Obstructions to chordal circular-arc graphs of small independence number

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Abstract

A *blocking quadruple* (BQ) is a quadruple of vertices of a graph such that any two vertices of the quadruple either miss (have no neighbours on) some path connecting the remaining two vertices of the quadruple, or are connected by some path missed by the remaining two vertices. This is akin to the notion of asteroidal triple used in the classical characterization of interval graphs by Lekkerkerker and Boland [10].

In this note, we first observe that blocking quadruples are obstructions for circular-arc graphs. We then focus on chordal graphs, and study the relationship between the structure of chordal graphs and the presence/absence of blocking quadruples.

Our contribution is two-fold. Firstly, we provide a forbidden induced subgraph characterization of chordal graphs without blocking quadruples. In particular, we observe that all the forbidden subgraphs are variants of the subgraphs forbidden for interval graphs [10]. Secondly, we show that the absence of blocking quadruples is sufficient to guarantee that a chordal graph with no independent set of size five is a circular-arc graph. In our proof we use a novel geometric approach, constructing a circular-arc representation by traversing around a carefully chosen clique tree.

Keywords: circular-arc graph, chordal graph, asteroidal triple, clique tree, obstruction, blocking quadruple, forbidden subgraph characterization

1 Introduction

The study of graph obstructions has a long tradition in graph theory. To understand the structure of graphs in a particular graph class, it is often useful (if not easier) instead to characterize all minimal graphs that are not in the class, usually known as *obstructions*. They often result in elegant characterization theorems and can be used as succinct certificates in certifying algorithms.

In this paper, we seek obstructions to *circular-arc graphs*, the intersection graphs of families of arcs of a circle. This problem dates back at least as far as the 1970's [9,12,13], and remains a challenging question capturing the interest of many researchers over the years [1,2,6,9,11,12,13].

Predating the study of circular arc graphs, the class of *interval graphs*, intersection graphs of families of intervals of the real line, was investigated. Interval graphs are a subclass of *chordal graphs*, graphs in which every cycle has a chord, as well as of circular-arc graphs. They are known to admit a number of interesting characterizations [8,10] and efficient recognition algorithms [4,5]. In particular, the result of Lekkerkerker and Boland [10] describes interval graphs in terms of forbidden induced subgraphs as well as forbidden substructures – chordless cycles and so-called *asteroidal triples*.

This result is the main motivation of our paper wherein we seek to describe analogous forbidden substructures for circular-arc graphs.

We remark in passing that, besides interval graphs, there are other subcases of circular-arc graphs that have already been characterized by the absence of simple obstructions. Namely, *unit circular-arc graphs* and *proper circular-arc graphs* in [13], *chordal proper circular-arc graphs* in [1], *cobipartite circular-arc graphs* in [12] and later in [6] (using so-called *edge-asteroids*), and *Helly circular-arc graphs* within circular-arc graphs in [11] (using so-called *obstacles*).

More recently, in [2], the authors gave forbidden induced subgraph characterizations for P_4-free circular-arc graphs, diamond-free circular-arc graphs, paw-free circular-arc graphs, and most relevant for this paper, they characterized claw-free chordal circular-arc graphs. Our results (namely Theorem 3.2) may be seen as complementing their work, since in this regard we give a forbidden induced subgraph characterization of $\overline{K_5}$-free chordal circular-arc graphs.

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2 Blocking quadruple

To build intuition, we start by recalling the definition of asteroidal triple. We say that a vertex x *misses* a path P in G if x has no neighbour on P.

Vertices x, y, z form an *asteroidal triple* of G if between any two of them, there is a path in G missed by the third vertex. It is easy to see that an interval graph cannot have an asteroidal triple [10].

We say that vertices x, y *avoid* vertices z, w in G if there exists an xy-path missed by both z and w, or there exists a zw-path missed by both x and y.

We say that vertices x, y, z, w form a *blocking quadruple* (BQ) of G if any two of them avoid the remaining two.

Lemma 2.1 *If G is a circular-arc graph, then G has no blocking quadruple.*

To see this, observe that a BQ is always an independent set of size four. Now, suppose that G has a circular-arc representation and the arcs representing vertices x, y, z, w appear in this circular order. Then no path between x and z can be missed by both y and w, and no path between y and w can be missed by x and z. In other words, the vertices x, z do not avoid y, w.

Let us now discuss various forms of blocking quadruples that one may encounter in graphs. One class of such examples arises from asteroidal triples: if a, b, c form an asteroidal triple of G and d is a vertex of degree zero in G, then a, b, c, d is a blocking quadruple. This can be seen in the first three graphs in Figure 1. Other ways of extending an asteroidal triple to a BQ are also illustrated in Figure 1. The vertices a, b, c in each of the graphs in the second row form an asteroidal triple while the vertices a, b, c, d form a blocking quadruple. For chordal graphs, these are all possible forms of BQs (see Theorem 3.1).

Unlike these examples, the two chordal graphs in Figure 2 do not contain blocking quadruples, and yet they are not circular-arc graphs. Thus the absence of blocking quadruples is not sufficient to guarantee that a (chordal)

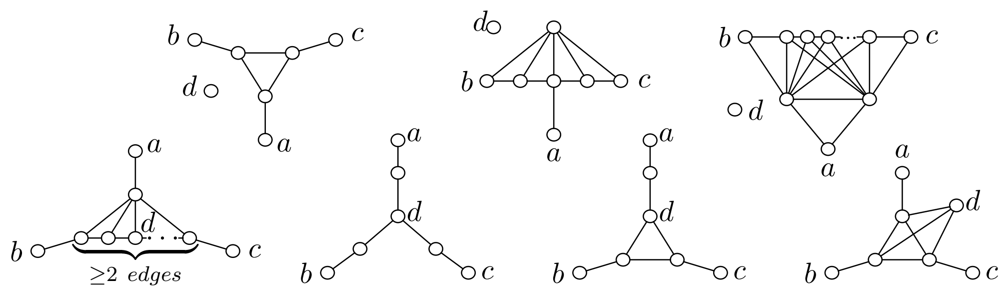


Fig. 1. Forbidden induced subgraph characterization of chordal graphs with no BQs.

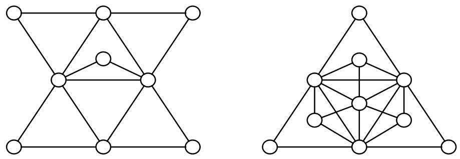


Fig. 2. Some minimal chordal non-circular-arc graphs with no BQs.

graph is a circular-arc graph. However, in some cases, it may be sufficient.

For instance, a result of [2] (Corollary 15) can be restated as follows.

Lemma 2.2 *A claw-free chordal graph is a circular-arc graph iff it has no BQ.*

We prove a similar statement for chordal graphs of independence number at most four (see Theorem 3.2). The absence of BQs therefore gives us a simple and uniform forbidden structure characterization of these classes, as opposed to more common forbidden induced subgraph characterizations [1,2,11,13].

3 Main results

In this section, we summarize the main theorems of this paper.

Firstly, we describe all minimal forbidden induced subgraphs characterizing chordal graphs with no BQs. These are the graphs depicted in Figure 1.

Theorem 3.1 *If G is chordal, then the following are equivalent.*
(i) G contains a blocking quadruple.
(ii) G contains an induced subgraph isomorphic to a graph in Figure 1.

In fact, the theorem holds for the more general class of *nearly chordal* graphs (a graph class defined in [3] generalizing both chordal and circular-arc graphs).

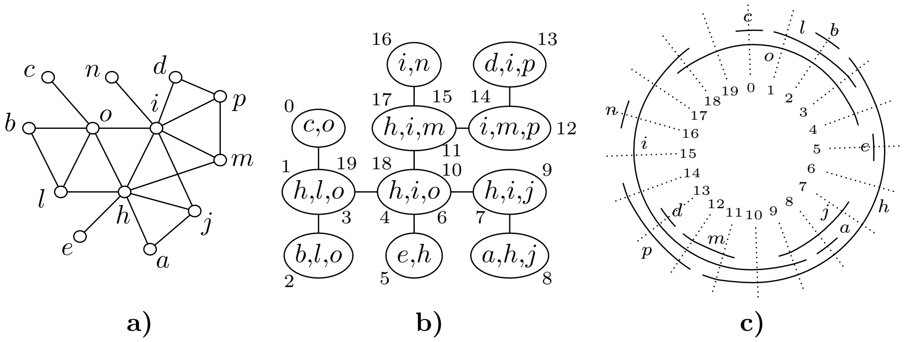
Secondly, we show that the absence of BQs is necessary and sufficient for a chordal graph of independence number $\alpha(G) \leq 4$ to be a circular-arc graph.

Theorem 3.2 *If G is chordal and $\alpha(G) \leq 4$, the following are equivalent.*
(i) G is a circular-arc graph.
(ii) G contains no blocking quadruple.

The theorem fails for chordal graphs G with $\alpha(G) \geq 5$ as Figure 2 shows.

4 Proof sketches

To prove Theorem 3.1, consider a chordal graph G with a blocking quadruple a, b, c, d. By symmetry, we may assume that G contains an ab-path P_b missed by both c and d, and an ac-path P_c missed by both b and d. If G also contains



v	$\phi(v)$	ℓ_v	r_v
a	8	7	9
b	2	1	3
c	0	19	1
d	13	12	14
e	5	4	6
h	6	2	12
i	15	8	19
j	8	6	10
l	2	0	4
m	12	10	13
n	16	15	17
o	1	17	5
p	13	11	15

Fig. 3. *a)* Example chordal graph, *b)* clique tree + an Euler tour, *c)* resulting arcs

a bc-path missed by a and d, then a, b, c is an asteroidal triple in $G - N[d]$. Thus by [10], G contains one of the graphs in the top row of Figure 1, or one of the first two graphs in the second row of Figure 1 (ignoring the label d). If this is not the case, then G contains an ad-path P_d missed by both b and c. We choose a, b, c, d so as to minimize $|P_b| + |P_c| + |P_d|$. It can be shown that this minimality combined with chordality of G implies that the union of vertices of the three paths P_b, P_c, P_d induces in G exactly one of the graphs in the second row of Figure 1. This is rather technical, so we omit further details.

For Theorem 3.2, let G be a chordal graph. The direction (i)$\Rightarrow$(ii) is proved as Lemma 2.1. For (ii)$\Rightarrow$(i), assume (ii). If $\alpha(G) \leq 2$, then G contains no asteroidal triple. So G is an interval graph by [10] which implies (i).

Suppose that $\alpha(G) = 3$. Note that (ii) is vacuously satisfied in this case. Let T be a clique tree of G (subtree intersection model, see [7]). Since G is a perfect graph, it has a clique cover by three cliques. Let Q_1, Q_2, Q_3 be these cliques. This implies that T has at most three leaves, and each leaf of T is one of Q_1, Q_2, Q_3. If T has two leaves, then G is an interval graph implying (i). So we may assume that T has exactly three leaves, namely Q_1, Q_2, Q_3.

From here, we proceed as follows. We fix some Euler tour $A_0, A_1, \ldots, A_{k-1}$ of T (considered as a digraph) where A_i are nodes of T. Using this tour, we construct a circular-arc representation of G. We start by placing k points $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ on the circle arranged in this order in the clockwise direction. Then for every vertex v of G we construct a circular-arc as follows.

Recall that $v \in Q_i$ for some $i \in \{1, 2, 3\}$, since Q_1, Q_2, Q_3 is a clique cover of G. We choose i such that $v \in Q_i$ and then choose in the Euler tour some occurrence of Q_i. Namely, we choose $\phi(v)$ such that $A_{\phi(v)} = Q_i$. Then we walk from $A_{\phi(v)}$ along the tour (in both directions) as far as possible so long as the cliques we encounter contain v. That is, we define indices ℓ_v, r_v such that

$$\ell_v = \phi(v) - \min\{i \geq 0 \mid v \notin A_{\phi(v)-i}\} \qquad r_v = \phi(v) + \min\{j \geq 0 \mid v \notin A_{\phi(v)+j}\}$$

We represent v by a clockwise arc from λ_{ℓ_v+1} to λ_{r_v-1} (indices taken modulo k).

We refer to Figure 3 for an illustration of this process. It can be shown that in case $\alpha(G) = 3$ this constructs a valid circular-arc representation of G.

So we may assume that $\alpha(G) = 4$. We proceed similarly as in the previous case but with more care. Let T be a clique tree of G. Since G is perfect, T has a clique cover by four cliques. Let Q_1, Q_2, Q_3, Q_4 be these cliques. It follows that T has at most four leaves and each leaf of T is one of Q_1, Q_2, Q_3, Q_4.

Assume first that T has exactly four leaves, namely Q_1, Q_2, Q_3, Q_4. Since T is a clique tree, for each $i \in \{1, 2, 3, 4\}$, there exists a vertex v_i that belongs to Q_i and no Q_j, $j \neq i$. Let H denote the graph on vertices $\{v_1, v_2, v_3, v_4\}$ where v_iv_j is an edge if and only if v_i, v_j avoid vertices $V(H) \setminus \{v_i, v_j\}$. We use H to guide us in the next steps. We choose an Euler tour of T satisfying ($\star$):

($\star$) if $v_iv_j \in E(H)$ and $\{v_{i'}, v_{j'}\} = V(H) \setminus \{v_i, v_j\}$, then the subtour of the Euler tour between Q_i and Q_j contains neither or both of $Q_{i'}$, $Q_{j'}$.

It can be shown that this is always possible for some clique tree T having Q_1, Q_2, Q_3, Q_4 as leaves. For this, we use the fact that H is either empty, or a 4-cycle, or a pair of disjoint edges, because v_1, v_2, v_3, v_4 is not a blocking quadruple of G by (ii). From this point we proceed as in the case $\alpha(G) = 3$.

Finally, if T has three leaves or less, we again have pairwise non-adjacent vertices v_1, v_2, v_3, v_4 in the cliques Q_1, Q_2, Q_3, Q_4, and we define the graph H as before. We use H to define ϕ (rather than to choose an Euler tour) using which we construct circular-arcs as in the case $\alpha(G) = 3$. We omit further details.

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