Maximum Flow Problems II.

Review: (directed) graph $G = (V, E)$, st-path, edge capacity $u : E \to \mathbb{R}_{\geq 0}$

$G = (V, E)$, edges in E are ordered pairs from $V \times V$

path = sequence $(v_0, v_1, \ldots, v_m)$ where v_i distinct and $v_{i-1}v_i \in E$ for $i \in \{1 \ldots m\}$

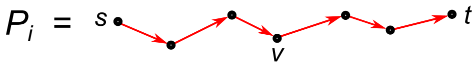
st-path = a path $(v_0, v_1, \ldots, v_m)$ with $v_0 = s$ and $v_m = t$

write $e \in P$ where e is an edge and $P = (v_0, \ldots, v_m)$ is a path
and say "the path P goes through (contains) e" if $e = v_{i-1}v_i$ for some $i \in \{1 \ldots m\}$

edge capacity $u : E \to \mathbb{R}_{\geq 0}$, capacitated graph/network $G = (V, E, u)$

Goal: $(P_1, \ldots, P_k)$ collection of st-paths

for $v \in V \setminus \{s, t\}$ and path P_i containing v, exactly <u>one</u> edge coming into v and <u>one</u> edge going out of v



$\Rightarrow$ #of paths P_i coming into v = #of paths P_i going out of v

Idea: instead of looking for paths directly we only count, for each edge e, the number of paths going through e $\rightsquigarrow$ we call this a <u>*flow*</u>

Formally:

(s, t)-*flow* or just *flow* = a function $x : E \to \mathbb{R}_{\geq 0}$ satisfying $\forall v \in V \setminus \{s, t\}$

$$\underbrace{\overbrace{\sum_{\substack{w \in V \\ wv \in E}} x_{wv}}^{\substack{\text{amount} \\ \text{coming } \underline{\text{into}}\ v}} - \overbrace{\sum_{\substack{w \in V \\ vw \in E}} x_{vw}}^{\substack{\text{amount going} \\ \underline{\text{out of}}\ v}}}_{\substack{\textit{net flow (excess)} \text{ at } v \\ \text{denoted by } f_x(v)}} = 0 \qquad \text{(Flow Conservation Law)}$$

- i.e., x is a flow if $f_x(v) = 0$ for all v except s and t.
- if x assigns only integral values (values from $\mathbb{Z}$) $\Rightarrow$ x is an *integral flow*

s a *source* and t a *target*

value of a flow x = the excess $f_x(t)$ at t (equal to $-f_x(s)$ by the conservation law)

feasible (s, t)-*flow* = an (s, t)-flow that respects capacities, i.e.,

$$0 \leq x_e \leq u_e \qquad \forall e \in E$$

Theorem 1. *The following statements are equivalent.*

(i) There exists a collection $(P_1, \ldots, P_k)$ of st-paths such that for each edge $e \in E$, the number of paths P_i containing e is at most u_e (in symbols, $|\{i \mid P_i \ni e\}| \leq u_e$).

(ii) There exists a feasible integral (s, t)-flow of value k.

Proof. ($\Rightarrow$) define $x_e = |\{i \mid P_i \ni e\}|$ for all $e \in E$, and note $0 \le x_e \le u_e$ by our assumption $\Rightarrow$ x is a feasible integral (s,t)-flow of value k.

($\Leftarrow$) let x be feasible integral (s,t)-flow of value k with smallest $\sum_{e\in E} x_e$. Recall:

$$f_x(v) = \sum_{\substack{w\in V\\ wv\in E}} x_{wv} - \sum_{\substack{w\in V\\ vw\in E}} x_{vw}, \qquad k = f_x(t) = -f_x(s), \quad f_x(v) = 0 \text{ for } v \in V \setminus \{s,t\}$$

Assume $k \ge 1$ (o/w done). We find an st-path $(v_0, \ldots, v_m)$ with $x_{v_{i-1}v_i} > 0 \;\; \forall i \in \{1 \ldots m\}$

Initially let $v_0 = s$ and since $k = -f_x(s) > 0 \Rightarrow \exists\, v_1$ with $x_{sv_1} > 0$.

Assume we have constructed $v_0, v_1, \ldots, v_j$ where $j \ge 1 \Rightarrow$ we find v_{j+1} or done.

- If $v_j = t$, then done $(m := j)$.
- If $v_j = s$, then $\forall i$ decrease $x_{v_{i-1}v_i}$ by 1 (recall $x_{v_{i-1}v_i} > 0$ and integral) $\Rightarrow$ feasible integral (s,t)-flow x with smaller $\sum_{e\in E} x_e$, a contradiction.
- Thus $v_j \in V \setminus \{s,t\}$ and $f_x(v_j) = 0$. Since $x_{v_{j-1}v_j} > 0 \Rightarrow \exists v_{j+1}$ with $x_{v_jv_{j+1}} > 0$.

$\Rightarrow$ add $(v_0, \ldots, v_m)$ to the collection of paths, and $\forall i$ decrease $x_{v_{i-1}v_i}$ by 1 $\Rightarrow$ a feasible integral (s,t)-flow of value $k-1$, repeat.

Maximum Flow Problem:

$$\begin{aligned} \text{Maximize} \;\; & f_x(t) = \sum_{\substack{w\in V\\ wt\in E}} x_{wt} - \sum_{\substack{w\in V\\ tw\in E}} x_{tw} \\ \text{subject to} \;\; & f_x(v) = \sum_{\substack{w\in V\\ wv\in E}} x_{wv} - \sum_{\substack{w\in V\\ vw\in E}} x_{vw} = 0 \qquad \forall v \in V \setminus \{s,t\} \\ & \qquad\qquad\qquad\qquad 0 \le x_e \le u_e \qquad \forall e \in E \end{aligned}$$