Maximum Flow Problems III.

Review: $G = (V, E)$; (s,t)-cut $\delta(A)$; edge capacity $u : E \to \mathbb{R}_{\geq 0}$; (s,t)-flow $x : E \to \mathbb{R}_{\geq 0}$

Goal: *Maximum Flow Problem*

$$\begin{aligned}
\text{Maximize} \quad f_x(t) &= \sum_{\substack{w\in V\\ wt\in E}} x_{wt} - \sum_{\substack{w\in V\\ tw\in E}} x_{tw} \\
\text{subject to} \quad f_x(v) &= \sum_{\substack{w\in V\\ wv\in E}} x_{wv} - \sum_{\substack{w\in V\\ vw\in E}} x_{vw} = 0 \qquad \forall v \in V \setminus \{s,t\} \\
& \qquad\qquad 0 \leq x_e \leq u_e \qquad \forall e \in E
\end{aligned}$$

1 Cuts

Recall $\delta(A) = \{vw \mid v \in A, w \in \overline{A}\}$, and (s,t)-cut $\delta(A)$ if $s \in A$ and $t \in \overline{A} = V \setminus A$

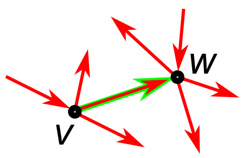
Theorem 1. *Every (s,t)-cut $\delta(A)$ and every (s,t)-flow x satisfy:*

$$\underbrace{\sum_{e\in\delta(A)} x_e - \sum_{e\in\delta(\overline{A})} x_e}_{\substack{\textit{flow across the cut } \delta(A)\\ \text{denoted by } x(\delta(A))}} = f_x(t)$$

Proof. x is a flow $\Rightarrow f_x(v) = 0$ for all $v \in V \setminus \{s,t\}$. Summing up over all $v \in \overline{A} \setminus \{t\}$:

$$\sum_{v\in\overline{A}} f_x(v) = f_x(t) + \overbrace{\sum_{v\in\overline{A}\setminus\{t\}} f_x(v)}^{=0} = f_x(t) \qquad \text{Recall: } f_x(v) = \overbrace{\sum_{\substack{w\in V\\ wv\in E}} x_{wv}}^{\text{incoming}} - \overbrace{\sum_{\substack{w\in V\\ vw\in E}} x_{vw}}^{\text{outgoing}}$$

Consider $e = vw \in E$



$+x_{vw}$ contribution to $f_x(w)$

$-x_{vw}$ contribution to $f_x(v)$

Contribution of vw to the left-hand-side (LHS)

$v \in A$	$w \in A$	none because LHS sums-up $f_x(v)$ for $v \in \overline{A}$	
$v \in A$	$w \in \overline{A}$	$+x_{vw}$ from $f_x(w)$	$e = vw \in \delta(A)$
$v \in \overline{A}$	$w \in A$	$-x_{vw}$ from $f_x(v)$	$e = vw \in \delta(\overline{A})$
$v \in \overline{A}$	$w \in \overline{A}$	$\underbrace{+x_{vw}}_{\text{from } f_x(v)} + \underbrace{-x_{vw}}_{\text{from } f_x(w)} = 0$	

$\square$

Corollary 1. *Every (s,t)-cut $\delta(A)$ and every feasible (s,t)-flow x satisfy:*

$$f_x(t) \leq \sum_{e\in\delta(A)} u_e$$

Proof. x is a feasible flow $\Rightarrow 0 \le x_e \le u_e$ for $\forall e \in E$. Thus, by Theorem 1 $\Rightarrow$

$$f_x(t) = \sum_{e\in\delta(A)} \underbrace{x_e}_{\le u_e} - \sum_{e\in\delta(\overline{A})} \underbrace{x_e}_{\ge 0} \le \underbrace{\sum_{e\in\delta(A)} u_e}_{\substack{\textit{capacity of}\\ \textit{the cut } \delta(A)\\ \text{denoted by } u(\delta(A))}}$$

$\square$

i.e., the value of a feasible flow is at most the capacity of a cut.

Theorem 2. *(Max-Flow Min-Cut Theorem)* [Ford-Fulkerson 1956], [Kotzig 1956]
The maximum value of a feasible (s,t)-flow is equal to the minimum capacity of an (s,t)-cut. If all capacities are integral, then there exists an integral maximum feasible flow.

2 Flow augmentation

Let x be a feasible (s,t)-flow of value k (for instance, $x = 0$ is a feasible flow of value 0)

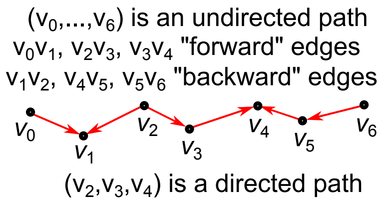
For an st-path $P = (v_0, \ldots, v_m)$ define <u>*x-width of P*</u> $= \min_{i\in\{1\ldots m\}} u_{v_{i-1}v_i} - x_{v_{i-1}v_i}$

If P is a path of x-width $\varepsilon > 0$, then $\forall i$ increase $x_{v_{i-1}v_i}$ by $\varepsilon \Rightarrow$ a feasible flow of value $k+\varepsilon$

(... just like in the proof of the flow-paths theorem. . .)

we may get stuck before reaching the maximum flow $\Rightarrow$ need to allow more general paths

Idea: use also backward edges



directed path = path (as defined before)

undirected path = a sequence $(v_0, \ldots, v_m)$ where v_i distinct and for all $i \in \{1 \ldots m\}$ either $v_{i-1}v_i \in E$ ("forward" edge) or $v_iv_{i-1} \in E$ ("backward" edge)

<u>*x-width*</u> of an undirected path $(v_0, \ldots, v_m) = \min_{i\in\{1\ldots m\}} \begin{cases} u_{v_{i-1}v_i} - x_{v_{i-1}v_i} & \text{if } v_{i-1}v_i \in E \\ x_{v_iv_{i-1}} & \text{if } v_iv_{i-1} \in E \end{cases}$

<u>*x-increasing path*</u> = undirected path of positive x-width

<u>*x-augmenting path*</u> = x-increasing path from s to t

If $P = (v_0, \ldots, v_m)$ is an x-augmenting path of width $\varepsilon > 0$, then $\forall i \in \{1 \ldots m\}$

$$\left.\begin{array}{r} \text{if } v_{i-1}v_i \in E, \text{ increase } x_{v_{i-1}v_i} \text{ by } \varepsilon, \\ v_iv_{i-1} \in E, \text{ decrease } x_{v_iv_{i-1}} \text{ by } \varepsilon. \end{array}\right\} \Rightarrow \text{a feasible flow of value } k+\varepsilon$$

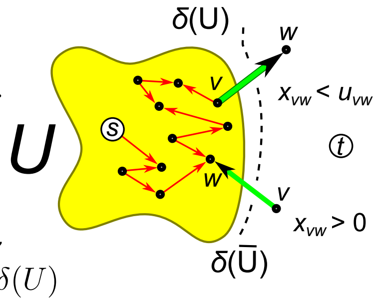
No x-augmenting path $\Rightarrow$ maximum flow (we now prove)

<u>*Proof of Max-Flow Min-Cut Theorem.*</u> Let x be a feasible flow of maximum falue.

Let $U = \{z \mid \exists$ an x-increasing path from s to $z\}$. Note that $s \in U$.

If $t \in U$, then $\exists$ an x-augmenting path $\Rightarrow x$ is not maximum flow, a contradiction.

So $s \in U$ and $t \in \overline{U} \Rightarrow \delta(U)$ is an (s,t)-cut. Moreover,



- every $e = vw \in \delta(U)$ satisfies $u_e - x_e = 0$, otherwise $w \in U$.
- every $e = vw \in \delta(\overline{U})$ satisfies $x_e = 0$, otherwise $v \in U$.

$$\underbrace{f_x(t)}_{\text{value of } x} = \sum_{e\in\delta(U)} \underbrace{x_e}_{=u_e} - \sum_{e\in\delta(\overline{U})} \underbrace{x_e}_{=0} = \sum_{e\in\delta(U)} u_e = \underbrace{u(\delta(U))}_{\text{capacity of } \delta(U)}$$

The value of x is equal to the capacity of $\delta(U)$. By Corollary 1, the value of a feasible (s,t)-flow is at most the capacity of an (s,t)-cut $\Rightarrow \delta(U)$ is a minimum cut.

Integral capacities $\Rightarrow$ integral widths of augmenting paths $\Rightarrow$ integral flow. □

3 Closing remarks

Notice the similarity of the above proof with that of the theorem about cuts and the existence of an st-path. This is no coincidence, as we shall see, and this correspondence will allow us to reduce the problem of finding augmenting paths to simple (s,t)-connectivity question on an auxiliary graph.

Advance note: similar situation occurs with the minimum-cost flow problem which reduces to maximum flow and iterations of shortest path question in an auxiliary graph with general weights (Bellman-Ford); this phenomenon is more generally captured by the so-called Primal-Dual method and is related to Linear Programming (LP) formulations of these problems...