on Discrete Mathematics and Applications (1999)
23. Mohar, B.: A polynomial time circle packing algorithm. Discrete Mathematics 117(1–3): 257–263 (1993)
24. Schaefer, M.: Complexity of Some Geometric and Topological Problems. In: Proceedings Graph Drawing 2009, LNCS 5849: 334–344 (2009)
25. Tucker, A.: An efficient test for circular-arc graphs, SIAM Journal on Computing 9 (1): 1–17 (1980)