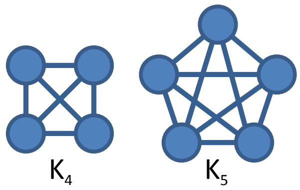
CS 137 - Graph Theory - Lectures 4-5
February 21, 2012

(further reading Rosen K. H.: *Discrete Mathematics and its Applications*, 5th ed., chapters 8.7, 8.8)

1.1. Summary

- Bipartite graphs
- Colouring vertices and edges
- Planar graphs

1.2. Graph substructures



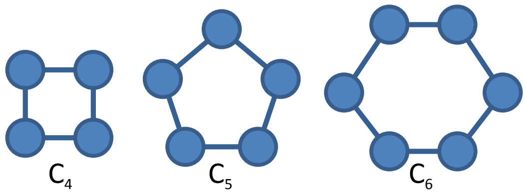
subgraph = G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$

independent set of G = set of pairwise non-adjacent vertices in G

clique of G = set of pairwise adjacent vertices in G

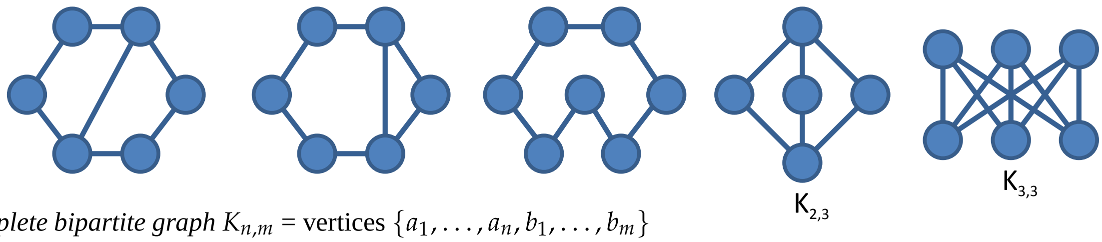
complete graph K_n

cycle C_n



2. Bipartite graphs

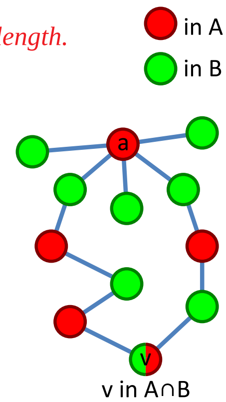
bipartite graph = vertex set can be partitioned into two independent sets



complete bipartite graph $K_{n,m}$ = vertices $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$

edges $\{\{a_i, b_j\} \mid i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$

Theorem 1. *A graph G is a bipartite graph if and only if it does not contain a cycle of odd length.*



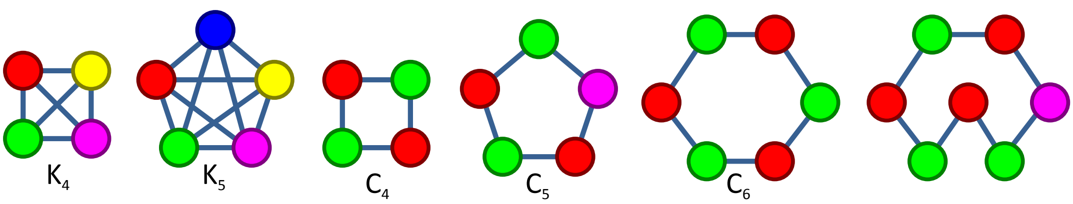
Proof. We may assume that G is connected (why?). Pick a vertex a and put it in A. Then repeatedly pick a vertex v in A and put its neighbours in B, or pick a vertex in B and put its neighbours in A. If a vertex is put both in A and in B (for the first time), we find an odd cycle. If this never happens, then the sets A, B form a partition of the vertices of G into two independent sets; i.e. G is a bipartite graph. $\square$

The proof suggests a notion of "colouring"... we used two colours for vertices in such a way that no two vertices of the same colour are adjacent... (the two colours represent the two independent sets we seek)

This can be generalized as follows.

3. Colouring

A *colouring* or a *vertex-colouring* of a graph G assigns colours to vertices so that no two adjacent vertices have the same colour. Smallest number of colours needed to colour G is the *chromatic number* of G, denoted by $\chi(G)$.



Example: If G is bipartite, assign 1 to each vertex in one independent set and 2 to each vertex in the other independent set. This constitutes a colouring using 2 colours.

Let G be a graph on n vertices. What is $\chi(G)$ if G is

– the complete graph
– the empty graph
– bipartite graph
– a cycle
– a tree

The largest degree of a vertex in G is denoted by $\Delta(G)$ and is called the *maximum degree* in G.

Theorem 2. $\chi(G) \leq \Delta(G) + 1$

"Greedy colouring": fix colours $\{1, \ldots, \Delta(G) + 1\}$ and iteratively colour every vertex using a colour that is not used by its neighbours $\Rightarrow$ always succeeds – there is always at least one available colour.

Notes:

– this bound is tight (why? consider K_n for any n and C_n for odd n)
– $\chi(G)$ can be arbitrarily far from $\Delta(G)$.

It turns out that complete graphs and odd cycles are the only graphs with $\chi(G) = \Delta(G) + 1$.

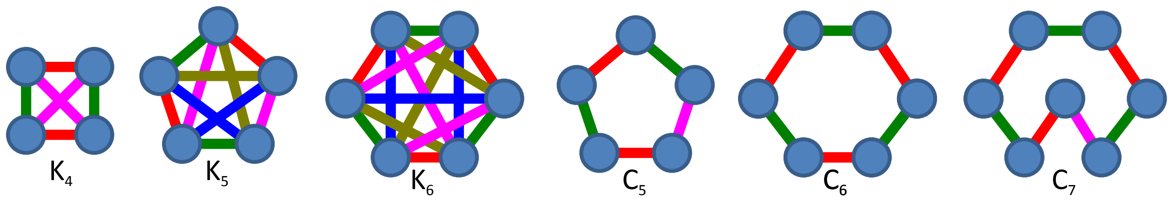
Theorem 3 (Brooks)**.** *$\chi(G) \leq \Delta(G)$ unless G is the complete graph or an odd cycle.*

Applications of colouring: schedulling, wireless communication, job assignment, and many more…

3.1. Edge-colouring

We can similarly colour edges of a graph.

An *edge-colouring* of G assigns colours to edges of G so that no edges that share an endpoint have the same colour. Smallest number of colours needed to edge-colour G is called the *chromatic index* of G, denoted by $\chi'(G)$.



Notes:

– observe that $\chi'(G) \geq \Delta(G)$
– "greedy" colouring gives $\chi'(G) \leq 2\Delta(G) - 1$.

Even better: the chromatic index can only be one of two values.

Theorem 4 (Vizing)**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

In case of bipartite graphs, the chromatic index is always $\Delta(G)$.

Theorem 5 (König)**.** *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

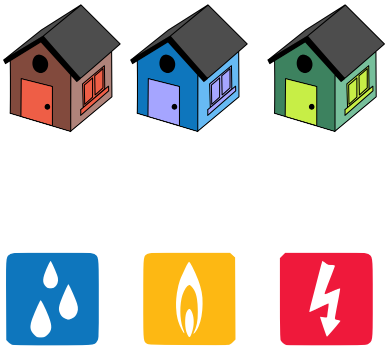
Proof. Fix colours $\{1, \ldots, \Delta(G)\}$ and greedily colour edges as long as possible. Suppose that at some point this process halts before colouring all edges. Let uv be an uncoloured edge.

Let col_u and col_v denote the sets of colours used by edges incident to u and v, respectively. Note that $|col_u| \leq \Delta(G) - 1$ and $|col_v| \leq \Delta(G) - 1$ since uv is not coloured and both u and v are incident to at most $\Delta(G)$ edges. Moreover, $|col_u \cup col_v| = \Delta(G)$ since we do not have a colour available to colour uv. Thus $col_u \setminus col_v \neq \emptyset$ and $col_v \setminus col_u \neq \emptyset$. In other words, some colour, say 1, is used on edges incident to u, but not on edges incident to v, and some other colour, say 2, is used on edges incident to v but not on edges incident to u. Consider the longest walk W in G starting from u that uses only edges coloured 1 and 2. Observe that W is a path (thus is finite), and W is unique. If W contains v, then it terminates in v because v is incident to no edge of colour 1. Adding the edge $\{u, v\}$ to W yields a cycle of odd length, which is impossible, since G is bipartite. So W does not contain v and we can exchange colours 1 and 2 on the edges of the walk W. This allows us to colour uv with colour 1.

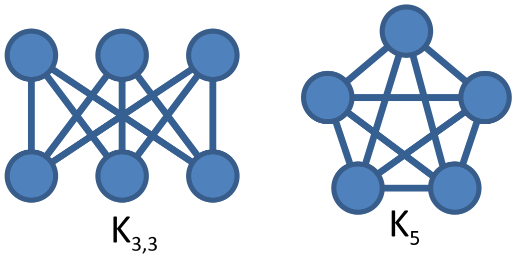
We continue this way and eventually all edges of G are coloured. $\square$

4. Planar graphs

A graph G is said to be *planar* if it can be drawn in the plane in such a way that no two edges cross one another. (We will not define this precisely as this is beyond the scope of this lecture.)



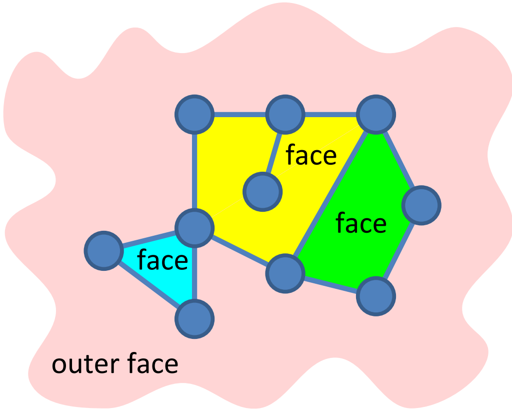
Example with 3 houses/3 utilities



Question: which of these graphs are planar ?

– the complete graph K_n
– the complete bipartite graph $K_{n,m}$
– trees

edges of a planar drawing divide the plane into *faces*



4 faces, 12 edges, 10 vertices

Theorem 6 (Jordan Curve Theorem)**.**
Any simple closed curve C divides the plane into two regions each having C as boundary

(simple means that the curve does not cross itself; such curve is also known as *Jordan curve*)

Theorem 7 (Euler's formula)**.** *Let G be a connected planar graph with n vertices and m edges and consider a planar drawing G having f faces. Then*

$$n - m + f = 2$$

Proof. By induction on the number of edges. If $m \leq n-1$, then $m = n-1$ and G is a tree; the drawing has exactly one face (because G has no cycles). So $f = 1$ and thus $n - m + f = n - (n-1) + 1 = 2$ as required.

So we may assume $m \geq n$ and thus G has a cycle C. We see that the edges of C form a closed curve of the plane. Pick any edge e of C and observe that e lies on the boundary of exactly two faces (the other possibility – that e lies on the boundary of only one face – is exluded by the Jordan Curve Theorem). Construct G' from G by removing e. Removing the edge e from the drawing yields a planar drawing of G' with $f-1$ faces. Since G' has $m-1$ edges (less than G), the inductive hypothesis can be applied to G' which yields $n - (m-1) + (f-1) = 2$. Thus $n - m + f = 2$ as required. $\square$

Theorem 8. *A connected planar graph G with $n \geq 4$ vertices and $m \geq 4$ edges has at most $3n-6$ edges. Moreover, if G has no triangles (cycles of length 3), then it has at most $2n-4$ edges.*

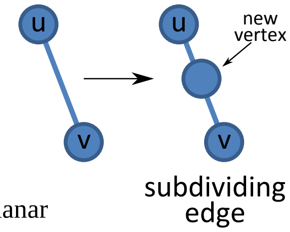
Proof. Consider a planar drawing of G and let f denote the number of faces in the drawing. Observe that every edge appears in at most two faces and every face is bounded by at least 3 edges (since $m \geq 3$). Thus $3f \leq 2m$. By Euler's formula, we have $m = n + f - 2 \leq n + 2m/3 - 2$. So $m/3 \leq n - 2$ and hence $n \leq 3m - 6$.

If G contains no triangles, then every face is bounded by at least 4 edges (since $m \geq 4$), and we have $4f \leq 2m$. This yields $m = n + f - 2 \leq n + m/2 - 2$ and thus $m/2 \leq n - 2$ which is $m \leq 2n - 4$ as required. $\square$

Notes: we can now show that K_5 and $K_{3,3}$ are not planar:

– K_5 has 10 edges but $10 > 3 * 5 - 6 = 9$
– $K_{3,3}$ has 9 edges and no triangle while $9 > 2 * 6 - 4 = 8$

- *subdividing* an edge = replace by a 2-edge path
- a *subdivision* of G = repeatedly subdivide edges of G
  observe that: G is planar if and only if every subdivision of G is also planar
- moreover, if we remove an edge from a planar graph, the resulting graph is also planar
  in other words: G is planar if and only if every subgraph of G is also planar
- put together: every graph that contains a subdivition of K_5 or $K_{3,3}$ as a subgraph is not planar



In fact, the reverse statement is also true as famously proved by Kuratowski in 1930's.

Theorem 9 (Kuratowski's theorem)**.**
A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

4.1. Colouring planar graphs (optional)

The famous "4-colour Theorem" proved by Appel and Haken (after almost 100 years of unsuccessful attempts) states that every planar graph G has a vertex colouring using 4 colours. If G has no triangles, then actually 3 colours are enough as proved by Grötzsch.

Theorem 10 (4-colour Theorem, Appel-Haken 1976)**.** *If G is planar, then $\chi(G) \leq 4$.*

Theorem 11 (Grötzsch's Theorem)**.** *If G is planar and has no triangles, then $\chi(G) \leq 3$.*

The proof of the 4-colour theorem is quite complicated and needs a computer to verify its correctness. A much simpler proof (though still non-trivial) is required to prove that every planar graph has a colouring with 5 colours.

Theorem 12 (5-colour Theorem, Heawood 1890)**.** *If G is planar, then $\chi(G) \leq 5$.*

To show that 6 colours are enough is actually quite easy.

Theorem 13 (6-colour Theorem)**.** *If G is planar, then $\chi(G) \leq 6$.*

Proof. As usual let n and m denote the number of vertices and edges in G. By Theorem 8, $m \leq 3n - 6$ while $2m = \sum_{v \in V(G)} deg(v)$ by Handshaking Theorem. This implies that $\sum_{v \in V(G)} deg(v) \leq 6n - 12 < 6n$. Therefore, G must have a vertex u_1 of degree at most 5. Remove this vertex and repeat; there will again be vertex u_2 of degree at most 5 and we remove it and continue until there are no more vertices. This produces an ordering $u_1, \ldots, u_n$ of all vertices of G in which each u_i has at most 5 neighbours among $u_{i+1}, \ldots, u_n$. To colour G with colours $\{1, 2, \ldots, 6\}$, we simply process the vertices from u_n to u_1, each time assigning to u_i a colour not used by its neighbours among $u_{i+1}, \ldots, u_n$. Since there are at most 5 such neighbours, there is always a colour available for u_i, since we use 6 colours altogether. Consequently, this way we succeed to colour G using 6 colours. □

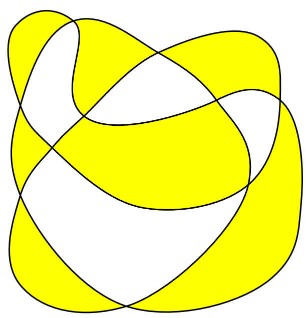
4.2. Doodling

Finally, what about colouring a planar graph with 2 colours?

Consider the following: put your pen down on the paper and draw a curve by moving your pen without lifting it so that you return to the starting point of the curve; colour each region with colours black and white so that no two neighbouring regions use the same colour – is this always possible ?

Answer: Yes ... why it works?

Hint: the dual[1] of a drawing of an Eulerian planar graph is always bipartite



[1] *dual* is a graph whose vertices are the regions of the drawing where two regions are adjacent if and only if they share a boundary – it is also a planar graph (can you see why?)