Constraint Satisfaction with Counting Quantifiers[*]

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Abstract. We initiate the study of *constraint satisfaction problems* (CSPs) in the presence of counting quantifiers, which may be seen as variants of CSPs in the mould of *quantified CSPs* (QCSPs).

We show that a **single** counting quantifier strictly between $\exists^{\geq 1} := \exists$ and $\exists^{\geq n} := \forall$ (the domain being of size n) already affords the maximal possible complexity of QCSPs (which have **both** $\exists$ and $\forall$), being Pspace-complete for a suitably chosen template.

Next, we focus on the complexity of subsets of counting quantifiers on clique and cycle templates. For cycles we give a full trichotomy – all such problems are in L, NP-complete or Pspace-complete. For cliques we come close to a similar trichotomy, but one class remains outstanding.

Afterwards, we consider the generalisation of CSPs in which we augment the extant quantifier $\exists^{\geq 1} := \exists$ with the quantifier $\exists^{\geq j}$ ($j \neq 1$). Such a CSP is already NP-hard on non-bipartite graph templates. We explore the situation of this generalised CSP on bipartite templates, giving various conditions for both tractability and hardness – culminating in a classification theorem for general graphs.

Finally, we use counting quantifiers to solve the complexity of a concrete QCSP whose complexity was previously open.

1   Introduction

The *constraint satisfaction problem* $\mathrm{CSP}(\mathcal{B})$, much studied in artificial intelligence, is known to admit several equivalent formulations, two of the best known of which are the query evaluation of primitive positive (pp) sentences – those involving only existential quantification and conjunction – on $\mathcal{B}$, and the homomorphism problem to $\mathcal{B}$ (see, e.g., [18]). The problem $\mathrm{CSP}(\mathcal{B})$ is NP-complete in general, and a great deal of effort has been expended in classifying its complexity for certain restricted cases. Notably it is conjectured [15, 6] that for all fixed $\mathcal{B}$,

the problem CSP($\mathcal{B}$) is in P or NP-complete. While this has not been settled in general, a number of partial results are known – e.g. over structures of size at most three [25, 5] and over smooth digraphs [16, 1].

A popular generalisation of the CSP involves considering the query evaluation problem for *positive Horn* logic – involving only the two quantifiers, $\exists$ and $\forall$, together with conjunction. The resulting *quantified constraint satisfaction problems* QCSP($\mathcal{B}$) allow for a broader class, used in artificial intelligence to capture non-monotonic reasoning, whose complexities rise to Pspace-completeness.

In this paper, we study counting quantifiers of the form $\exists^{\geq j}$, which allow one to assert the existence of at least j elements such that the ensuing property holds. Thus on a structure $\mathcal{B}$ with domain of size n, the quantifiers $\exists^{\geq 1}$ and $\exists^{\geq n}$ are precisely $\exists$ and $\forall$, respectively. Counting quantifiers have been extensively studied in finite model theory (see [11, 22]), where the focus is on supplementing the descriptive power of various logics. Of more general interest is the majority quantifier $\exists^{\geq n/2}$ (on a structure of domain size n), which sits broadly midway between $\exists$ and $\forall$. Majority quantifiers are studied across diverse fields of logic and have various practical applications, e.g. in cognitive appraisal and voting theory [10]. They have also been studied in computational complexity, e.g., in [19].

We study variants of CSP($\mathcal{B}$) in which the input sentence to be evaluated on $\mathcal{B}$ (of size $|B|$) remains positive conjunctive in its quantifier-free part, but is quantified by various counting quantifiers.

For $X \subseteq \{1, \ldots, |B|\}$, $X \neq \emptyset$, the X-CSP($\mathcal{B}$) takes as input a sentence given by a conjunction of atoms quantified by quantifiers of the form $\exists^{\geq j}$ for $j \in X$. It then asks whether this sentence is true on $\mathcal{B}$. The idea to study $\{1, \ldots, |B|\}$-CSP($\mathcal{B}$) is originally due to Andrei Krokhin.

In Section 3, we consider the power of a single quantifier $\exists^{\geq j}$. We prove that for each $n \geq 3$, there is a template $\mathcal{B}_n$ of size n, such that $\exists^{\geq j}$ ($1 < j < n$) already has the full complexity of QCSP, i.e., $\{j\}$-CSP($\mathcal{B}_n$) is Pspace-complete.

In Section 4, we go on to study the complexity of subsets of our quantifiers on clique and cycle templates, $\mathcal{K}_n$ and $\mathcal{C}_n$, respectively. We derive the following classification theorems.

Theorem 1. *For $n \in \mathbb{N}$ and $X \subseteq \{1, \ldots, n\}$:*
  (i) *X-CSP($\mathcal{K}_n$) is in L if $n \leq 2$ or $X \cap \{1, \ldots, \lfloor n/2 \rfloor\} = \emptyset$.*
  (ii) *X-CSP($\mathcal{K}_n$) is NP-complete if $n > 2$ and $X = \{1\}$.*
  (iii) *X-CSP($\mathcal{K}_n$) is Pspace-complete if $n > 2$ and either $j \in X$ for $1 < j < n/2$ or $\{1, j\} \subseteq X$ for $j \in \{\lceil n/2 \rceil, \ldots, n\}$.*

This is a near trichotomy – only the cases where n is even and we have the quantifier $\exists^{\geq n/2}$ remain open. For cycles, however, the trichotomy is complete.

Theorem 2. *For $n \geq 3$ and $X \subseteq \{1, \ldots, n\}$, the problem X-CSP($\mathcal{C}_n$) is either in L, is NP-complete or is Pspace-complete. Namely:*
  (i) *X-CSP($\mathcal{C}_n$) $\in$ L if $n = 4$, or $1 \notin X$, or n is even and $X \cap \{2, \ldots, n/2\} = \emptyset$.*
  (ii) *X-CSP($\mathcal{C}_n$) is NP-complete if n is odd and $X = \{1\}$.*
  (iii) *X-CSP($\mathcal{C}_n$) is Pspace-complete in all other cases.*

In Section 5, we consider $\{1, j\}$-CSP$(\mathcal{H})$, for $j \neq 1$ on graphs. The CSP is already NP-hard on non-bipartite graph templates. We explore the situation of this generalised CSP on bipartite graph templates, giving various conditions for both tractability and hardness, using and extending results of Section 4. We are most interested here in the distinction between P and NP-hard. To understand which of these cases are Pspace-complete would include as a subclassification the Pspace-complete cases of QCSP$(\mathcal{H})$, a question which has remained open for five years [21]. We give a classification theorem for graphs in fragments of the logic involving bounded use of $\exists^{\geq 2}$ followed by unbounded use of $\exists$. In the case of QCSP ($\exists^{\geq n}$ instead of $\exists^{\geq 2}$), this is perfectly natural and is explored with bounded alternations in, e.g., [8, 9, 17], and with bounded use of $\forall = \exists^{\geq n}$ in [7]. We prove that either there exists such a fragment in which the problem is NP-hard or for all such fragments the problem is in P.

Afterwards in Section 6, we use counting quantifiers to solve the complexity of QCSP$(\mathcal{C}_4^*)$, where $\mathcal{C}_4^*$ is the reflexive 4-cycle, whose complexity was previously open. Finally, in Section 7, we give some closing remarks and open problems.

2   Preliminaries

Let $\mathcal{B}$ be a finite structure over a finite signature σ whose domain B is of cardinality $|B|$. For $1 \leq j \leq |B|$, the formula $\exists^{\geq j} x \; \phi(x)$ with *counting quantifier* should be interpreted on $\mathcal{B}$ as stating that there exist at least j distinct elements $b \in B$ such that $\mathcal{B} \models \phi(b)$. Counting quantifiers generalise existential ($\exists := \exists^{\geq 1}$), universal ($\forall := \exists^{\geq |B|}$) and (weak) majority ($\exists^{\geq |B|/2}$) quantifiers. Counting quantifiers do not in general commute with themselves, viz $\exists^{\geq j} x \exists^{\geq j} y \neq \exists^{\geq j} y \exists^{\geq j} x$ (in contrast, $\exists$ and $\forall$ do commute with themselves, but not with one another).

For $\emptyset \neq X \subseteq \{1, \ldots, |B|\}$, the X-CSP$(\mathcal{B})$ takes as input a sentence of the form $\Phi := Q_1 x_1 Q_2 x_2 \ldots Q_m x_m \; \phi(x_1, x_2, \ldots, x_m)$, where ϕ is a conjunction of positive atoms of σ and each Q_i is of the form $\exists^{\geq j}$ for some $j \in X$. The set of such sentences forms the logic X-pp (recall the pp is primitive positive). The yes-instances are those for which $\mathcal{B} \models \Phi$. Note that all problems X-CSP$(\mathcal{B})$ are trivially in Pspace, by cycling through all possible evaluations for the variables.

The problem $\{1\}$-CSP$(\mathcal{B})$ is better-known as just CSP$(\mathcal{B})$, and $\{1, |B|\}$-CSP$(\mathcal{B})$ is better-known as QCSP$(\mathcal{B})$. We will consider also the logic $[2^m 1^*]$-pp and restricted problem $[2^m 1^*]$-CSP$(\mathcal{B})$, in which the input $\{1, 2\}$-pp sentence has prefix consisting of no more than m $\exists^{\geq 2}$ quantifiers followed by any number of $\exists$ quantifiers (and nothing else).

A homomorphism from $\mathcal{A}$ to $\mathcal{B}$, both σ-structures, is a function $h : A \rightarrow B$ such that $(a_1, \ldots, a_r) \in R^{\mathcal{A}}$ implies $(h(a_1), \ldots, h(a_r)) \in R^{\mathcal{B}}$, for all relations R of σ. A frequent role will be played by the *retraction* problem Ret$(\mathcal{B})$ in which one is given a structure $\mathcal{A}$ containing $\mathcal{B}$, and one is asked if there is a homomorphism from $\mathcal{A}$ to $\mathcal{A}$ that is the identity on $\mathcal{B}$. It is well-known that retraction problems are special instances of CSPs in which the constants of the template are all named [12].

In line with convention we consider the notion of hardness reduction in proofs to be polynomial many-to-one (though logspace is sufficient for our results).

2.1 Game characterisation

There is a simple game characterisation for the truth of sentences of the logic X-pp on a structure $\mathcal{B}$. Given a sentence Ψ of X-pp, and a structure $\mathcal{B}$, we define the following game $\mathscr{G}(\Psi, \mathcal{B})$. Let $\Psi := Q_1 x_1 Q_2 x_2 \ldots Q_m x_m \ \psi(x_1, x_2, \ldots, x_m)$. Working from the outside in, coming to a quantified variable $\exists^{\geq j} x$, the *Prover* (female) picks a subset B_x of j elements of B as witnesses for x, and an *Adversary* (male) chooses one of these, say b_x, to be the value of x. Prover wins iff $\mathcal{B} \models \psi(b_{x_1}, b_{x_2}, \ldots, b_{x_m})$. The following comes immediately from the definitions.

Lemma 1. *Prover has a winning strategy in the game $\mathscr{G}(\Psi, \mathcal{B})$ iff $\mathcal{B} \models \Psi$.*

We will often move seemlessly between the two characterisations of Lemma 1. One may alternatively view the game in the language of homomorphisms. There is an obvious bijection between σ-structures with domain $\{1, \ldots, m\}$ and conjunctions of positive atoms in variables $\{v_1, \ldots, v_m\}$. From a structure $\mathcal{B}$ build the conjunction $\phi_{\mathcal{B}}$ listing the tuples that hold on $\mathcal{B}$ in which element i corresponds to variable v_i. Likewise, for a conjunction of positive atoms ψ, let $\mathcal{D}_\psi$ be the structure whose relation tuples are listed by ψ, where variable v_i corresponds to element i. The relationship of $\mathcal{B}$ to $\phi_{\mathcal{B}}$ and ψ to $\mathcal{D}_\psi$ is very similar to that of *canonical query* and *canonical database* (see [18]), except there we consider the conjunctions of atoms to be existentially quantified. For example, $\mathcal{K}_3$ on domain $\{1, 2, 3\}$ gives rise to $\phi_{\mathcal{K}_3} := \exists v_1, v_2, v_3 \ E(v_1, v_2) \wedge E(v_2, v_1) \wedge E(v_2, v_3) \wedge E(v_3, v_2) \wedge E(v_3, v_1) \wedge E(v_3, v_1)$. The Prover-Adversary game $\mathscr{G}(\Psi, \mathcal{B})$ may be seen as Prover giving j potential maps for element x in $\mathcal{D}_\psi$ (ψ is quantifier-free part of Ψ) and Adversary choosing one of them. The winning condition for Prover is now that the map given from $\mathcal{D}_\psi$ to $\mathcal{B}$ is a homomorphism.

In the case of QCSP, i.e. $\{1, |B|\}$-pp, each move of a game $\mathscr{G}(\Psi, \mathcal{B})$ is trivial for one of the players. For $\exists^{\geq 1}$ quantifiers, Prover gives a singleton set, so Adversary's choice is forced. In the case of $\exists^{\geq |B|}$ quantifiers, Prover must advance all of B. Thus, essentially, Prover alone plays $\exists^{\geq 1}$ quantifiers and Adversary alone plays $\exists^{\geq |B|}$ quantifiers.

3 Complexity of a single quantifier

In this section we consider the complexity of evaluating X-pp sentences when X is a singleton, i.e., we have at our disposal only a single quantifier.

Theorem 3.
(1) $\{1\}$-CSP($\mathcal{B}$) (i.e. CSP($\mathcal{B}$)) is in NP for all $\mathcal{B}$. For each $n \geq 2$, there exists a template $\mathcal{B}_n$ of size n such that $\{1\}$-CSP($\mathcal{B}_n$) is NP-complete.
(2) $\{|B|\}$-CSP($\mathcal{B}$) is in L for all $\mathcal{B}$.
(3) For each $n \geq 3$, there exists a template $\mathcal{B}_n$ of size n such that $\{j\}$-CSP($\mathcal{B}_n$) is Pspace-complete for all $1 < j < n$.

Proof. Parts (1) and (2) are well-known (see [23], resp. [20]). For (3), let $\mathcal{B}_{\text{NAE}}$ be the Boolean structure on domain $\{0, 1\}$ with a single ternary not-all-equal relation $R_{\text{NAE}} := \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$. To show Pspace-completeness, we reduce from QCSP($\mathcal{B}_{\text{NAE}}$), the *quantified not-all-equal-3-satisfiability* (see [23]). We distinguish two cases.

Case I: $j \leq \lfloor n/2 \rfloor$. Define $\mathcal{B}_n$ on domain $\{0, \ldots, n-1\}$ with a single unary relation U and a single ternary relation R. Set $U := \{0, \ldots, j-1\}$ and set

$\quad R := \{0, \ldots, n-1\}^3 \setminus \{(a,b,c) : a,b,c \text{ either all odd or all even}\}.$

The even numbers will play the role of false 0 and odd numbers the role of true 1.

Case II: $j > \lfloor n/2 \rfloor$. Define $\mathcal{B}_n$ on domain $\{0, \ldots, n-1\}$ with a single unary relation U and a single ternary relation R. Set $U := \{0, \ldots, j-1\}$ and set

$R := \{0, \ldots, n-1\}^3 \setminus \{(a,b,c) : a,b,c \leq n-j \text{ and either all odd or all even}\}.$

In this case even numbers $\leq n-j$ play the role of false 0 and odd numbers $\leq n-j$ play the role of true 1. The $j-1$ numbers $n-j+1, \ldots, n-1$ are somehow universal and will always satisfy any R relation.

The reduction we use is the same for Cases I and II. We reduce $\mathrm{QCSP}(\mathcal{B}_{\mathrm{NAE}})$ to $\{j\}\text{-CSP}(\mathcal{B}_n)$. Given an input $\Psi := Q_1 x_1 Q_2 x_2 \ldots Q_m x_m \ \psi(x_1, x_2, \ldots, x_m)$ to the former (i.e. each Q_i is $\exists$ or $\forall$) we build an instance Ψ' for the latter. From the outside in, we convert quantifiers $\exists x$ to $\exists^{\geq j} x$. For quantifiers $\forall x$, we convert also to $\exists^{\geq j} x$, but we add the conjunct $U(x)$ to the quantifier-free part ψ.

We claim $\mathcal{B}_{\mathrm{NAE}} \models \Psi$ iff $\mathcal{B}_n \models \Psi'$. For the $\exists$ variables of Ψ, we can see that any j witnesses from the domain B_n for $\exists^{\geq j}$ must include some element playing the role of either false 0 or true 1 (and the other $j-1$ may always be found somewhere). For the $\forall$ variables of Ψ, U forces us to choose both 0 and 1 among the $\exists^{\geq j}$ (and the other $j-2$ will come from $2, \ldots, j-1$). The result follows. $\square$

4 Counting quantifiers on cliques and cycles

4.1 Cliques: proof of Theorem 1

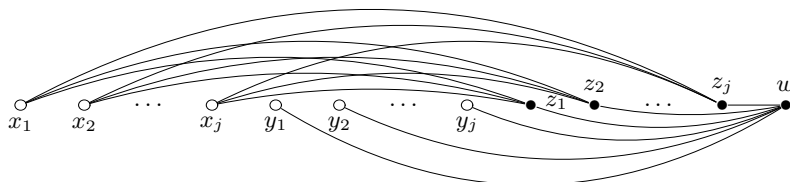Recall that $\mathcal{K}_n$ is the complete irreflexive graph on n vertices.

Fig. 1. The gadget $\mathcal{G}_j$.

Proposition 1. *If $1 < j$, then $\{j\}$-CSP($\mathcal{K}_{2j+1}$) is* Pspace-*complete.*

Proof. By reduction from $\mathrm{QCSP}\big(\mathcal{K}_{\binom{2j+1}{j}}\big)$, *quantified $\binom{2j+1}{j}$-colouring*, which is Pspace-complete by [4]. The key part of our proof involves the gadget $\mathcal{G}_j$, in Figure 1, having vertices $x_1, \ldots, x_j, y_1, \ldots, y_j, z_1, \ldots, z_j, w$ and all possible edges between $\{x_1, \ldots, x_j\}$ and $\{z_1, \ldots, z_j\}$, and between w and $\{y_1, \ldots, y_j, z_1, \ldots, z_j\}$. The left $2j$ vertices represent free variables $x_1, \ldots, x_j, y_1, \ldots, y_j$. Observe that $\exists^{\geq j} z_1, \ldots, z_j, w \ \phi_{\mathcal{G}_j}$ is true on $\mathcal{K}_{2j+1}$ iff $|\{x_1, \ldots, x_j\} \cap \{y_1, \ldots, y_j\}| < j$. If $|\{x_1, \ldots, x_j\}| = |\{y_1, \ldots, y_j\}| = j$, this is equivalent to $\{x_1 \ldots x_j\} \neq \{y_1 \ldots y_j\}$. Thus this gadget will help us to encode the edge relation on $\mathcal{K}_{\binom{2j+1}{j}}$ where we represent vertices by sets $\{a_1, \ldots, a_j\} \subset \{1, \ldots, 2j+1\}$ with $|\{a_1, \ldots, a_j\}| = j$.

Consider an instance Ψ of $\mathrm{QCSP}\big(\mathcal{K}_{\binom{2j+1}{j}}\big)$. We construct the instance Ψ' of $\{j\}\text{-CSP}(\mathcal{K}_{2j+1})$ as follows. From the graph $\mathcal{D}_\psi$, build $\mathcal{D}_{\psi'}$ by transforming each vertex v into an independent set of j vertices $\{v^1, \ldots, v^j\}$, and each edge uv of $\mathcal{D}_\psi$ to an instance of the gadget $\mathcal{G}_j$ in which the $2j$ free variables correspond to $u^1, \ldots, u^j, v^1, \ldots, v^j$. The other variables of the gadget $\{z_1, \ldots, z_j, w\}$ are unique to each edge and are quantified innermost in Ψ' in the order $z_1, \ldots, z_j, w$.

It remains to explain the quantification of the variables of the form $v^1, \ldots, v^j$. We follow the quantifier order of Ψ. Existentially quantified variables $\exists v$ of Ψ are quantified as $\exists^{\geq j} v^1, \ldots, v^j$ in Ψ'. Universally quantified variables $\forall v$ of Ψ are also quantified $\exists^{\geq j} v^1, \ldots, v^j$ in Ψ', but we introduce additional variables $v^{1,1}, \ldots, v^{1,j+1}, \ldots, v^{j,1}, \ldots, v^{j,j+1}$ before $v^1, \ldots, v^j$ in the quantifier order of Ψ', and for each $i \in \{1, \ldots, j\}$, we join $v^{i,1}, \ldots, v^{i,j+1}$ into a clique with v^i.

It is now not difficult to verify that $\mathcal{K}_{\binom{2j+1}{j}} \models \Psi$ iff $\mathcal{K}_{2j+1} \models \Psi'$. $\qquad\square$

Corollary 1. *If* $1 < j < n/2$*, then* $\{j\}\text{-CSP}(\mathcal{K}_n)$ *is* Pspace-*complete.*

Proof. We reduce from $\{j\}\text{-CSP}(\mathcal{K}_{2j+1})$ and appeal to Proposition 1. Given an input Ψ for $\{j\}\text{-CSP}(\mathcal{K}_{2j+1})$, we build an instance Ψ' for $\{j\}\text{-CSP}(\mathcal{K}_n)$ by adding an $(n - 2j - 1)$-clique on new variables, quantified outermost in Ψ', and link by an edge each variable of this clique to every other variable. Adversary chooses $n - 2j - 1$ elements of the domain for this clique, effectively reducing the domain size to $2j + 1$ for the rest. Thus $\mathcal{K}_n \models \Psi'$ iff $\mathcal{K}_{2j+1} \models \Psi$ follows. $\quad\square$

Proposition 2. *If* $1 < j \leq n$*, then* $\{1, j\}\text{-CSP}(\mathcal{K}_n)$ *is* Pspace-*complete.*

Proof. By reduction from $\mathrm{QCSP}(\mathcal{K}_n)$. We simulate existential quantification $\exists v$ by itself, and universal quantification $\forall v$ by the introduction of $(n - j + 1)$ new variables $v^1, \ldots, v^{n-j}$, joined in a clique with v, and quantified by $\exists^{\geq j}$ before v (which is also quantified by $\exists^{\geq j}$). The argument follows as in Proposition 1. $\quad\square$

Define the n-star $\mathcal{K}_{1,n}$ to be the graph on vertices $\{0, 1, \ldots, n\}$ with edges $\{(0, j), (j, 0) \ : \ j \geq 1\}$ where 0 is called the *centre* and the remainder are *leaves.*

Proposition 3. *If* $X \cap \{1, \ldots, \lfloor n/2 \rfloor\} = \emptyset$*, then* $X\text{-CSP}(\mathcal{K}_n)$ *is in* L.

Proof. Instance Ψ of $X\text{-CSP}(\mathcal{K}_n)$ of the form $\exists^{\geq \lambda_1} x_1 \ldots \exists^{\geq \lambda_m} x_m \ \psi(x_1, \ldots, x_m)$ induces the graph $\mathcal{D}_\psi$, which we may consider totally ordered (the order is given left-to-right ascending by the quantifiers). We claim that $\mathcal{K}_n \models \Psi$ iff $\mathcal{D}_\psi$ does not contain as a subgraph (not necessarily induced) a $(n - \lambda_i + 1)$-star in which the $n - \lambda_i + 1$ leaves all come before the centre x_i in the ordering.

($\Rightarrow$) If $\mathcal{D}_\psi$ contains such a star, then Ψ is a no-instance, as we may give a winning strategy for Adversary in the game $\mathscr{G}(\Psi, \mathcal{K}_n)$. Adversary should choose distinct values for the variables associated with the $n - \lambda_i + 1$ leaves of the star (can always be done as each of the possible quantifiers assert existence of $> n/2$ elements and $n - \lambda_i < n/2$), whereupon there is no possibility for Prover to choose λ_i witnesses to the variable x_i associated with the centre.

($\Leftarrow$) If $\mathcal{D}_\psi$ does not contain such a star, then we give the following winning strategy for Prover in the game $\mathscr{G}(\Psi, \mathcal{K}_n)$. Whenever a new variable comes up, its corresponding vertex in $\mathcal{D}_\psi$ has $l < n - \lambda_i + 1$ adjacent predecessors, which were

answered with $b_1, \ldots, b_l$. Prover suggests any set of size λ_i from $B \setminus \{b_1, \ldots, b_l\}$ (which always exists) and the result follows. $\qquad\square$

Proof of Theorem 1. For $n \leq 2$ see [21], and for (ii) see [16]. The remainder of (i) is proved as Proposition 3 while Corollary 1 and Proposition 2 give (iii). $\square$

4.2 Cycles: proof of Theorem 2

Recall that $\mathcal{C}_n$ denotes the irreflexive symmetric cycle on n vertices. We consider $\mathcal{C}_n$ to have vertices $\{0, 1, \ldots, n-1\}$ and edges $\big\{(i,j) \; : \; |i - j| \in \{1, n-1\}\big\}$.

In the forthcoming proof, we use the following elementary observation from additive combinatorics. Let $n \geq 2$, $j \geq 1$, and A, B be sets of integers. Define:

- $A +_n B = \big\{(a+b) \bmod n \mid a \in A, b \in B\big\}$
- $j \times_n A = \underbrace{A +_n \ldots +_n A}_{j \text{ times}}$

Lemma 2. *Let $n \geq 3$ and $2 \leq j < n$. Then*

$$\Big| j \times_n \{-1, +1\} \Big| = \begin{cases} j + 1 & n \text{ is odd} \\ \min\{j+1, n/2\} & n \text{ is even} \end{cases}$$

$$\Big| n \times_n \{-1, +1\} \Big| = \Big| n \times_n \{-2, 0, +2\} \Big| = \begin{cases} n & n \text{ is odd} \\ n/2 & n \text{ is even} \end{cases}$$

Proposition 4. *If $n \geq 3$, then X-$CSP(\mathcal{C}_n)$ is in L if $n = 4$, or $1 \notin X$, or n is even and $X \cap \{2, 3 \ldots, n/2\} = \emptyset$,*

Proof. Let Ψ be an instance of X-CSP$(\mathcal{C}_n)$. Recall that $\mathcal{D}_\psi$ is the graph corresponding to the quantifier-free part of Ψ. We write $x \prec y$ if x, y are vertices of $\mathcal{D}_\psi$ (i.e., variables of ψ) such that x is quantified before y in Ψ. For an edge xy of $\mathcal{D}_\psi$ where $x \prec y$, we say that x is a *predecessor* of y. Note that a vertex can have several predecessors. The following restricts the yes-instances of X-CSP$(\mathcal{C}_n)$.

Let x be a vertex of $\mathcal{D}_\psi$ quantified in Ψ by $\exists^{\geq j}$ for some j. If $\mathcal{C}_n \models \Psi$ then

(1a) if $j \geq 3$, then x has no predecessors,

(1b) if n is even and $j > n/2$, then x is the first vertex (w.r.t. $\prec$) of some connected component of $\mathcal{D}_\psi$, and

(1c) if $n \neq 4$ and $j = 2$, then all predecessors of x except for its first predecessor (w.r.t. $\prec$) are quantified by $\exists^{\geq 1}$.

Using these we prove the proposition. First, we consider the case $n = 4$. We show that $\{1, 2, 3, 4\}$-CSP$(\mathcal{C}_4)$ is in L. This will imply that X-CSP$(\mathcal{C}_4)$ is in L for every X. Observe that if $\mathcal{D}_\psi$ contains a vertex x quantified by $\exists^{\geq 3}$ or $\exists^{\geq 4}$, then by (1b) this vertex is the first in its component (if Ψ is not a trivial no-instance). Thus by symmetry replacing its quantification by $\exists^{\geq 1}$ does not change the truth of Ψ. So we may assume that Ψ is an instance of $\{1, 2\}$-CSP$(\mathcal{C}_4)$. We now claim that $\mathcal{C}_4 \models \Psi$ if and only if $\mathcal{D}_\psi$ is bipartite. Clearly, if $\mathcal{D}_\psi$ is not bipartite, it has no homomorphism to $\mathcal{C}_4$ and hence $\mathcal{C}_4 \not\models \Psi$. Conversely, assume that $\mathcal{D}_\psi$ is bipartite with bipartition (A, B). Our strategy for Prover offers the set $\{0, 2\}$ or its subsets for the vertices in A and offers $\{1, 3\}$ or its subsets for every vertex in B. It is easy to verify that this is a winning strategy for Prover. Thus $\mathcal{C}_4 \models \Psi$. The complexity now follows as checking (1b) and checking if a graph is bipartite is in L by [24].

Now, we may assume $n \neq 4$, and next we consider the case $1 \notin X$. If also $2 \notin X$, then by (1a) the graph $\mathcal{D}_\psi$ contains no edges (otherwise Ψ is a trivial

no-instance). This is clearly easy to check in L. Thus $2 \in X$. We claim that if we satisfy (1a) and (1c), then $\mathcal{C}_n \models \Psi$. We provide a winning strategy for Prover. Namely, for a vertex x, if x has no predecessors, offer any set for x. If x has a unique predecessor y for which the value i was chosen, then x is quantified by $\exists^{\geq 2}$ (or $\exists$) by (1a) and we offer $\{i-1, i+1\}$ (mod n) for x. There are no other cases by (1a) and (1c). It follows that Prover always wins with this strategy. In terms of complexity, it suffices to check (1a) and (1c) which is in L.

Finally, suppose that n is even and $X \cap \{2 \ldots n/2\} = \emptyset$. Note that every vertex of $\mathcal{D}_\psi$ is either quantified by $\exists^{\geq 1}$ or by $\exists^{\geq j}$ where $j > n/2$. Thus, using (1b), unless Ψ is a trivial no-instance, we can again replace every $\exists^{\geq j}$ in Ψ by $\exists^{\geq 1}$ without changing the truth of Ψ. Hence, we may assume that Ψ is an instance of $\{1\}$-CSP$(\mathcal{C}_n)$. Thus, as n is even, $\mathcal{C}_n \models \Psi$ if and only if $\mathcal{D}_\psi$ is bipartite. The complexity again follows from [24]. That concludes the proof. $\qquad\square$

Proposition 5. *Let $n \geq 3$. Then X-CSP$(\mathcal{C}_n)$ is Pspace-complete if $n \neq 4$ and $\{1, j\} \subseteq X$: where $j \in \{2, \ldots, n\}$ if n is odd and $j \in \{2, \ldots, n/2\}$ if n is even.*

Proof. By reduction, namely a reduction from QCSP$(\mathcal{C}_n)$ for odd n, and from QCSP$(\mathcal{K}_{n/2})$ for even n. Both problems are known to be Pspace-hard [4].

First, consider the case of odd n. Let Ψ be an instance of QCSP$(\mathcal{C}_n)$. In other words, Ψ is an instance of $\{1, n\}$-CSP$(\mathcal{C}_n)$. Clearly, $j < n$ otherwise we are done.

We modify Ψ by replacing each universally-quantified variable x of Ψ by a path. Namely, let π_x denote the pp-formula that encodes that

$$x_1^1, x_2^1, \ldots, x_{j-1}^1, \ x_1^2, x_2^2, \ldots, x_{j-1}^2, \quad \ldots \quad , x_1^n, x_2^n, \ldots, x_{j-1}^n, \ x$$

is a path in that order (all but x are new variables). We replace $\forall x$ by

$$Q_x = \exists^{\geq j} x_1^1 \ \exists^{\geq j} x_1^2 \ldots \ \exists^{\geq j} x_1^n \ \exists^{\geq j} x \ \exists^{\geq 1} x_2^1 \ldots \exists^{\geq 1} x_{j-1}^1 \ \ldots \ \exists^{\geq 1} x_2^n \ldots \exists^{\geq 1} x_{j-1}^n$$

and append π_x to the quantifier-free part of the formula. Let Ψ' denote the final formula after considering all universally quantified variables. Note that Ψ' is an instance of $\{1, j\}$-CSP$(\mathcal{C}_n)$. We claim that $\mathcal{C}_n \models \Psi$ if and only if $\mathcal{C}_n \models \Psi'$.

To do this, it suffices to show that Ψ' correctly simulates the universal quantifiers of Ψ. Namely, that $\mathcal{C}_n \models Q_x \pi_x$, and for each $\ell \in \{0 \ldots n-1\}$, Adversary has a strategy on $Q_x \pi_x$ that evaluates x to ℓ. (We omit further details.)

It remains to investigate the case of even n. Recall that $n \geq 6$ and $j \leq n/2$. We show a reduction from QCSP$(\mathcal{K}_{n/2})$ to $\{1, j\}$-QCSP$(\mathcal{C}_n)$. The reduction is a variant of the construction from [13] for the problem of retraction to even cycles.

Let Ψ be an instance of QCSP$(\mathcal{K}_{n/2})$, and define $r = (-n/2 - 2) \bmod (j - 1)$. We construct a formula Ψ' from Ψ as follows. First, we modify Ψ by replacing universal quantifiers exactly as in the case of odd n. Namely, we define Q_x and π_x as before, replace each $\forall x$ by Q_x, and append π_x to the quantifier-free part of the formula. After this, we append to the formula a cycle on n vertices $v_0, v_1, \ldots, v_{n-1}$ with a path on $r + 1$ vertices $w_0, w_1, \ldots, w_r$. (See the black vertices in Figure 2.) Then, for each edge xy of $\mathcal{D}_\psi$, we replace $E(x, y)$ in Ψ by the gadget depicted in Figure 2 (consisting of the cartesian product of $\mathcal{C}_n$ and a path on $3n/2$ vertices together with two attached paths on $n/2 - 2$, resp. $r + 1$ vertices). The vertices x and y represent the variables x and y while all
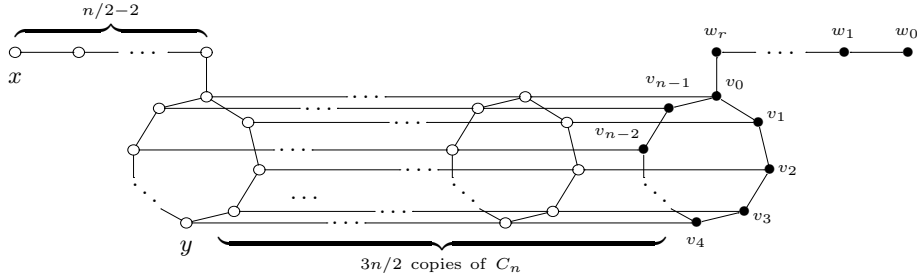
Fig. 2. The gadget for the case of even n where $r = (-n/2 - 2) \bmod (j-1)$.

other white vertices are new variables, and the black vertices are identified with $v_0, \ldots, v_{n-1}, w_0, \ldots, w_r$ introduced in the previous step.

Finally, we prepend the following quantification to the formula:

$$\exists^{\geq 1} w_0 \; \exists^{\geq j} v_{j-r-2} \; \exists^{\geq j} v_{2j-r-3} \ldots \exists^{\geq j} v_{(k \cdot j - r - k - 1)} \ldots \exists^{\geq j} v_{n/2+1}$$

followed by $\exists^{\geq 1}$ quantification of all the remaining variables of the gadgets.

We prove that $\mathcal{K}_{n/2} \models \Psi$ if and only if $\mathcal{C}_n \models \Psi'$. First, we show that Ψ' correctly simulates the universal quantification of Ψ. The argument for this is essentially the same as in the case of odd n. Next, we need to analyse possible assignments to the vertices $v_0, \ldots, v_{n-1}$. There are two cases: either the values chosen for $v_0, \ldots, v_{n-1}$ are all distinct, or not. In the former, we show that Prover can complete the homomorphism to $\mathcal{C}_n$ if and only if $\mathcal{K}_{n/2} \models \Psi$. All other cases are degenerate and have to be addressed separately. (We omit the details.) $\quad\square$

Proof of Theorem 2. The case (i) is proved as Proposition 4, and the case (ii) follows from [16]. Finally, the case (iii) is proved as Proposition 5. $\quad\square$

5 Extensions of the CSP

In this section we consider single-quantifier extensions of the classical $\mathrm{CSP}(\mathcal{B})$, i.e., the evaluation of X-pp sentences, where $X := \{1, j\}$ for some $1 < j \leq |B|$.

5.1 Bipartite graphs

In the case of (irreflexive, undirected) graphs, it is known that $\{1\}\text{-}\mathrm{CSP}(\mathcal{H}) = \mathrm{CSP}(\mathcal{H})$ is in L if $\mathcal{H}$ is bipartite and is NP-complete otherwise [16] (for membership in L, one needs also [24]). It is also known that something similar holds for $\{1, |H|\}\text{-}\mathrm{CSP}(\mathcal{H}) = \mathrm{QCSP}(\mathcal{H})$ – this problem is in L if $\mathcal{H}$ is bipartite and is NP-hard otherwise [21]. Of course, the fact that $\{1, j\}\text{-}\mathrm{CSP}(\mathcal{H})$ is hard on non-bipartite $\mathcal{H}$ is clear, but we will see that it is not always easy on bipartite $\mathcal{H}$.

First, we look at complete bipartite graphs (in a more general statement).

Proposition 6. *Let $\mathcal{K}_{k,l}$ be the complete bipartite graph with partite sets of size k and l. Then $\{1, \ldots, k+l\}\text{-}CSP(\mathcal{K}_{k,l})$ is in* L.

Proof. We reduce to $\mathrm{QCSP}(\mathcal{K}_2^1)$, where $\mathcal{K}_2^1$ indicates $\mathcal{K}_2$ with one vertex named by a constant, say 1. $\mathrm{QCSP}(\mathcal{K}_2^1)$ is equivalent to $\mathrm{QCSP}(\mathcal{K}_2)$ (identify instances of 1 to a single vertex) and both are well-known to be in L (see, e.g., [21]). Let

Ψ be input to $\{1, \ldots, k + l\}$-CSP$(\mathcal{K}_{k,l})$. Produce Ψ' by substituting quantifiers $\exists^{\geq j}$ with $\exists$, if $j \leq \min\{k, l\}$, or with $\forall$, if $j > \max\{k, l\}$. Variables quantified by $\exists^{\geq j}$ for $\min\{k, l\} < j \leq \max\{k, l\}$ should be replaced by the constant 1. It is easy to see that $\mathcal{K}_{k,l} \models \Psi$ iff $\mathcal{K}_2 \models \Psi'$, and the result follows. $\square$

Proposition 7. *For each j, there exists m s.t. $[2^m 1^*]$-CSP$(\mathcal{C}_{2j})$ is NP-complete.*

Proof. Membership in NP follows because m is bounded – one may try all possible evaluations to the $\exists^{\geq 2}$ variables. NP-hardness follows as in the proof of case (*iii*) of Theorem 2, but we are reducing from CSP$(\mathcal{K}_{n/2})$ not QCSP$(\mathcal{K}_{n/2})$. As a consequence, the only instances of $\exists^{\geq 2}$ we need to consider are those used to isolate the cycle $\mathcal{C}_{2j}$ (one may take $m := j + 3$). $\square$

Corollary 2. *If $\mathcal{H}$ is bipartite and (for $j \geq 3$) contains some $\mathcal{C}_{2j}$ but no smaller cycle, then exists m s.t. $[2^m 1^*]$-CSP$(\mathcal{H})$ is NP-complete.*

Proof. Membership and reductions for hardness follow similarly to Proposition 7. The key part is in isolating a copy of the cycle, but we can not do this as easily as before. If d is the diameter of $\mathcal{H}$ (the maximum of the minimal distances between two vertices) then we begin the sentence Ψ' of the reduction with $\exists^{\geq 2} v_1, \ldots, v_{d+1}$, and then, for each $i \in \{1, \ldots, d - j + 1\}$ we add $\exists^{\geq 2} x_i, x_i', \ldots, x'^{\cdots'}$ ($j - 1$ dashes) and join $v_i, \ldots, v_{i+j}, x_i'^{\cdots'}, \ldots, x_i$ in a $2i$-cycle (with $E(x_i, v_i)$ also). For each of these $d - j + 1$ cycles $\mathcal{C}_{2j}$ we build a separate copy of the rest of the reduction. We can not be sure which of these cycles is evaluated truly on some $\mathcal{C}_{2j}$, but at least one of them must be. $\square$

In passing, we note the following simple propositions.

Proposition 8. *If $j \in \{2, \ldots, n - 3\}$ then one may exhibit a bipartite $\mathcal{H}_j$ of size n such that $\{1, j\}$-CSP$(\mathcal{H}_j)$ is Pspace-complete.*

Proof. The case $j = 2$ follows from Theorem 2; assume $j \geq 3$. Take the graph $\mathcal{C}_6$ and construct $\mathcal{H}_j$ as follows. Augment $\mathcal{C}_6$ with $j - 3$ independent vertices each with an edge to vertices 1, 3 and 5 of $\mathcal{C}_6$. Apply the proof of Theorem 2 with $\mathcal{H}_j$. $\square$

Proposition 9. *Let $\mathcal{H}$ be bipartite with largest partition in a connected component of size $< j$. Then $\{1, j\}$-CSP$(\mathcal{H})$ is in L.*

Proof. We will consider an input Ψ to $\{1, j\}$-CSP$(\mathcal{H})$ of the form $Q_1 x_1 \, Q_2 x_2 \ldots Q_m x_m \, \psi(x_1, x_2, \ldots, x_m)$. An instance of an $\exists^{\geq j}$ variable is called *trivial* if it has neither a path to another (distinct) $\exists^{\geq j}$ variable, nor a path to an $\exists$ variable that precedes it in the natural order on $\mathcal{D}_\psi$. The key observation here is that any non-trivial $\exists^{\geq j}$ variable *must* be evaluated on more than one partition of a connected component. If in Ψ there is a non-trivial $\exists^{\geq j}$ variable, then Ψ must be a no-instance (as $\exists^{\geq j}$s must be evaluated on more than one partition of a connected component, and a path can not be both even and odd in length). All other instances are readily seen to be satisfiable. Detecting if Ψ contains a non-trivial $\exists^{\geq j}$ variable is in L by [24], and the result follows. $\square$

We note that Proposition 8 is tight, namely in that $\{1, j\}$-CSP$(\mathcal{H})$ is in L if $j \in \{1, |H| - 2, |H| - 1, |H|\}$. (For space restrictions, we omit the details.)

Proposition 10. *If $\mathcal{H}$ is bipartite and contains $\mathcal{C}_4$, then $\Psi \in \{1,2\}$-CSP$(\mathcal{C}_4)$ iff the underlying graph $\mathcal{D}_\psi$ of Ψ is bipartite. In particular, $\{1,2\}$-CSP$(\mathcal{H})$ is in L.*

Proof. Necessity is clear; sufficiency follows by the canonical evaluation of $\exists^{\geq 1}$ and $\exists^{\geq 2}$ on a fixed copy of $\mathcal{C}_4$ in $\mathcal{H}$. Membership in L follows from [24]. $\quad\square$

Proposition 11. *Let $\mathcal{H}$ be a forest, then $[2^m 1^*]$-CSP$(\mathcal{H})$ is in P for all m.*

Proof. We evaluate each of the m variables bound by $\exists^{\geq 2}$ to all possible pairs, and what we obtain in each case is an instance of CSP$(\mathcal{H}')$ where $\mathcal{H}'$ is an expansion of $\mathcal{H}$ by some constants, i.e., equivalent to the retraction problem. It is known that Ret$(\mathcal{H})$ is in P for all forests $\mathcal{H}$ [14], and the result follows. $\quad\square$

We bring together some previous results into a classification theorem.

Theorem 4. *Let $\mathcal{H}$ be a graph. Then*

– *$[2^m 1^*]$-CSP$(\mathcal{H}) \in$ P for all m, if $\mathcal{H}$ is a forest or a bipartite graph containing $\mathcal{C}_4$*
– *$[2^m 1^*]$-CSP$(\mathcal{H})$ is NP-complete from some m, if otherwise.*

Proof. Membership of NP follows since m is fixed. The cases in P follow from Propositions 11 and 10. Hardness for non-bipartite graphs follows from [16] and for the remaining bipartite graphs it follows from Corollary 2. $\quad\square$

6 The complexity of QCSP$(\mathcal{C}_4^*)$

Let $\mathcal{C}_4^*$ be the reflexive 4-cycle. The complexities of Ret$(\mathcal{C}_6)$ and Ret$(\mathcal{C}_4^*)$ are both hard (NP-complete) [13, 12], and retraction is recognised to be a "cousin" of QCSP (see [2]). The problem QCSP$(\mathcal{C}_6)$ is known to be in L (see [21]), but the complexity of QCSP$(\mathcal{C}_4^*)$ was hitherto unknown. Perhaps surprisingly, we show that is is markedly different from that of QCSP$(\mathcal{C}_6)$, being Pspace-complete.

Proposition 12. $\{1,2,3,4\}$-CSP$(\mathcal{C}_4^*)$ *is* Pspace-*complete.*

Corollary 3. *QCSP$(\mathcal{C}_4^*)$ is* Pspace-*complete.*

The proofs of these claims are based on the hardness of the retraction problem to reflexive cycles [12] and are similar to our proof of the even case of Proposition 5.

While QCSP$(\mathcal{C}_4^*)$ has different complexity from QCSP$(\mathcal{C}_6)$, we remark that the better analog of the retraction complexities is perhaps that $\{1, |C_4^*|\}$-CSP$(\mathcal{C}_4^*)$ and $\{1, |C_6|/2\}$-CSP$(\mathcal{C}_6)$ **do** have the same complexities (recall the reductions to Ret$(\mathcal{C}_4^*)$ and Ret$(\mathcal{C}_6)$ involved CSP$(\mathcal{K}_{|C_4^*|})$ and CSP$(\mathcal{K}_{|C_6|/2})$, respectively.

7 Conclusion

We have taken first important steps to understanding the complexity of CSPs with counting quantifiers, even though several interesting questions have resisted solution. We would like to close the paper with some open problems.

In Section 4.1, the case $n = 2j$ remains. When $j = 1$ and $n = 2$, we have $\{1\}$-CSP$(\mathcal{K}_2)$=CSP$(\mathcal{K}_2)$ which is in L by [24]. For higher j, the question of the complexity of $\{j\}$-CSP$(\mathcal{K}_{2j})$ is both challenging and open.

We would like to prove the following more natural variants of Theorem 4, whose involved combinatorics appear to be much harder.

Conjecture 1. Let $\mathcal{H}$ be a graph. Then

- $[2^*1^*]$-CSP$(\mathcal{H})$ is in P, if $\mathcal{H}$ is a forest or a bipartite graph containing $\mathcal{C}_4$,
- $[2^*1^*]$-CSP$(\mathcal{H})$ is NP-hard, if otherwise.

Conjecture 2. Let $\mathcal{H}$ be a graph. Then

- $\{1,2\}$-CSP$(\mathcal{H})$ is in P, if $\mathcal{H}$ is a forest or a bipartite graph containing $\mathcal{C}_4$,
- $\{1,2\}$-CSP$(\mathcal{H})$ is NP-hard, if otherwise.

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