The Complexity of the Comparator Circuit Value Problem

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Joint work with Yuval Filmus and Dai Tri Man Lê

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Outline of the talk

1. Define comparator circuits

2. Define CC as the class of problems reducible to $C_{CV}$ (the comparator circuit value problem)

3. Give interesting complete problems for CC

4. Introduce universal comparator circuits, with resulting robustness properties of CC.

5. Support the conjecture that CC and NC are incomparable using oracle separations.
Comparitor Circuits

- Originally invented for *sorting*, e.g.,
  - Batcher’s $O(\log^2 n)$-depth sorting networks (’68)
  - Ajtai-Komlós-Szemerédi (AKS) $O(\log n)$-depth sorting networks (’83)
- Can also be considered as Boolean circuits.

**Comparator gate**

\[
\begin{align*}
  p & \quad x \quad p \land q \\
  q & \quad y \quad p \lor q
\end{align*}
\]

**Example**

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Comparator Circuit Value (CCV) Problem (decision)

Given a comparator circuit with specified Boolean inputs, determine the output value of a designated wire.
Comparator Circuit Value ($\text{CCv}$) Problem (decision)

Given a comparator circuit with specified Boolean inputs, determine the output value of a designated wire.

Comparator Circuit complexity class

1. $\text{CC} = \{\text{decision problems AC}^0 \text{ many-one-reducible to CCv}\}$
### Comparator Circuit Value (Ccv) Problem (decision)

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### Comparator Circuit complexity class

1. \( \text{CC} = \{ \text{decision problems AC}^0 \ \text{many-one-reducible to Ccv} \} \)
2. Subramanian ['90] Defined CC using log space many-one reducibility
3. We introduce universal comparator circuits and use them to show that the two definitions coincide.
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Given a comparator circuit with specified Boolean inputs, determine the output value of a designated wire.

Comparator Circuit complexity class

1. CC = \{decision problems AC^0 many-one-reducible to CCv\}
2. Subramanian ['90] Defined CC using log space many-one reducibility
3. We introduce universal comparator circuits and use them to show that the two definitions coincide.
4. Subramanian showed \( NL \subseteq CC \subseteq P \)

NL is nondeterministic log space
Recall $\text{NL} \subseteq \text{CC} \subseteq \text{P}$

But also $\text{NL} \subseteq \text{NC} \subseteq \text{P}$
where $\text{NC}$ (the parallel class) contains the problems solvable by uniform polysize polylog depth Boolean circuit families.

$\text{NC}$ contains all context-free languages, and matrix powering and determinants over $\mathbb{Z}, \mathbb{Q}$ etc.
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**Conjecture**

$NC$ and $CC$ are incomparable. (So in particular $CC \not\subseteq P$.)
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**Conjecture**

$\text{NC}$ and $\text{CC}$ are incomparable. (So in particular $\text{CC} \subsetneq \text{P}$.)

Intuitively, we think $\text{CC} \subsetneq \text{P}$ because each of the two comparator gate outputs in a comparator circuit is limited to fan-out one. (More later...)
If our conjecture (that NC and CC are incomparable) is correct then none of these complete problems has an efficient parallel algorithm.
Example Complete Problems for $CC$

- Ccv
- Stable Marriage Problem
- Lexicographical first maximal matching
- Telephone connection problem
- Others . . .

If our conjecture (that NC and CC are incomparable) is correct then none of these complete problems has an efficient parallel algorithm.
Stable Marriage Problem (search version) (Gale-Shapley ’62)

- Given \( n \) men and \( n \) women together with their preference lists.
- Find a stable marriage between men and women, i.e.,
  1. a perfect matching
  2. satisfies the stability condition: no two people of the opposite sex like each other more than their current partners
  3. A stable marriage always exists, but may not be unique.

The Stable Marriage problem has been used to pair medical interns with hospital residencies in the USA.
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Lex-first maximal matching problem (CC-Complete)

Lex-first maximal matching

- Let $G$ be a bipartite graph.
- Successively match the bottom nodes $x, y, z, \ldots$ to the least available top node.
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Lex-first maximal matching

- Let $G$ be a bipartite graph.
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Edge: Is a given edge $\{u, v\}$ in the lex-first maximal matching of $G$?

Vertex: Is a given (top) vertex $v$ in the lex-first maximal matching of $G$?

The problems are equivalent.
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Diagram:

```
  a -- b -- c
  |    |    |
  x    y    z
      -- w
```
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![Graph Diagram]

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Lex-first maximal matching decision problems

- **Edge** Is a given edge $\{u, v\}$ in the lex-first maximal matching of $G$?
- **Vertex** Is a given (top) vertex $v$ in the lex-first maximal matching of $G$?
- The problems are equivalent.
Reducing vertex lex-first maximal matching to $C_{cv}$
Reducing $C_{CV}$ to lex-first maximal matching

\[ p_0 \quad \rightarrow \quad p_1 \]
\[ q_0 \quad \rightarrow \quad q_1 \]
Reducing $\mathcal{C}_{\text{cv}}$ to lex-first maximal matching

$p_0 \quad 1 \quad \bullet \quad 1 \quad p_1$
$q_0 \quad 1 \quad \bullet \quad 1 \quad q_1$

$p_0 \quad q_0 \quad p_1 \quad q_1
x \quad y$
Reducing $\text{Cov}$ to lex-first maximal matching

$p_0 \quad 1 \quad \uparrow \quad 1 \quad p_1$
$q_0 \quad 1 \quad \downarrow \quad 1 \quad q_1$

$x \quad q_0 \quad p_0 \quad p_1 \quad q_1 \quad y$
Reducing $C_{CV}$ to lex-first maximal matching

\[ p_0 \quad 0 \quad \rightarrow \quad 1 \quad p_1 \]
\[ q_0 \quad 1 \quad \rightarrow \quad 0 \quad q_1 \]
Reducing $C_{CV}$ to lex-first maximal matching

$p_0 \quad 0 \quad \rightarrow \quad 1 \quad p_1$
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\[\begin{array}{ccc}
p_0 & q_0 & p_1 \\
x & y & q_1 \end{array}\]
This result is due to Feder [1992].
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Show $\text{stCONN} \leq_{m}^{\text{AC}^0} \text{CCV}$.

May assume that the given directed graph $G = (V, E)$ has edges of the form $(u_i, u_j)$, where $i < j$. 

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May assume that the given directed graph \( G = (V, E) \) has edges of the form \( (u_i, u_j) \), where \( i < j \).
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Notation

- $x, y, z, \ldots$ denote elements of $\mathbb{N}$ (presented in unary)
- $X, Y, Z, \ldots$ denote binary strings
- $|X|$ denotes the length of $X$.
- A complexity class is a set of relations of the form $R(\vec{x}, \vec{X})$
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- A complexity class is a set of relations of the form $R(\vec{x}, \vec{X})$
- $\mathsf{AC}^0$ many-one reducibility
  
  $R_1(X) \leq^\mathsf{AC}^0_m R_2(X)$ iff there exists an $\mathsf{AC}^0$ function $F(X)$ such that
  
  $R_1(X) \leftrightarrow R_2(F(X))$
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- $\text{AC}^0$ many-one reducibility
  \[ R_1(X) \leq_{m}^{\text{AC}^0} R_2(X) \text{ iff there exists an } \text{AC}^0 \text{ function } F(X) \text{ such that} \]
  \[ R_1(X) \iff R_2(F(X)) \]
- Thus $\text{CC}$ is the class of relations $R(\vec{x}, \vec{X})$ that are $\text{AC}^0$ many-one reducible to $\text{C}_{\text{cv}}$. 
Function Classes

- Given a class $C$ of relations, we associate a class $FC$ of functions as follows.

- A function $F$ taking strings to strings is in $FC$ iff
  1. $|F(X)| = |X|^{O(1)}$ (p-bounded)
  2. The bit graph $B_F(i, X)$ is in $C$

- Here $B_F(i, X)$ holds iff the $i$th bit of $F(X)$ is 1.
Is FCC closed under composition?

- This question was left open in our earlier paper in CSL 2011 paper (before Yuval Filmus joined our project)
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- Suppose $F(X) = G(H(X))$. Let $Y = H(X)$.

- The bit graph of $G(Y)$ is $AC^0$-reducible to $CCV$.

- Thus the circuit computing $G(Y)$ is described by $Y' = AC^0(Y)$. 
Is FCC closed under composition?

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- Suppose \( F(X) = G(H(X)) \). Let \( Y = H(X) \).
- The bit graph of \( G(Y) \) is \( \text{AC}^0 \)-reducible to \( \text{C}_{\text{CV}} \).
- Thus the circuit computing \( G(Y) \) is described by \( Y' = \text{AC}^0(Y) \).
- But \( Y = H(X) \) is the output of another comparator circuit.
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- The bit graph of \( G(Y) \) is \( AC^0 \)-reducible to \( CCV \).

- Thus the circuit computing \( G(Y) \) is described by \( Y' = AC^0(Y) \).

- But \( Y = H(X) \) is the output of another comparator circuit.

So we need a universal comparator circuit, taking \( Y' \) as input, to compute \( G(Y) \).
Here is a gadget which allows a conditional application of a comparator to two of its inputs $x, y$, depending on whether $b$ is 0 or 1.
Universal comparator circuits [Filmus]

Here is a gadget which allows a conditional application of a comparator to two of its inputs $x, y$, depending on whether $b$ is 0 or 1.

![Diagram of the gadget]

Operation of the gadget:
Universal comparator circuits

In order to simulate a single arbitrary comparator in a circuit with $m$ wires we put in $m(m-1)$ gadgets in a row, for the $m(m-1)$ possible comparators.
Universal comparator circuits

- In order to simulate a single arbitrary comparator in a circuit with \( m \) wires we put in \( m(m - 1) \) gadgets in a row, for the \( m(m - 1) \) possible comparators.

- Simulating \( n \) comparators requires \( m(m - 1)n \) gadgets.
Universal comparator circuits

- In order to simulate a single arbitrary comparator in a circuit with $m$ wires we put in $m(m-1)$ gadgets in a row, for the $m(m-1)$ possible comparators.
- Simulating $n$ comparators requires $m(m-1)n$ gadgets.
- Thus there is an $AC^0$ function $UNIV$ such that if $m, n$ are arbitrary parameters, then
  \[
  U = UNIV(m, n) = \langle m', n', U' \rangle
  \]
  is a universal circuit with $m'$ wires and $n'$ gates which simulates all comparator networks with at most $m$ wires and at most $n$ comparators.

\[
\begin{align*}
  m' & = 2m(m-1)n + m \\
n' & = 4m(m-1)n
\end{align*}
\]
Applications of universal comparator circuits

- FCC is closed under composition.
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- FCC is closed under composition.
- CC is closed under (many-one) log-space reducibility.
Applications of universal comparator circuits

- FCC is closed under composition.
- CC is closed under (many-one) log-space reducibility.
- This is because $NL \subseteq CC$, so FCC includes all log space functions. And FCC is closed under composition.
- If $R(X) \leftrightarrow CCV(F(X))$, where $F$ is log-space computable, then

$$\chi_R(X) = \chi_{CCV}(F(X))$$

where $\chi_R$ is the characteristic function of $R$. 
Applications of universal comparator circuits Cont’d

- $R(X)$ is in CC iff there is an $\text{AC}^0$-uniform family $\{C^R_k\}_{k \in \mathbb{N}}$ of comparator circuits, where $C_k$ computes $R(X)$ for $|X| = k$.

The direction $\Rightarrow$ is immediate.

Proof of direction $\Leftarrow$: This is clear if $R(X)$ is in $\text{AC}^0$. (An $\text{AC}^0$ circuit converts into a polysize tree circuit, which converts to a comparator circuit.) If $R(X) \in \text{CC}$, then $R(X) \leftrightarrow C_{\text{cv}}(F(X))$ for some $\text{AC}^0$ function $F(X)$. Apply a universal circuit to the output of $F(X)$.
Applications of universal comparator circuits Cont’d

- $R(X)$ is in CC iff there is an $\text{AC}^0$-uniform family $\{C_k^R\}_{k \in \mathbb{N}}$ of comparator circuits, where $C_k$ computes $R(X)$ for $|X| = k$.

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Applications of universal comparator circuits Cont’d

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Applications of universal comparator circuits Cont’d

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- If \( R(X) \in \text{CC} \), then

\[
R(X) \leftrightarrow \text{CcV}(F(X))
\]

for some \( \text{AC}^0 \) function \( F(X) \). Apply a universal circuit to the output of \( F(X) \).
The circuit $C_k$ computing $R(X)$ for $|X| = k$

Compute $F(X)$

$F(X) = \text{INPUT}(\text{CIR}^R(X), \text{INP}^R(X))$

$\text{UNIV}(m_k, n_k)$
Conjecture: NC and CC are incomparable

- Lex-First Max Matching ($L_{FMM}$) is in CC.

Conjecture

$\mathbf{L_{FMM}}$ is not in NC.
(The obvious algorithm for $\mathbf{L_{FMM}}$ is sequential.)
Conjecture: \textbf{NC and CC are incomparable}

- Lex-First Max Matching (\texttt{LFMM}) is in CC.

\begin{Verbatim}
Conjecture
LFMM is not in NC.
(The obvious algorithm for LFMM is sequential.)
\end{Verbatim}

- The function $A \leadsto A^n$ (where $A$ is an $n \times n$ integer matrix) is in NC$^2$, but we do not know how to put it in CC.
Why do we think $\text{NC}^2 \subsetneq \text{CC}$?

- $\text{NC}^2$-gates have multiple fan-out, but each end of a comparator gate has fan-out one.
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  Thus comparator gates are 1-Lipschitz.
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- But flipping an input to an $\text{NC}^2$-gate can generate many parallel flip-paths.
Relativized CC and NC are incomparable

Oracle gates for comparator circuits

- The oracle $\alpha : \{0, 1\}^* \to \{0, 1\}^*$ is length preserving.
- $\alpha_n : \{0, 1\}^n \to \{0, 1\}^n$ is the restriction of $\alpha$ to $n$.
- An oracle gate $\alpha_n$ can be inserted anywhere in a relativized comparator circuit: select any $n$ wires as inputs to the gate and any $n$ wires as outputs.
Relativized $\textbf{CC}$ and $\textbf{NC}$ are incomparable

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- To make $\alpha_n$ gates look more like comparator gates, we require that $\alpha_n$ have the 1-Lipschitz property.
Relativized CC and NC are incomparable

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- We allow $\neg$ gates in relativized $\text{CC}(\alpha)$ circuits. (We can allow them in comparator circuits without changing CC.)
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- An oracle gate $\alpha_n$ can be inserted anywhere in a relativized comparator circuit: select any $n$ wires as inputs to the gate and any $n$ wires as outputs.
- To make $\alpha_n$ gates look more like comparator gates, we require that $\alpha_n$ have the 1-Lipschitz property.
- We allow $\neg$ gates in relativized $\text{CC}(\alpha)$ circuits. (We can allow them in comparator circuits without changing $\text{CC}$.)
- Changing one input to one $\alpha_n$ gate produces a unique flip path in the circuit from that gate to the outputs of the circuit.
Theorem

There is a relation $R_1(\alpha)$ computable by a polysize family of comparator oracle circuits by which cannot be computed by any $\text{NC}(\alpha)$ circuit family (even when $\alpha$ is restricted to be 1-Lipschitz).

Proof Idea.

- $\alpha^k_n(\vec{0})$ is easily computed by relativized comparator circuits, but requires depth $k$ circuits [ACN 07].
Theorem

There is a relation $R_1(\alpha)$ computable by a polysize family of comparator oracle circuits by which cannot be computed by any $\text{NC}(\alpha)$ circuit family (even when $\alpha$ is restricted to be 1-Lipschitz).

Proof Idea.

- $\alpha_n^k(0)$ is easily computed by relativized comparator circuits, but requires depth $k$ circuits [ACN 07].
- The hard part is proving the depth lower bound when $\alpha$ is 1-Lipschitz.
Theorem

There is a relation $R_2(\alpha)$ computable by an $\text{NC}^3(\alpha)$ circuit family but not computable by any polysize family of comparator oracle circuits (even when $\alpha$ is restricted to be 1-Lipschitz).
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There is a relation $R_2(\alpha)$ computable by an $\text{NC}^3(\alpha)$ circuit family but not computable by any polysize family of comparator oracle circuits (even when $\alpha$ is restricted to be 1-Lipschitz).

Proof Idea where $\alpha$ is weakly 1-Lipschitz

(At most one output bit flips when one input bit flips.)

- Let $a^k_i : \{0, 1\}^{dn} \to \{0, 1\}$ be a Boolean oracle.
- Let $A^k = (a^k_1, \ldots, a^k_n)$
- Define a function $y = f[A^1, \ldots, A^m]$ as follows:

\[
x^k_i = a^k_i(x^{k+1}_1, \ldots, x^{k+1}_n), \quad k \in [m], \ i \in [n],
\]
\[
x^m+1_i = 0,
\]
\[
y = x^1_1 \oplus \cdots \oplus x^1_n.
\]
$x_i^k = a_i^k(x_1^{k+1}, \ldots, x_1^{k+1}, \ldots, x_n^{k+1}, \ldots, x_n^{k+1})$, \quad k \in [m], \ i \in [n], \ i \in [n],
\begin{align*}
x_i^{m+1} &= 0, \\
y &= x_1^1 \oplus \cdots \oplus x_n^1.
\end{align*}

- $a_i^k$ has $dn$ inputs and one output.
- Add $dn - 1$ zeros as extra outputs for each $a_i^k$.
- Each $a_i^k$ computes a weakly 1-Lipschitz function.
- Let $X^k = (x_1^k, \ldots, x_n^k)$ \quad $A^k = (a_1^k, \ldots, a_n^k)$
- $y = f[A^1, \ldots, A^m]$
For $k \in [m], i \in [n], x_i^{m+1} = 0,
\begin{align*}
x_i^k &= a_i^k(x_1^{k+1}, \ldots, x_1^{k+1}, \ldots, x_n^{k+1}, \ldots, x_n^{k+1}), \\
y &= x_1^1 \oplus \cdots \oplus x_n^1.
\end{align*}

- $a_i^k$ has $dn$ inputs and one output.
- Add $dn - 1$ zeros as extra outputs for each $a_i^k$.
- Each $a_i^k$ computes a weakly 1-Lipschitz function.
- Let $X^k = (x_1^k, \ldots, x_n^k)$, $A^k = (a_1^k, \ldots, a_n^k)$
- $y = f[A^1, \ldots, A^m]$
- Set $m = \log^2 n$ and $d = 4$
- Then a depth $\log^2 n \text{ NC}^3$ oracle circuit computes $f$
\[ x_i^k = a_i^k(x_1^{k+1}, \ldots, x_1^{k+1}, \ldots, x_n^{k+1}, \ldots, x_n^{k+1}), \quad k \in [m], \; i \in [n], \]
\[ x_i^{m+1} = 0, \quad i \in [n], \]
\[ y = x_1^1 \oplus \cdots \oplus x_n^1. \]

- \( a_i^k \) has \( dn \) inputs and one output.
- Add \( dn - 1 \) zeros as extra outputs for each \( a_i^k \).
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- Set \( m = \log^2 n \) and \( d = 4 \)
- Then a depth \( \log^2 n \) \( \text{NC}^3 \) oracle circuit computes \( f \)

**Claim:** Every oracle comparator circuit computing \( f[A^1, \ldots, A^m] \) has at least \( \min(2^n, (d - 2)^{m-1}) \) gates. (Superpolynomial size)
Proof outline of Claim:

Every oracle comparator circuit computing \( f[A^1, \ldots, A^m] \) has at least 
\( \min(2^n, (d - 2)^{m-1}) \) gates.

Fix an oracle comparator circuit \( C \) computing \( y = f[A^1, \ldots, A^m] \)

- Def’n: An input to an oracle \( a^k_i \) is regular if it has the form 
  \( (b_1)^d \cdots (b_n)^d \).
  We say oracle \( a^k_i \) is regular if \( a^k_i(Z) = 0 \) for all irregular inputs \( Z \).
Proof outline of Claim:

Every oracle comparator circuit computing $f[A^1, \ldots, A^m]$ has at least 
$\min(2^n, (d-2)^{m-1})$ gates.

Fix an oracle comparator circuit $C$ computing $y = f[A^1, \ldots, A^m]$

- Def’n: An input to an oracle $a_{ik}$ is *regular* if it has the form 
  $(b_1)^d \cdots (b_n)^d$.
  We say oracle $a_{ik}$ is *regular* if $a_{ik}(Z) = 0$ for all irregular inputs $Z$.

- Let $g$ be the total number of any of the gates $a_{ik}$ in $C$.
  Given an assignment to the oracles, we say a particular gate $a_{ik}$ is *active* if its input is correct.
Proof outline of Claim:

Every oracle comparator circuit computing $f[A^1, \ldots, A^m]$ has at least $\min(2^n, (d - 2)^{m-1})$ gates.

Fix an oracle comparator circuit $C$ computing $y = f[A^1, \ldots, A^m]$.

- Def’n: An input to an oracle $a^k_i$ is regular if it has the form $(b_1)^d \cdots (b_n)^d$.
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- Let $g$ be the total number of any of the gates $a^k_i$ in $C$.
  Given an assignment to the oracles, we say a particular gate $a^k_i$ is active if its input is correct.

- Let $g_k$ be the expected total number of active gates $a^k_1, \ldots, a^k_n$ in $C$ under a uniformly random regular setting of all oracles.

- $g_1 \geq n$ (because $y = x^1_1 \oplus \cdots \oplus x^1_n$)
Proof outline of Claim:

Every oracle comparator circuit computing \( f[A^1, \ldots, A^m] \) has at least \( \min(2^n, (d - 2)^{m-1}) \) gates.

Fix an oracle comparator circuit \( C \) computing \( y = f[A^1, \ldots, A^m] \)

- **Def’n:** An input to an oracle \( a^k_i \) is *regular* if it has the form \((b_1)^d \cdots (b_n)^d\).
  We say oracle \( a^k_i \) is *regular* if \( a^k_i(Z) = 0 \) for all irregular inputs \( Z \).

- Let \( g \) be the total number of any of the gates \( a^k_i \) in \( C \).
  Given an assignment to the oracles, we say a particular gate \( a^k_i \) is *active* if its input is correct.

- Let \( g_k \) be the expected total number of active gates \( a^k_1, \ldots, a^k_n \) in \( C \) under a uniformly random *regular* setting of all oracles.

- \( g_1 \geq n \) (because \( y = x^1_1 \oplus \cdots \oplus x^1_n \))

- It suffices to show \( g_{k+1} \geq (d - 2)(g_k - g/2^n) \)
Proof idea of final Claim:

\[ g_{k+1} \geq (d - 2)(g_k - g/2^n) \]

- Consequence of weakly 1-Lipschitz: If we change the definition of some gate \( a_i^k \) at its current input in \( C \), this generates a unique flip-path which may end at some copy of some other gate, in which case we say that the latter gate *consumes* the flip-path.
**Proof idea of final Claim:**

\[ g_{k+1} \geq (d - 2)(g_k - g/2^n) \]

- Consequence of weakly 1-Lipschitz: If we change the definition of some gate \( a^k_i \) at its current input in \( C \), this generates a unique flip-path which may end at some copy of some other gate, in which case we say that the latter gate *consumes* the flip-path.

- Let \( G_1, \ldots, G_{2^n} \) be a Gray code listing all strings in \( \{0, 1\}^n \), starting at \( G_1 = X_{k+1} \). We change the definition of the output of \( A^{k+1} \) (at its active input) successively from \( G_1 \) to \( G_{2^n} \) and count the number of flip paths generated.

- The Claim follows because every time a particular \( a^k_i \) gate is updated from one active input to the next, it will absorb as least \( d - 2 \) flip paths.
Conclusion

The complexity class CC is interesting because

- It is robust: It has several alternative characterizations.
- It has interesting complete problems.
- It appears to be a proper subset of $P$ and incomparable with $NC$ (and $SC$).
Conclusion

The complexity class CC is interesting because

- It is robust: It has several alternative characterizations.
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Open Problems:
Are any of the following problems in CC?

- Integer matrix powering?
- All context free languages?
- Maximum matching in graphs?