Incompleteness Revisited

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Review: Theory, Consequences, Theorems

The *theory of a structure* $\mathcal{M}$ is

$$\mathrm{Th}(\mathcal{M}) = \{A : A \text{ is a sentence and } \mathcal{M} \models A\}.$$

The set of *theorems* of Σ is

$$\mathrm{Thm}(\Sigma) = \{A : A \text{ is a sentence and } \Sigma \vdash A\}.$$

The set of *consequences* of Σ is

$$\mathrm{Cn}(\Sigma) = \{A : A \text{ is a sentence and } \Sigma \models A\}.$$

Soundness and completeness imply $\mathrm{Thm}(\Sigma) = \mathrm{Cn}(\Sigma)$.

Axiomatization

A *theory* T is a set of sentences in a fixed language.

A set of sentences Γ *axiomatizes* T if

$$\mathrm{Cn}(\Gamma) = \mathrm{Cn}(T).$$

T is *effectively axiomatizable* if such a Γ can be chosen Turing-recognizable: $\Gamma \in \mathbf{TR}$.

Effective axioms

If Γ is an effective axiomatization of T, then $\mathrm{Thm}(T) \in \mathbf{TR}$.

Section 1

Incompleteness Revisited

Incompleteness v1 (Informal)

No formal system can simultaneously have all four of the following properties:

1. **Completeness**. Every true mathematical statement is provable.
2. **Soundness**. Every provable statement is true.
3. **Verifiable proofs**. An algorithm can decide whether a proposed proof is valid.
4. **Expressivity**. The system can describe enough of mathematics, including the operations of a Turing machine.

Incompleteness v1 (Formal)

Let Σ be a set of sentences, and $\mathcal{N}$ be the standard model (more on this later). Then, the following cannot be satisfied simultaneously.

1. **Completeness**. $\mathcal{N} \models A \implies \Sigma \vdash A$
2. **Soundness**. $\Sigma \vdash A \implies \mathcal{N} \models A$
3. **Recognizable axioms**. $\Sigma \in \mathbf{TR}$
4. **Expressivity**. For each TM M, there is a formula φ_M such that $\mathcal{N} \models \varphi_M$ if and only if M halts on the emptystring.

Proof

By contradiction, suppose that all 4 conditions hold. Let $T = \mathrm{Thm}(\Sigma)$ **Claim 1.** T is decidable **Claim 2.** T is not decidable

Claim 1. T is decidable

Claim 2. T is not decidable

Section 2

Arithmetic

Incompleteness v1 (Formal, stronger)

Instead of assuming the expressivity property, we'll show that expressing arithmetic suffices.

Let Σ be a set of sentences in the language of arithmetic, and $\mathcal{N}$ be the standard model. Then, the following cannot be satisfied simultaneously.

1. **Completeness**. For all A, $\Sigma \vdash A$ or $\Sigma \vdash \neg A$
2. **Soundness**. $\mathcal{N} \models \Sigma$
3. **Recognizable axioms**. $\Sigma \in \mathbf{TR}$

The goal of the rest of this section is to construct φ_M as a sentence in this language. Satisfying the property we needed previously: $\mathcal{N} \models \varphi_M$ if and only if M halts on the empty string.

Note that the completeness and soundness conditions have also changed slightly but are implied by the conditions from the previous statement.

Language of Arithmetic

$$\mathcal{L}_E = \{0, S, <, +, \cdot, E\}.$$

- Constant: 0.
- Unary function: S (successor).
- Binary functions: $+$, $\cdot$, and E (exponentiation).
- Binary relation: $<$.

We may be loose and write x^y for $E(x, y)$, and $x + 1$ for $S(x)$ etc...

Question: What are the terms in this language?

Numerals

Standard Model

The *standard model of arithmetic* $\mathcal{N}$ has domain $\mathbb{N} = \{0, 1, 2, \ldots\}$, with all functions and relations taking their standard meanings. E.g. $+^{\mathcal{N}}$ is actual addition in $\mathbb{N}$.

We take $m^0 = 1$, including $0^0 = 1$.

Section 3

Arithmetization

Goal

Construct φ_M is a sentence in the language of arithmetic

φ_M loosely

One step of a TM

What we need

Essentially we need to show how to represent sequences and operations on those sequences using just numbers!

Sequences over a fixed alphabet

Sequences over an arbitrary numbers

Selected operations