3-colouring AT-free graphs in polynomial time

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Abstract. Determining the complexity of the colouring problem on AT-free graphs is one of long-standing open problems in algorithmic graph theory. One of the reasons behind this is that AT-free graphs are not necessarily perfect unlike many popular subclasses of AT-free graphs such as interval graphs or co-comparability graphs. In this paper, we resolve the smallest open case of this problem, and present a polynomial time algorithm for 3-colouring of AT-free graphs.

1 Introduction

The colouring problem is one of the most studied problems on graphs. It is also one of the first problems known to be NP-hard [4]. In other words, it is unlikely that there is a polynomial time algorithm for solving this problem. This is true even in very special cases such as in planar graphs, line graphs, graphs of bounded degree or if the number of colours k is fixed and at least three. On the other hand, for $k = 2$ the problem is polynomially solvable, as is the general problem for many structured classes of graphs such as interval graphs, chordal graphs, comparability graphs, and more generally for perfect graphs [5]. In these cases, the special structure of the classes in question allows for polynomial algorithms.

We study the colouring problem in the class of *AT-free graphs*, i.e., graphs with no *asteroidal triple* (a triple of vertices such that between any two vertices of the triple there is a path disjoint from the closed neighbourhood of the third vertex). This class is a generalization of interval and co-comparability graphs as well as some non-perfect graphs such as the complements of triangle-free graphs. Unlike other standard optimization problems such as the independent set or the clique problem whose complexity on AT-free graphs is known (the former is solvable in polynomial time, while the latter is NP-hard [2]), the complexity of colouring is not known on AT-free graphs.

As a first step towards resolving this, we propose in this paper a polynomial time algorithm for the 3-colouring problem on AT-free graphs. In particular, we prove the following theorem.

Theorem 1. *There is an $O(n^4)$ time algorithm to decide, given an AT-free graph G, if G is 3-colourable and to construct a 3-colouring of G if it exists.*

We show this in three stages:

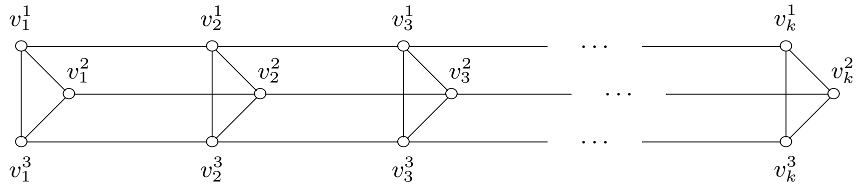

Fig. 1. The triangular strip of order k.

(1) we reduce the problem to AT-free graphs with no induced diamonds,
(2) we show how to decompose every AT-free graph with no induced diamond and no K_4 into triangular strips (see Figure 1) using stable cutsets, and
(3) we prove that we are allowed to contract minimal stable separators without changing the answer to the problem.

This reduces the problem to graphs whose blocks are triangular strips which are all clearly 3-colourable. If at any stage we encounter K_4, a clique on four vertices, we declare the graph not 3-colourable. A sketch of an algorithm resulting from this is presented below as Algorithm 1. (Note that $G/_S$ denotes the graph we obtain from G by contracting the set of vertices S into a single vertex.)

Input: An AT-free graph G.
Output: A 3-colouring of G or "G is not 3-colourable".

1 **if** G contains K_4 **then**
2   **return** "G is not 3-colourable"
  / Now G contains no K_4 */*
3 **if** G contains adjacent vertices u, v with $|N(u) \cap N(v)| \geq 2$ **then**
4   Recursively find a 3-colouring of $G/_{N(u)\cap N(v)}$.
  / Now G contains no induced diamond and no K_4 */*
5 **if** G contains a cutpoint or is disconnected **then**
6   Recursively colour all blocks of G.
  / Now G is 2-connected and contains no induced diamond and no K_4 */*
7 **if** G contains a minimal stable separator S with $|S| \geq 2$ **then**
8   Recursively find a 3-colouring of $G/_S$.
  / Now G is a triangular strip */*
9 Construct a 3-colouring of G.

Algorithm 1: Find a 3-colouring of an AT-free graph

Note that, in the above algorithm, once Line 5 is reached, the graph is guaranteed to be 3-colourable. This follows from the fact that AT-free graphs with no induced diamond and no K_4 are 3-colourable (we prove this as Theorem 3). Hence, to obtain a decision algorithm, one can modify the procedure in Algorithm 1 to announce that "G is 3-colourable" once Line 5 is reached.

We remark that, for instance, in graphs with no induced path on five [11] or six [10] vertices the problem of 3-colouring is also known to be polynomially solvable even though to compute the chromatic number is NP-hard in both classes [8]. (In fact, in the former case, the k-colouring problem for every fixed k is polynomially solvable [7].) The main reason behind this is that in these cases we are able to reduce the problem of 3-colouring to an instance of 2-satisfiability which is solvable in polynomial time. Our approach for AT-free graphs differs from this in that it instead focuses on efficient decomposition of AT-free graphs to graphs for which 3-colouring can be decided in polynomial time.

In the following sections, we examine the main ingredients to the proof of correctness of our algorithm which are summarized in the following two theorems.

Theorem 2. *Let G be an AT-free graph with at least three vertices and no induced diamond or K_4. Then either*

(i) G is a triangular strip, or
(ii) G contains a stable cutset.

Theorem 3. *Every AT-free graph G with no induced diamond and no K_4 is 3-colourable. Moreover, if G contains a minimal stable separator S, then there is a 3-colouring of G in which all vertices of S have the same colour.*

In the final section, we explain implementation details needed to guarantee the running time $O(n^4)$.

2 Notation

In this paper, a graph is always simple, undirected, and loopless.

For a vertex v of a graph G, we denote by $N_G(v)$ the set of vertices adjacent to v in G, and write $N_G[v] = N_G(v) \cup \{v\}$. We drop the index G and write $N(v)$ and $N[v]$ whenever it is clear from context. For $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced by X, and write $G - X$ for the subgraph of G induced by $V(G) \setminus X$. A set $X \subseteq V(G)$ is *stable*, if $G[X]$ contains no edges, and X is a *clique*, if $G[X]$ has all possible edges. As usual, K_n denotes the complete graph on n vertices, and *diamond* is the (unique) graph on four vertices with five edges.

We say that a path P of a graph G is *missed* by a vertex x if no vertex of P is adjacent to x. A triple of vertices x, y, z of a graph G is *asteroidal* if between any two vertices of the triple there exists a path missed by the third vertex.

We write $G/_S$ for the graph we obtain from G by *contracting* (i.e., identifying) all vertices in S into a single vertex. That is,

$$V(G/_S) = (V(G) \setminus S) \cup \{s\} \text{ where } s \notin V(G),$$
$$E(G/_S) = \Big\{ xy \in E(G) \;\Big|\; x, y \notin S \Big\} \cup \Big\{ sy \;\Big|\; xy \in E(G) \wedge x \in S \wedge y \notin S \Big\}.$$

A set $S \subseteq V(G)$ *disconnects* vertices a, b in G if a and b are in different connected components of $G - S$. We say that S is a *cutset* of G if it disconnects some vertices a, b. We say that S is a *minimal separator* of G if there exist vertices a and b such that S disconnects a and b, but no proper subset of S disconnects them. (Note that a minimal separator is not necessarily an inclusion-wise minimal cutset; for example, consider a 4-cycle with a pendant vertex.)

For a complete terminology, see [13].

3 Removing diamonds

In this section, we explain how to reduce the problem to the case of AT-free graphs with no induced diamonds. We show that if we have a diamond in G, i.e., we have adjacent vertices u, v such that their common neighbourhood contains two non-adjacent vertices, then we can contract any maximal set S of pair-wise non-adjacent common neighbours of u, v and the resulting graph remains AT-free. It is also 3-colourable if and only if G is, since in any 3-colouring of G all vertices of S must have the same colour. Thus we show the following theorem.

Theorem 4. *If u, v are adjacent vertices of an AT-free graph G and S is a maximal stable set in $N(u) \cap N(v)$, then $G/_S$ is AT-free. Moverover, G is 3-colourable if and only if $G/_S$ is.*

To prove this, we use a more general tool that allows contracting specific sets in G without creating asteroidal triples. We say that a set $S \subseteq V(G)$ is *externally connected* in G, if for each $x \in V(G)$ with $N[x] \cap S = \emptyset$, the set S is contained in a (single) connected component of $G - N[x]$.

Lemma 1. *Let G be an AT-free graph and $S \subseteq V(G)$ be an externally connected set in G. Then $G/_S$ is AT-free.*

Proof. Let s denote the vertex of $G/_S$ to which we contracted the vertices of S, and suppose that $G/_S$ contains an asteroidal triple $\{x, y, z\}$. Let P be a path in $G/_S$ from y to z missed by x. If s is not on P, then P is also a path in G, and if $x = s$, then every vertex of S misses P in G. So, suppose that s belongs to P and is not one of the endpoints of P. Let u, v be the two neighbours of s on P. By the construction of $G/_S$, there exist vertices $a, b \in S$, such that $ua, vb \in E(G)$. Since $xs \notin E(G/_S)$, we have $N_G[x] \cap S = \emptyset$, and since S is externally connected in G, we conclude that a and b, and hence, u and v are in the same connected component of $G - N_G[x]$. Consequently, there is a path in G from y and z missed by x. Similarly, if s is one of the endpoints of P, say $y = s$, then we conclude that there exists a path in G missed by x between z and each vertex of S.

This proves that if s is not one of x, y, z, then x, y, z is an asteroidal triple of G, and otherwise, if say $x = s$, then a, y, z is an asteroidal triple of G for every $a \in S$, a contradiction. $\square$

From this lemma, we immediately obtain a proof of Theorem 4 as well as two other corollaries that we make use of later.

Lemma 2. *If G is AT-free and $G[S]$ is connected, then $G/_S$ is AT-free.*

Proof. By Lemma 1, it suffices to show that S is externally connected. This is obvious, since S induces a connected subgraph in $G - N[x]$ for $N[x] \cap S = \emptyset$. $\square$

Lemma 3. *If G is AT-free and S is a minimal separator, then $G/_S$ is AT-free.*

Proof. Again, we show that S is externally connected. Consider $x \in V(G)$ with $N[x] \cap S = \emptyset$, and let K denote the connected component of $G - S$ that contains x. Since S is a minimal separator, there exists a connected component K' of $G - S$ different from K such that each vertex of S has a neighbour in K'. Therefore, $G[K' \cup S]$ is connected, and so, S belongs to a connected component of $G - N[x]$, since clearly $N[x] \cap (K' \cup S) = \emptyset$. This proves that S is externally connected, and so the claim follows from Lemma 1. $\square$

Proof of Theorem 4. For the first part of the claim, it again suffices to prove that S is externally connected. Consider $x \in V(G)$ with $N[x] \cap S = \emptyset$. Therefore x is not adjacent to any vertex of S implying that $S \cup \{x\}$ is a stable set. By the maximality of S, x is non-adjacent to one of u, v. By symmetry, suppose that $xu \notin E(G)$. Then $S \cup \{u\}$ is in a connected component of $G - N[x]$ since $G[S \cup \{u\}]$ is connected. So, we conclude that S is indeed externally connected.

For the second part of the claim, let s be the vertex of $G/_S$ to which we contracted S. If we have a 3-colouring of $G/_S$, then we can extend this colouring of G by colouring all vertices of S with the colour of s. Conversely, if we have a 3-colouring of G, then u, v have different colours in this colouring, and hence, all vertices of S must have the same colour. So, we use this colour for s and colour all other vertices of $G/_S$ as in G. This clearly yields a 3-colouring of $G/_S$. $\square$

4 Structural decomposition

In this section, we prove Theorem 2 asserting that every AT-free graph with no induced diamond and no K_4 decomposes into triangular strips via stable cutsets.

The *triangular strip* of order k is the graph formed by taking three disjoint paths $P^1 = v_1^1, v_2^1, \ldots, v_k^1$, $P^2 = v_1^2, v_2^2, \ldots, v_k^2$, $P^3 = v_1^3, v_2^3, \ldots, v_k^3$ and adding a triangle on v_i^1, v_i^2, v_i^3 for each $i = 1 \ldots k$. In other words, the triangular strip of order k is the cartesian product of an induced path on k vertices and a triangle. (See Figure 1 for an illustration.) We say that the triangles v_1^1, v_1^2, v_1^3 and v_k^1, v_k^2, v_k^3 of the triangular strip of order k are the *end-triangles*.

We say that G is a triangular strip if G is isomorphic to the triangular strip of order k for some k. Clearly, every triangular strip is AT-free and contains no induced diamond or K_4. Note that triangular strips have no stable cutsets; in other words, the two conditions of Theorem 2 are mutually exclusive.

Let G be an AT-free graph with $|V(G)| \geq 3$, no induced diamond, and no K_4. First, we observe that it suffices to prove Theorem 2 for 2-connected graphs G, since any cutpoint (and also the empty set) forms a stable cutset of G. Since G contains no diamond and no K_4, no two triangles of G share an edge. We show that, actually, no two triangles share a vertex provided G is 2-connected.

Lemma 4. *Let G be a 2-connected AT-free graph with no induced diamond and no K_4. Then every vertex of G is in at most one triangle.*

Proof. Let x be a vertex that belongs to two different triangles, namely, a triangle x, a, b and a triangle x, u, v. Clearly, $\{u, v\} \cap \{a, b\} = \emptyset$, since otherwise x, u, v, a, b induces a diamond or a K_4 in G. For the same reason, there is no edge between vertices u, v and a, b.

Since G is 2-connected, $G-x$ is connected, and hence, there is a path between vertices u, v and a, b in $G - x$. Let P be a shortest such path. Without loss of generality, P is a path from u to a. Let y be the second vertex on P (after u). Clearly, $yv \notin E(G)$ and $xy \notin E(G)$, since otherwise y, v, u, x induces a diamond or a K_4. Also, y is not adjacent to one of a, b, since otherwise y, a, b, x induces a diamond. In particular, $yb \notin E(G)$, since otherwise $ya \notin E(G)$ and u, y, b is a shorter path from u, v to a, b which contradicts the minimality of P.

We now show that $\{y, v, b\}$ is an asteroidal triple of G. Indeed, v, x, b is a path from v to b missed by y, and v, u, y is a path from v to y missed by b. Finally, $P' = P \setminus \{u\} \cup \{b\}$ is a path from y to b missed by v, since $vy \notin E(G)$ and v has no neighbour on $P \setminus \{u, y\}$ by the minimality of P. $\square$

By the above lemma, every vertex of G is in at most one triangle. If some vertex v is in no triangle, then $N(v)$ is a stable cutset of G unless $V(G) = N[v]$ in which case v is a cutpoint because G is assumed to have at least three vertices.

This implies that we may assume that every vertex of G is in exactly one triangle. In other words, G contains a triangular strip (of order 1). We show that by taking a maximal such strip, we either get the whole graph G or find a stable cutset in G, thus proving Theorem 2. To simplify the proof of this, we need the following technical lemma.

Lemma 5. *Let G be an AT-free graph with no induced diamond and no K_4, and let H be a (not necessarily induced) subgraph of G isomorphic to a triangular strip. Then (i) H is induced in G, and (ii) no vertex of H has a neighbour in $G - V(H)$ except for the vertices of the end-triangles of H.*

Proof. Let v^i_j for $i = 1, 2, 3$ and $j = 1 \ldots k$ for some k be the vertices of H. Suppose that H is not induced in G, and let $v^i_j v^{i'}_{j'}$ be an edge not in H such that $j < j'$ and $j' - j$ is smallest possible. By symmetry, we may assume that $i' = 1$, and $i \in \{1, 2\}$. Clearly, $j \neq j'$.

First, we observe that v^i_j is not adjacent to $v^2_{j'}$ and $v^3_{j'}$, since otherwise $v^i_j, v^1_{j'}, v^2_{j'}, v^3_{j'}$ induces a diamond or K_4 in G. This also implies $j' - j \geq 2$. By the choice of j, j' and the fact that $j' - j \geq 2$, we conclude that v^i_j is not adjacent to $v^3_{j+1}, v^3_{j+2}, \ldots, v^3_{j'}$, and v^3_{j+1} is not adjacent to $v^1_{j'}$. By the same token, $v^2_{j'}$ is not adjacent to v^1_{j+1} and v^3_{j+1}. We show that $\{v^i_j, v^3_{j+1}, v^2_{j'}\}$ is an asteroidal triple in G. Indeed, the path $v^i_j, v^1_{j'}, v^2_{j'}$ is missed by v^3_{j+1}, and the path $v^3_{j+1}, v^3_{j+2}, \ldots, v^3_{j'}, v^2_{j'}$ is missed by v^i_j. Finally, $v^2_{j'}$ is non-adjacent to at least one of v^1_j, v^3_j otherwise $v^1_j, v^2_j, v^3_j, v^2_{j'}$ induces a diamond or K_4 in G. If $v^3_j v^2_{j'} \notin E(G)$, then the path v_j^i, v_j^3, v_{j+1}^3 is missed by $v_{j'}^2$. Otherwise, $v_j^1 v_{j'}^2 \notin E(G)$ in which case $v_j^i, v_j^1, v_{j+1}^1, v_{j+1}^3$ is a path (or walk) missed by $v_{j'}^2$. This proves (i).

For (ii), let $x \notin V(H)$ be a vertex adjacent to v_j^i for some $i \in \{1,2,3\}$ and $j \in \{2 \ldots k-1\}$. By symmetry, we may assume $i = 1$. Clearly, x is non-adjacent to both v_j^2 and v_j^3, otherwise x, v_j^1, v_j^2, v_j^3 induces a diamond or K_4 in G. First, suppose that x is also adjacent to v_{j+1}^1. Then x is non-adjacent to both v_{j+1}^2 and v_{j+1}^3, since otherwise $x, v_{j+1}^1, v_{j+1}^2, v_{j+1}^3$ induces a diamond or a K_4. But now $\{x, v_j^3, v_{j+1}^2\}$ is an asteroidal triple in G. Indeed, the path x, v_j^1, v_j^3 is missed by v_{j+1}^2, the path $v_j^3, v_{j+1}^3, v_{j+1}^2$ is missed by x, and the path v_{j+1}^2, v_{j+1}^1, x is missed by v_j^3. So, we may assume $xv_{j+1}^1 \notin E(G)$, and by symmetry, also $xv_{j-1}^1 \notin E(G)$.

Suppose that x is non-adjacent to both v_{j+1}^2 and v_{j-1}^3. Then $\{x, v_{j+1}^2, v_{j-1}^3\}$ is an asteroidal triple in G. Indeed, the path $x, v_j^1, v_j^2, v_{j+1}^2$ is missed by v_{j-1}^3, the path $v_{j+1}^2, v_j^2, v_j^3, v_{j-1}^3$ is missed by x, and the path $v_{j-1}^3, v_j^3, v_j^1, x$ is missed by v_{j+1}^2. So x has at least one neighbour among v_{j+1}^2, v_{j-1}^3. By the same token, x has at least one neighbour among v_{j+1}^3, v_{j-1}^2. Clearly, x cannot be adjacent to both v_{j+1}^2, v_{j+1}^3 or to both v_{j-1}^2, v_{j-1}^3, since we get an induced diamond in G on $x, v_{j+1}^1, v_{j+1}^2, v_{j+1}^3$, or on $x, v_{j-1}^1, v_{j-1}^2, v_{j-1}^3$. So, by symmetry, we may assume that x is adjacent to v_{j-1}^2 and v_{j+1}^2 and non-adjacent to v_{j-1}^3 and v_{j+1}^3. But then $\{x, v_{j-1}^3, v_{j+1}^3\}$ is an asteroidal triple in G. Indeed, the path x, v_{j+1}^2, v_{j+1}^3 is missed by v_{j-1}^3, the path $v_{j+1}^3, v_j^3, v_{j-1}^3$ is missed by x, and the path v_{j-1}^3, v_{j-1}^2, x is missed by v_{j+1}^3. That concludes the proof of (ii). $\square$

Now, we are finally ready to prove Theorem 2.

Proof of Theorem 2. As remarked in the discussion above, we may assume that G is 2-connected, and contains a triangle (triangular strip).

Let H be the largest triangular strip induced in G. If $V(H) = V(G)$, then G is a triangular strip, and we are done. Otherwise, there exists a vertex $v \in V(G) \setminus V(H)$ adjacent to a vertex of H. By Lemma 5, v is adjacent to a vertex c of an end-triangle of H; let a, b be the other two vertices of this triangle. Clearly, $va, vb \notin E(G)$ since otherwise v, a, b, c induces a diamond or K_4 in G.

First, we note that $N(b) \setminus \{a\}$ and $N(a) \setminus \{b\}$ are stable sets, since otherwise a or b is in two triangles which is not possible by Lemma 4. Also, the sets $N(a) \cap N(v)$ and $N(b) \cap N(v)$ are both stable sets of G, because otherwise we have an induced diamond in G. Moreover, we prove that there are no edges between the two sets. Suppose otherwise, and let $u \in N(a) \cap N(v)$ and $w \in N(b) \cap N(v)$ be adjacent. We observe that if $u \in V(H)$, then u belongs to a triangle in H and the triangle u, v, w. But these triangles are different since $v \notin V(H)$ contradicting Lemma 4. Hence, $u \notin V(H)$, and by the same token, $w \notin V(H)$. So, $G[V(H) \cup \{u, v, w\}]$ contains a spanning triangular strip which is, by Lemma 5, induced in G. This, however, contradicts the maximality of H.

Now, suppose that there are no edges between $N(b) \setminus \{a\}$ and $N(a) \cap N(v)$. In other words, $S = (N(b) \setminus \{a\}) \cup (N(a) \cap N(v))$ is a stable set. We show that S is a stable cutset of G separating a from v. Suppose otherwise, and let P be a shortest path in $G - S$ from a to v. Let z be the second vertex on P (after a). Since $N(a) \cap N(v) \subseteq S$, we conclude $zv \notin E(G)$. Also, $zc \notin E(G)$, since otherwise a, b, c, z induces a diamond or K_4 in G. By the same token, $zb \notin E(G)$. We show that $\{b, v, z\}$ is an asteroidal triple in G which will yield a contradiction. Indeed, the path v, c, b is missed by z, the path z, a, b is missed by v, and $P \setminus \{a\}$ is a path from z to v missed by b, since all neighbours of b except for a are in S.

Similarly, if there are no edges between $N(a) \setminus \{b\}$ and $N(b) \cap N(v)$, we conclude that G contains a stable cutset. So, we may assume that there exists $x \in N(a) \cap N(v)$ adjacent to some $y \in N(b) \setminus \{a\}$, and $x' \in N(b) \cap N(v)$ adjacent to some $y' \in N(a) \setminus \{b\}$. We show that this is impossible. Clearly, $y, y' \notin N(v)$, since there are no edges between $N(a) \cap N(v)$ and $N(b) \cap N(v)$. Also, y is not adjacent to any of a, c, x', since otherwise b is in two triangles which is impossible by Lemma 4. By the same token, y' is not adjacent to any of b, c, x. We show that G contains an asteroidal triple. Suppose that $yy' \notin E(G)$. Then $\{y, y', v\}$ is an asteroidal triple of G. Indeed, the path y, x, v is missed by y', the path y', x', v is missed by y, and the path y, b, a, y' is missed by v. So, $yy' \in E(G)$ in which case $\{x, c, x'\}$ is an asteroidal triple of G. Indeed, the path x, a, c is missed by x', the path c, b, x' is missed by x, and the path x, y, y', x' is missed by c.

That concludes the proof. $\square$

5 Proof of Theorem 3

The proof is by induction on $|V(G)|$. Let G be an AT-free graph with no induced diamond and no K_4. If G has at most 2 vertices, the claim is trivially satisfied.

Therefore, we may assume $|V(G)| \geq 3$. If G has a stable cutset, then by (possibly) removing some of its vertices, we can find a minimal stable separator in G. So, if G has no minimal stable separator, then it must be, by Theorem 2, a triangular strip with vertices v_j^i for $i = 1, 2, 3$ and $j = 1 \ldots k$ for some k. We obtain a 3-colouring of G by assigning each v_j^i the colour $((i + j) \bmod 3) + 1$.

So, we may assume that G contains a minimal stable separator S. If S is empty, then G is disconnected and we obtain a 3-colouring of G by independently 3-colouring its connected components by induction. If S has one element, then G has a cutpoint and we obtain a 3-colouring of G by 3-colouring its blocks by induction, and permuting the colours in blocks so that they match on cutpoints. In both cases, all vertices in S have the same colour. So, we may assume $|S| \geq 2$.

To prove the claim, it now suffices to show that for every connected component K of $G - S$, there exists a 3-colouring of $G[K \cup S]$ in which all vertices of S have the same colour.

Let K be a (fixed) connected component of $G - S$. Let S' denote the set of vertices of S with at least one neighbour in K. If $S' \neq S$, then S' is a minimal stable separator in $G' = G - (S \setminus S')$. By induction, there exists a 3-colouring of G' in which all vertices of S' have the same colour. By restricting this colouring to $K \cup S$ and colouring the vertices of $S \setminus S'$ with the common colour of the vertices of S', we obtain the required 3-colouring. (Note that the vertices of $S \setminus S'$ are isolated in $G[K \cup S]$.)

Hence, we may assume that every vertex of S has a neighbour in K. Further, since S is a minimal separator, there exists a connected component K' of $G-S$ different from K such that each vertex of S also has a neighbour in K'. Let G' denote the graph $G[K \cup K' \cup S]/_{K'}$, and let x be the vertex of G' to which we contracted K'. By Lemma 2, G' is AT-free. Moreover, G' contains no induced diamond or K_4, since any such subgraph is either in G, or contains x, but x belongs to no triangle of G'. Also, S is a minimal separator in G'. Hence, if G' has fewer vertices than G, then, by induction, there exists a 3-colouring of G' in which all vertices of S have the same colour. This colouring when restricted to the vertices $K \cup S$ yields the required 3-colouring.

It follows that we may assume that $G - S$ has exactly two connected components, one of which is K, the other consists of a single vertex x, and every vertex of S is adjacent to x and has a neighbour in K.

Now, let S^* be a smallest subset of S such that $\bigcup_{u\in S^*} N(u) = \bigcup_{u\in S} N(u)$. Suppose that S^* contains three distinct vertices u, v, w. By the minimality of S^*, there exist vertices u', v', w' such that $u' \in N(u) \setminus (N(v) \cup N(w))$, $v' \in N(v) \setminus (N(u) \cup N(w))$ and $w' \in N(w) \setminus (N(u) \cup N(v))$. Clearly, $u', v', w' \in K$ since S is a stable set and u, v, w are adjacent to x. Suppose that $u'v' \notin E(G)$. Then $\{u', v', x\}$ is an asteroidal triple of G. Indeed, the path u', u, x is missed by v', the path v', v, x is missed by u', and x misses any path in K between u' and v'. Hence, we must conclude $u'v' \in E(G)$, and by the same token, $u'w', v'w' \in E(G)$. However, then $\{u, v, w\}$ is an asteroidal triple in G. Indeed, the path u, u', v', v is missed by w, the path u, u', w', w is missed by v, and the path v, v', w', w is missed by u. We therefore conclude $|S^*| \leq 2$.

If $S^* \neq S$, then we consider the graph $G' = G - (S \setminus S^*)$. Clearly, S^* is a minimal separator in G', and therefore, there exists, by induction, a 3-colouring of G' in which all vertices of S^* have the same colour. We extend this colouring to G by colouring all vertices of $S \setminus S^*$ with the common colour of the vertices of S^*. By the definition of S^*, this yields the required 3-colouring.

Hence, we may assume that S consists of exactly two vertices u and v. We let $A = N(u) \setminus N(v)$, $B = (N(u) \cap N(v)) \setminus \{x\}$, and $C = N(v) \setminus N(u)$. By the minimality of S^*, we have $A \neq \emptyset$ and $C \neq \emptyset$. Moreover, each vertex $a \in A$ is adjacent to every vertex $c \in C$, since otherwise $\{a, c, x\}$ is an asteroidal triple of G. Indeed, the path a, u, x is missed by c, the path c, v, x is missed by a, and any path between a and c in K is missed by x. Furthermore, A is a stable set in G, since any adjacent $a, a' \in A$ yield an induced diamond u, a, a', c for any vertex $c \in C$. By the same token, C is a stable set. Finally, B is a stable set, since any adjacent $b, b' \in B$ yield an induced diamond b, b', u, v in G.

Suppose that there is $b \in B$ adjacent to some $a \in A$, and let $c \in C$. We show that $N(b) \setminus \{u\} \subseteq N(c)$. Suppose otherwise and let $w \in N(b) \setminus \{u\}$ be such that $wc \notin E(G)$. Clearly, $bc \notin E(G)$ since otherwise u, a, b, c induces a diamond in G. Also, $wu, wa \notin E(G)$ since otherwise w, a, b, u induces a diamond or K_4 in G. Finally, $wx \notin E(G)$, since w is not one of u, v and x is only adjacent to u, v. It follows that $\{w, x, c\}$ is an asteroidal triple in G. Indeed, the path w, b, a, c is missed by x, the path c, a, u, x is missed by w, and the path x, u, b, w is missed by c. This proves that $N(b) \setminus \{u\} \subseteq N(c)$. Now, by induction, there exists a 3-colouring of $G - b$ in which u and v have the same colour. We extend this colouring to G by assigning b the colour of c. Clearly, b and u have different colours in this colouring, since otherwise c, u, v have the same colour, impossible since cv is an edge in $G - b$. Also, b has colour different from its other neighbours, since $N(b) \setminus \{u\} \subseteq N(c)$. So, this gives the required 3-colouring.

It follows that we may assume that there are no edges between A and B. In other words, $A \cup B$ is a stable set. It is also a minimal separator of $G[K \cup S]$ separating u from v, since u, a, c, v and u, b, v are paths from u to v for each $a \in A$, $b \in B$, and $c \in C$. So, by induction, there is a 3-colouring of $G[K \cup S]$ in which all vertices of $A \cup B$ have the same colour. If $B \neq \emptyset$, then by recolouring u with the colour of v, we obtain the required 3-colouring. So, we conclude $B = \emptyset$.

Now, suppose that there is $a \in A$ with $N(a) \subseteq C \cup \{u\}$. If $|A| \geq 2$, then u, v is a minimal separator in $G - a$, and hence, there exists, by induction, a 3-colouring of $G - a$ in which u, v have the same colour. Recall that $N(a') \supseteq C \cup \{u\}$ for all $a' \in A$. So, by assigning a the colour of any vertex in $A \setminus \{a\}$, we obtain the required 3-colouring. Hence, we may assume $A = \{a\}$, and we observe that C is a minimal separator of $G - u$ separating a from v. By induction, there is a 3-colouring of $G - u$ in which all vertices of C have the same colour. To obtain the required 3-colouring, we colour u with the colour of v and recolour a with the colour different from the colour of u and the common colour of the vertices of C.

Hence, we may assume that there exists $w \in N(a) \setminus (C \cup \{u\})$ for some $a \in A$, and by symmetry, we also have $z \in N(c) \setminus (A \cup \{v\})$ for some $c \in C$. We show that G contains an asteroidal triple which will lead to a contradiction. Clearly, w and z are both different from and non-adjacent to all of u, v, x. If $wc \in E(G)$ or $wz \in E(G)$, then $\{w, u, v\}$ is an asteroidal triple. Indeed, the path w, a, u is missed by v, the path w, c, v or w, z, c, v is missed by u, and the path u, x, v is missed by w. So, $wc, wz \notin E(G)$, and by symmetry, $za \notin E(G)$. But now $\{z, w, x\}$ is an asteroidal triple. Indeed, the path w, a, c, z is missed by x, the path w, a, u, x is missed by z, and the path z, c, v, x is missed by w.

That concludes the proof. $\square$

6 The algorithm

In this section, we finally prove Theorem 1 by showing that Algorithm 1 is correct and can be implemented to run in time $O(n^4)$.

The correctness follows easily from Theorems 2, 3, 4 and Lemma 3. We therefore focus on the details of $O(n^4)$ implementation.

First, we note that the complexity is easily seen to be polynomial, since all the tests in Algorithm 1 are polynomial including the test in Line 7 which follows from [1]. Also, the algorithm makes at most n recursive calls, since each call reduces the graph by at least one vertex. So, to get the running time $O(n^4)$, it suffices to explain how to implement each test in time $O(n^3)$.

The test in Line 3 has a straightforward implementation of complexity $O(n^3)$. Similarly, the test in Line 5 can be clearly implemented in time $O(n^2)$ by the standard algorithm of [12]. Also, we can recognize and colour triangular strips in Line 9 in time $O(n^2)$ by iteratively removing triangles on vertices of degree 3.

For the test in Line 1, we do the following. If we execute Line 1 for the first time, we test if G contains a K_4 by trying all possible 4-sets of vertices in time $O(n^4)$. If we reach Line 1 by recursion to $G/_S$ and s is the vertex of $G/_S$ to which we contracted S, then we only test if the neighbourhood of s in $G/_S$ contains a triangle. This requires only $O(n^3)$ time, and it is enough to verify that $G/_S$ contains no K_4, since before contracting S, the graph G was assumed to contain no K_4 (because we have reached at least Line 3 before the recursive call).

Therefore, it remains to show that we can implement the test in Line 7 in time $O(n^3)$. This requires a little more work. As remarked earlier, if the neighbourhood of some vertex x is a stable set, then either $N(x)$ is a stable cutset of G, or x is a cutpoint of G, or $|V(G)| \leq 2$. It turns out that a partial converse of this is also true as shown in the following lemma.

Lemma 6. *If S is a minimal stable separator of an AT-free graph G, then there exists a vertex $x \in V(G)$ with $N(x) \supseteq S$.*

Proof. Let S be a counterexample to the claim and let S^* be a smallest subset of S for which there is no vertex x with $N(x) \supseteq S^*$. Clearly, $|S^*| \geq 2$.

First, suppose that $|S^*| = 2$. Hence, $S^* = \{x, y\}$ for some vertices x, y. Since S is a minimal separator, there are connected components K, K' of $G - S$ such that each vertex of S has a neighbour in both K and K'. In particular, we have $u \in N(x) \cap K$ and $v \in N(x) \cap K'$. Clearly, $uy, vy \notin E(G)$ by the minimality of S^*. This implies that $\{u, v, y\}$ is an asteroidal triple of G. Indeed, the path u, x, v is missed by y. Also, since S is a minimal separator, we have a path P in $G[K \cup \{y\}]$ from y to u, and a path P' in $G[K' \cup \{y\}]$ from y to v. Clearly, P is missed by v and P' is missed by u.

We therefore conclude $|S^*| \geq 3$, and let x, y, z be any three vertices of S^*. By the minimality of S^*, there exist vertices a, b, c such that $N(a) \supseteq S^* \setminus \{x\}$, $N(b) \supseteq S^* \setminus \{y\}$, and $N(c) \supseteq S^* \setminus \{z\}$, and also $ax, by, cz \notin E(G)$. Therefore, $\{x, y, z\}$ is an asteroidal triple of G. Indeed, the path x, c, y is missed by z, the path y, a, z is missed by x, and the path z, b, x is missed by y.

That concludes the proof. $\square$

We further need the following observation which is easy to check.

Observation 7. *If S is a stable cutset of a connected graph G, and $S' \supseteq S$ is a stable set, then S' is also a stable cutset of G.* $\square$

Now, if G contains a minimal stable separator S, then all we have to do, by Lemma 6, is to find a vertex x with $N(x) \supseteq S$. Since G is assumed to have no induced diamond and no K_4 in Line 7, $N(x)$ contains, by Lemma 4, at most two maximal stable sets one of which contains S. But then this set is also a stable cutset of G by Observation 7. So, to find a minimal stable separator in G, we test for each vertex x if $N(x)$ or $N(x) \setminus \{u\}$ or $N(x) \setminus \{v\}$ is a stable cutset where uv (if exists) is the unique edge in $G[N(x)]$. This can be accomplished in time

$O(n^2)$ by a standard graph search, so altogether $O(n^3)$ for all x. If a stable cutset S is found, we reduce it to a minimal stable separator by iteratively removing vertices of S and testing if the resulting set is a still a cutset. Again, $O(n^3)$ time, since it suffices to test each vertex only once.

7 Conclusion

In this paper, we have shown how to find in polynomial time a 3-colouring of a given AT-free graph if one exists. To this end, we used a nice structural decomposition of AT-free graphs without diamonds. Note that similar structural results are also known for other restrictions of AT-free graphs [3, 6].

Finally, after submitting the paper for review, we learned that Haiko Müller et al. announced that for every fixed k, the k-colouring problem on AT-free graphs is solvable in polynomial time [9]. Their result is yet to be published.

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