A Decomposition Theorem for Chordal Graphs and its Applications

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Abstract

We introduce a special decomposition, the so-called split-minors, of the reduced clique graphs of chordal graphs. Using this notion, we characterize asteroidal sets in chordal graphs and clique trees with minimum number of leaves.

Keywords: chordal graph, asteroidal set, split-minor, leafage, polynomial time

1 Introduction

In this paper, graph is always simple, undirected and loopless.

For a vertex v of a graph G, we denote by $N(v)$ the neighbourhood of v in G, i.e., the set of vertices u such that $uv \in E(G)$; we denote by $N[v]$ the set $N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, we denote by $G[X]$ the graph induced on X, i.e., graph $G[X] = (X, E \cap X \times X)$, and denote by $G - X$ the graph $G[V(G) \setminus X]$. A *complete subgraph* or a *clique* of G is a (not necessarily maximal) set of pairwise adjacent vertices of G. (For a complete terminology, see [5].)

A set A of vertices of a graph G is *asteroidal*, if for each vertex v of A, the set $A \setminus \{v\}$ belongs to one connected component of $G - N[v]$.

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The *asteroidal number* $a(G)$ of G is the size of a largest asteroidal set of G. Computing $a(G)$ is NP-hard already in planar graphs [3], but it is efficiently solvable in chordal graphs [3] (and other structured classes of graphs).

The *leafage* $l(G)$ of a connected chordal graph G is the least number of leaves in a clique tree of G. If G is disconnected, $l(G)$ is defined as the maximum leafage over all connected components of G. While testing $l(G) \leq k$ for $k \in \{2, 3\}$ is polynomial [4], the complexity of computing $l(G)$ is not known.

In this paper, we introduce a notion of a split-minor of the reduced clique graph of a chordal graph (Section 3). This novel tool allows us to obtain a total decomposition of the reduced clique graph well suited for algorithmic use. We apply this tool (Section 4) to characterize asteroidal sets of chordal graphs, and give a partial condition for the leafage.

2 The reduced clique graph

Let G be a connected chordal graph. A *clique tree* of G is any tree T whose vertices are the maximal cliques of G such that for every two maximal cliques C, C', each clique on the path from C to C' in T contains $C \cap C'$.

Two cliques C, C' of G form a *separating pair*, if every path from a vertex of $C \setminus C'$ to a vertex of $C' \setminus C$ contains a vertex of $C \cap C'$.

The *reduced clique graph* $\mathcal{C}_r(G)$ of G is a graph whose vertices are the maximal cliques of G, and whose edges CC' are between cliques C, C' forming separating pairs. In addition, each edge CC' of $\mathcal{C}_r(G)$ is labeled by $C \cap C'$.

The following is a fundamental result about reduced clique graphs.

Theorem 2.1 [1] *Every clique tree of G is a maximum weight spanning tree of $\mathcal{C}_r(G)$ where the weight of each edge CC' is defined as $|C \cap C'|$. Moreover, the reduced clique graph $\mathcal{C}_r(G)$ is precisely the union of all clique trees of G.*

If G is disconnected, the reduced clique graph $\mathcal{C}_r(G)$ of G is defined as the disjoint union of the reduced clique graphs of its connected components.

3 Split-minors

Let H be a graph, and let e be an edge of H. By $H/_e$ we denote the graph obtained from H by *contracting* the edge e to a new vertex v_e (i.e., removing x and y, adding v_e, and connecting all neighbours of x and neighbours of y to v_e).

Let $X \cup Y$ be a partition of $V(H)$. We say that $X \cup Y$ is a *split* of H if every vertex of X with a neighbour in Y has the same neighbourhood in Y.

We say that a graph H' is a *split-minor* of H, if H' can be obtained from H by performing a sequence of the following three operations:

(S1) if v is an isolated vertex, remove v.
(S2) if e is an edge, contract e.
(S3) if $X \cup Y$ is a split, remove all edges between X and Y.

Now, suppose that H has labeled edges. We say that an edge $e = xy$ of H is *permissible*, if for every triangle x, y, z, the edges xz and yz have the same label. If e is a permissible edge of H, we denote by $H/_e$ the graph with labeled edges obtained by contracting e to a new vertex v_e and assigning labels as follows: for every neighbour z of v_e, the edge $v_e z$ is labeled using the label of xz if $xz \in E(H)$, or using the label of yz if $yz \in E(H)$, and all other edges are labeled using the same label they have in H. We say that a split $X \cup Y$ of H is *permissible*, if all edges between X and Y have the same label.

We say that a graph H' with labeled edges is a *labeled split-minor* of H if H' can be obtained from H by a sequence of the following operations:

(L1) if v is an isolated vertex, remove v.
(L2) if e is a permissible edge, contract e.
(L3) if $X \cup Y$ is a permissible split, remove all edges between X and Y.

If $e = CC'$ is an edge of $\mathcal{C}_r(G)$, we denote by $G/\!\!/_e$ the graph obtained from G by adding all possible edges between the vertices of C and C'.

Theorem 3.1 *If H' is a labeled split-minor of $H = \mathcal{C}_r(G)$, then there exists a graph G' such that $H' = \mathcal{C}_r(G')$.*

Proof. (Sketch) By induction. Let H' be the graph obtained from $\mathcal{C}_r(G)$ by one of the three operations. Suppose we apply (L1) to a vertex C of $\mathcal{C}_r(G)$. Then C forms a connected component of G, and hence, $\mathcal{C}_r(G - C) = H'$. Suppose we apply (L3) to a split $X \cup Y$. Then we let $V_X \subseteq V(G)$, respectively $V_Y \subseteq V(G)$, be the union of all maximal cliques that are elements of X, respectively Y. (Note that $V_X \cap V_Y \neq \emptyset$.) We let G' be the disjoint union of $G[V_X]$ and $G[V_Y]$, and it follows that $\mathcal{C}_r(G') = H'$. Finally, suppose we apply (L2) to an edge $e = CC'$. Then it is not difficult to show that $\mathcal{C}_r(G/\!\!/_e) = H'$.□

An edge e of H is *maximal*, resp. *minimal*, if there is no edge e' in H whose label strictly contains ($\supsetneq$), resp. is strictly contained ($\subsetneq$) in, the label of e.

Observation 3.2 *Every maximal edge of $\mathcal{C}_r(G)$ is permissible, and every minimal edge of $\mathcal{C}_r(G)$ is an edge between sets X, Y of a permissible split $X \cup Y$.*

As a consequence of this observation we obtain the following theorem.

Theorem 3.3 (Split-minor decomposition) *Every reduced clique graph is totally decomposable with respect to* (L1),(L3) *and also with respect to* (L2).

4 Applications

A vertex v of H is *S-dominated*, if S is a subset of the label of every edge incident to v. An edge $e = xy$ with label S is *good*, if e is permissible in H, and no vertex of H is S-dominated unless at least one of x, y is S-dominated.

We say that H' is a *good split-minor* of H, if H' can be obtained from H by a sequence of operations (L1),(L3), and the following:

(L2′) if e is a good edge, contract e.

Theorem 4.1 *If H' is a good split-minor of $\mathcal{C}_r(G)$, and G' is the graph from Theorem 3.1 such that $H' = \mathcal{C}_r(G')$, then $a(G') \leq a(G)$.*

Proof. (Sketch) By induction. The case of (L1) and (L3) is easy to handle. Suppose that $\mathcal{C}_r(G')$ is obtained by contracting a good edge $e = CC'$ using (L2′). Let $S = C \cap C'$. Then $G' = G /\!\!/_e$ is obtained from G by making $C \cup C'$ a clique. Let A be a largest asteroidal set of G'. Suppose that A is not asteroidal in G. We may assume $|A| \geq 3$. Hence, there exist $a_1, a_2, a_3 \in A$ such that a_1 and a_3 are in different connected components of $G - N_G[a_2]$. Since A is asteroidal in G', we deduce $S \subseteq N_G[a_2]$, and there is no other vertex $a \in A$ with $S \subseteq N_G[a]$. Hence, we let C^* be a maximal clique that contains $\{a_2\} \cup S$. It now follows that both CC^* and $C'C^*$ are edges of $\mathcal{C}_r(G)$ with label S.

Next, we let $A' = A \setminus \{a_2\}$ and observe that A' is asteroidal in both G' and G. After that, we deduce that both C and C' are not S-dominated, and hence, since e is a good edge, C^* is also not S-dominated. Therefore, there exists a maximal clique C^{**} such that C^*C^{**} is an edge of $\mathcal{C}_r(G)$ with label $S^* \not\supseteq S$. We let a_2'' be any vertex of $C^{**} \setminus C^*$, and let $A'' = A' \cup \{a_2''\}$.

Finally, we show that A'' is an asteroidal set of both G' and G. □

A *k-star* is a graph obtained by taking a vertex with k neighbours forming an independent set. A *labeled k-star* is a k-star whose edges are labeled.

Theorem 4.2 *$a(G) < k$ iff no labeled k-star is a good split-minor of $\mathcal{C}_r(G)$.*

Proof. (Sketch) If a labeled k-star $\mathcal{C}_r(G')$ is a good split-minor of $\mathcal{C}_r(G)$, it is not difficult to show that the label of no edge of $\mathcal{C}_r(G')$ is contained in another edge. Hence, $a(G') = k$, and by Theorem 4.1, we deduce $a(G) \geq k$.

Now, let $A = \{a_1, \ldots, a_k\}$ be an asteroidal set of G. Let $C_1, \ldots, C_k$ be maximal cliques of G containing $a_1, \ldots, a_k$, respectively. Let T be a clique tree of G, and T' be the subtree of T formed by taking all paths between $C_1, \ldots, C_k$. Since A is asteroidal, $C_1, \ldots, C_k$ are leaves of T'. Let T be chosen so that T' is smallest possible. Let C_iC_i' be the (unique) edge incident to C_i in T'.

We observe that removal of any minimal edge of T yields two trees whose vertices form a permissible split of $\mathcal{C}_r(G)$. Hence, by minimality of T', we can remove or contract all edges whose label does not appear on T'. Finally, we contract all edges of T' other than C_iC_i', and obtain a k-star. $\square$

Corollary 4.3 *If $\mathcal{C}_r(G) \cong \mathcal{C}_r(G')$ as labeled graphs, then $a(G') = a(G)$.*

For the leafage of chordal graphs, we have the following similar statement (proof omitted). We say that a vertex v of H is *S-bounded*, if v is incident to an edge labeled with S, and the label of every other edge incident to v is a subset ($\subseteq$) of S. An edge $e = xy$ with label S is *nice*, if e is maximal in H and no vertex of H is S-bounded unless at least one of x, y is S-bounded.

We say that H' is a *nice split-minor* of H, if H' can be obtained from H by a sequence of operations (L1),(L3), and the following:

(L2$''$) if e is a nice edge, contract e.

Theorem 4.4 *If $\mathcal{C}_r(G')$ is a nice split-minor of $\mathcal{C}_r(G)$, then $l(G') \leq l(G)$.*

Unfortunately, we do not have a characterization of $l(G)$ similar to Theorem 4.2, since the minimal forbidden split-minors for the leafage are not easy to describe. However, we can describe other conditions that allow computing the leafage of G from its reduced clique graph. (More in [2].)

We close by noting that the restrictions introduced in the operations (L2$'$) and (L2$''$) still allow for a total decomposition of reduced clique graphs.

Theorem 4.5 *Every reduced clique graph is totally decomposable with respect to* (L2$'$) *and also with respect to* (L2$''$).

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