Penalty Methods for Nonlinear HJB PDEs

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Outline

- Overview
- Problem Descriptions
- Numerical Methods
- Numerical Results
- Conclusion

Black-Scholes

The Black-Scholes PDE [Black and Scholes, 1973] framework models many pricing problems in finance. It is given by

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV \equiv \mathcal{L}(\sigma, r)V \quad (1)$$

- S – asset price variable
- τ – backward time variable from maturity T ($\tau = T - t$)
- σ – volatility of asset price
- r – interest rate

Some non-vanilla option pricing problems are obtained by adding terms or modifying existing terms in Equation (1). Then we have

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \max\{\mathcal{L}_1 V, \mathcal{L}_2 V, 0\} + \rho \max\{V^* - V, 0\} \quad (2)$$

with $\mathcal{L}_i$ being linear second order differential operators, and ρ is a large value (for American options) or zero (European options).
For some problems, first max is replaced with min.

Hamilton-Jacobi-Bellman (HJB) equations

Hamilton-Jacobi-Bellman (HJB) equations model many nonlinear pricing problems in finance.
General form of HJB equations in finance:

$$V_\tau = \sup_{Q,\mu} \left\{ a(S,\tau,Q)V_{SS} + b(S,\tau,Q)V_S + c(S,\tau,Q)V + d(S,\mu) \right\} \quad (3)$$

- Q, μ – control variables (μ for American)
- $aV_{SS} + bV_S + cV + d$ is $\mathcal{L}(\cdot)V$ with additional and/or modified terms
- The above is for short positions. For long positions, sup is replaced by inf.

Example Problems

We study the following nonlinear pricing problems in computational finance under the Black-Scholes framework

- Stock Borrowing Fee problem [Duffie et al., 2002] with American-style exercise rights [Forsyth and Labahn, 2007]
- Uncertain Volatility Models [Avellaneda et al., 1995]
- Transaction Cost Models [Leland, 1985]

formulated as HJB equations and as nonlinear PDEs.

We consider the solution of the HJB equations with policy iteration [Forsyth and Labahn, 2007] which we improve for problems with American exercise rights.

We derive penalty-like (penalty) iteration algorithms for the solution of the nonlinear PDEs with max and min terms, inspired by [Forsyth and Vetzal, 2002, Chen and Christara, 2021].

Outline

Control problem:

$$V_\tau = \sup_{\mu} \inf_{Q} \left\{ \frac{\sigma^2 S^2}{2} V_{SS} + q_3 q_1 (S V_S - V) + (1 - q_3)((r_l - r_f) S V_S - q_2 V) + \mu \frac{V^* - V}{\epsilon} \right\}, \tag{4}$$

with $Q = (q_1, q_2, q_3)$, $q_1 \in \{r_l, r_b\}$, $q_2 \in \{r_l, r_b\}$, $q_3 \in \{0, 1\}$, $\mu \in \{0, 1\}$.
PDE problem:

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_b (S V_S - V) + \min\{(r_l - r_b)(S V_S - V), -(r_b - r_l + r_f) S V_S, 0\} + \rho \max\{V^* - V, 0\} \tag{5}$$

Initial condition of PDE (“straddle payoff"):

$$V(t = 0, S) = \max(K - S, S - K) \tag{6}$$

Note that $r_b > r_l > r_f \geq 0$

Uncertain Volatility problem (best case)

Control problem:

$$V_\tau = \sup_{q \in \{\sigma_{\min}, \sigma_{\max}\}} \left\{ \frac{q^2 S^2}{2} V_{SS} + rSV_S - rV \right\} \tag{7}$$

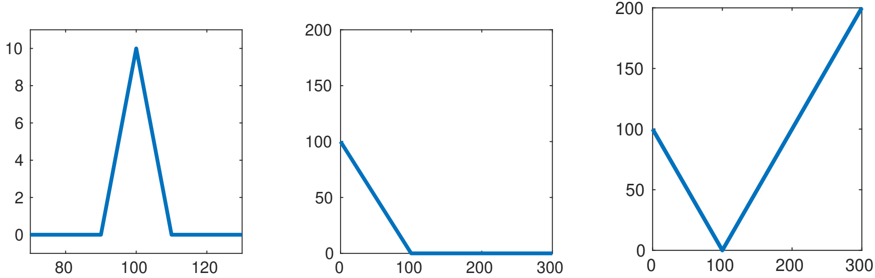
PDE problem:

$$V_\tau = \frac{\sigma_{\min}^2 S^2}{2} V_{SS} + rSV_S - rV + \max\left\{ \frac{(\sigma_{\max}^2 - \sigma_{\min}^2) S^2}{2} V_{SS}, 0 \right\} \tag{8}$$

Initial condition of PDE (“butterfly spread"):

$$V(t=0, S) = (S - K_1)^+ - 2(S - K)^+ + (S - K_2)^+ \text{ where } X^+ \equiv \max(X, 0) \tag{9}$$

Plot of Payoffs



From left to right: Butterfly Spread payoff, Put payoff, Straddle payoff.

Transaction Cost problem

Control problem:

$$V_\tau = \inf_{q\in\{-\kappa,\kappa\}} \left\{ \left(\frac{\sigma^2}{2} + q \right) S^2 V_{SS} + rSV_S - rV \right\} \tag{10}$$

With American exercise rights

$$V_\tau = \sup_{\mu\in\{0,1\}} \inf_{q\in\{-\kappa,\kappa\}} \left\{ \left(\frac{\sigma^2}{2} + q \right) S^2 V_{SS} + rSV_S - rV + \mu \frac{V^* - V}{\epsilon} \right\} \tag{11}$$

PDE problem:

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\} \tag{12}$$

With American exercise rights

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\} + \rho \max\{V^* - V, 0\} \tag{13}$$

Put (convex) and Butterfly Spread (nonconvex) payoff are used as initial conditions.

Nonlinearity arising from transaction cost disappears in convex/concave case.

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Discussion on Spatial Grid

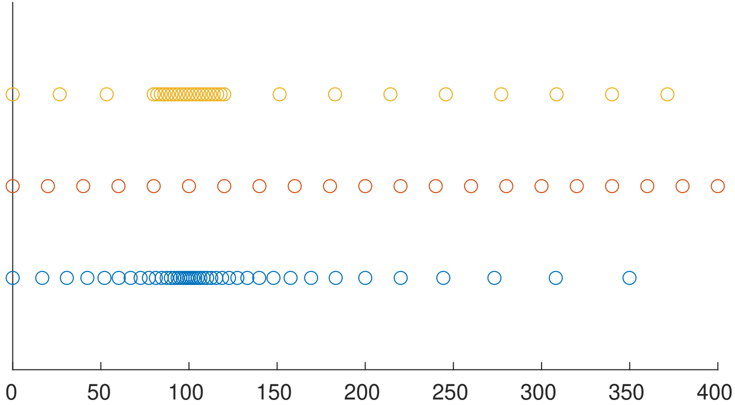

Figure 1: Red: uniform grid. Blue: Nonuniform grid for put and straddle payoffs. Yellow: Nonuniform grid for Butterfly Spread payoff

Penalty Matrices and nonlinear discretization

We use penalty matrices to discretize nonlinear terms involving max or min.

Let A denote the matrix that computes the spatial discretization of $\mathcal{L}$. For penalty it is constant, but for policy iteration it depends on the control Q.

Let T_1, T_2 denote the tridiagonal matrices used to compute the finite difference approximations of the first and second derivatives, and let D_S denote a diagonal matrix with the S_i (gridpoints) on its diagonal.

When we have an American exercise right, the value function cannot be less than the payoff. Hence we use the same penalty matrix as in [Forsyth and Vetzal, 2002] to enforce this restriction:

$$P_A(v) = \begin{cases} \rho & \text{if } v^* > v \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

The penalty-like matrix for each problem is then computed based on a maximum of several terms involving T_1, T_2, and D_S. Details to follow.

Penalty Matrix for Stock Borrowing Fees

We introduce a penalty-like matrix to compute $\min\{(r_l - r_b)(SV_S - V), -(r_b - r_l + r_f)SV_S, 0\}$.

Let $P_1 = (r_l - r_b)(D_S T_1 - I)$ and $P_2 = -(r_b - r_l + r_f)D_S T_1$ be the tridiagonal matrices arising from the discretization of $(r_l - r_b)(SV_S - V)$ and $-(r_b - r_l + r_f)SV_S$, respectively.

Define the tridiagonal penalty matrix $P = P(v^j)$ by

$$P_{i,:} = \begin{cases} 0 & \text{if } [P_1 v^j]_i \geq 0 \text{ and } [P_2 v^j]_i \geq 0 \\ [P_1]_{i,:} & \text{if } [P_1 v^j]_i < 0 \text{ and } [P_1 v^j]_i < [P_2 v^j]_i \\ [P_2]_{i,:} & \text{if } [P_2 v^j]_i < 0 \text{ and } [P_1 v^j]_i \geq [P_2 v^j]_i. \end{cases} \quad (15)$$

For convenience, we have borrowed the colon notation from matlab.

Penalty Matrix for Uncertain Volatility

We introduce a penalty matrix to handle the nonlinear term

$$\max\left\{\frac{(\sigma_{\text{max}}^2 - \sigma_{\text{min}}^2)S^2}{2} V_{SS}, 0\right\} \tag{16}$$

Define the matrix P by

$$P_{i,:} = \begin{cases} \frac{1}{2}(\sigma_d)[D_S^2 T_2]_{i,:} & \text{if } [D_S^2 T_2 v^j]_i > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

where $\sigma_d = \sigma_{\text{max}}^2 - \sigma_{\text{min}}^2$.

Penalty Matrix for Transaction costs

The tridiagonal penalty matrix $P = P(v^j)$ to handle the term $\min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\}$ in (12) is defined by

$$P_{i,:} = \begin{cases} -\kappa [D_S^2 T_2]_{i,:} & \text{if } [D_S^2 T_2 v^j]_i > 0 \\ \kappa [D_S^2 T_2]_{i,:} & \text{otherwise.} \end{cases} \tag{18}$$

All these matrices are defined in a way consistent with the discretization of the nonlinear terms involved. We apply a Crank-Nicolson discretization for timestepping which gives us our algorithms.

When European options are considered, we use a uniform Crank-Nicolson timestepping throughout. Where American options are considered, we use a variable CN timestepping proposed in [Forsyth and Vetzal, 2002].

In both cases we use Rannacher smoothing, where the first two timesteps are split into four half-size fully implicit timesteps for smoothing the initial conditions sufficiently.

Double-Penalty Iteration

Algorithm 1 Double-penalty iteration at step j, with θ-timestepping
Works for both European and American; if European set $P_A = 0$

Require: Solve $[(I - \theta\Delta\tau(A + P(v^j))) + P_A(v^j)]v^j = g^j + P_A(v^j)v^*$
where $g^j = (I + (1-\theta)\Delta\tau(A + P(v^{j-1})))v^{j-1}$

Initialize $v^{j,0} = v^{j-1}$, $P^{j,0} = P(v^{j-1})$, and $P_A^{j,0} = P_A(v^{j-1})$
for $k = 1, \ldots, maxit$ **do**
  Solve $[(I - \theta\Delta\tau(A + P^{j,k-1})) + P_A^{j,k-1}]v^{j,k} = g^j + P_A^{j,k-1}v^*$
  if first stopping criterion satisfied **then**
    Break
  end if
  Compute $P^{j,k} = P(v^{j,k})$, $P_A^{j,k} = P_A(v^{j,k})$
  if second stopping criterion satisfied **then**
    Break
  end if
end for
Set $v^j = v^{j,k}$

Improved Policy Iteration

Algorithm 2 Policy Iteration for HJB PDEs at step j, with θ timestepping
Works for both European and American; if European set $R = 0$

Require: Solve $(I - \theta \Delta\tau A^j + R^j) v^j = g^{j-1} + R^j v^*$
where $g^{j-1} = (I + (1-\theta)\Delta\tau A^{j-1}) v^{j-1}$ and $R = \mathrm{diag}(\mu_i/\epsilon)$.
subject to $Q_i^j = \arg\sup_{Q \in \hat{Q}} [A(Q) v^j]_i$ and $\mu_i^j = \arg\sup_{\mu \in \{0,1\}} [R(\mu)(v^* - v)]_i$

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Initialize v^{j,0} = v^{j-1}, μ^{j,0} = μ^{j-1}, and Q^{j,0} = Q^{j-1}
for k = 1, ..., maxit do
    Solve (I - θΔτ A^{j,k-1} + R^{j,k-1}) v^{j,k} = g^{j-1} + R^{j,k-1} v^*
    if first stopping criterion satisfied then
        Break
    end if
    Compute Q_i^{j,k} = arg sup_{Q ∈ Q̂} [A(Q) v^{j,k}]_i, μ_i^{j,k} = arg sup_{μ ∈ {0,1}} [R(μ)(v^* - v^{j,k})]_i
    if second stopping criterion satisfied then
        Break
    end if
end for
Set v^j = v^{j,k}, μ^j = μ^{j,k}, and Q^j = Q^{j,k}
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Stopping Criteria

The first stopping criterion for both penalty and policy iteration is

$$\max_i \{ \frac{|v_i^{j,k} - v_i^{j,k-1}|}{\max(scale, |v_i^{j,k}|)} \} < tol \tag{19}$$

The second stopping criterion for penalty iteration is

$$\max_i \{ \frac{|[P^{j,k} v^{j,k} - P^{j,k-1} v^{j,k}]_i|}{\max(scale, |[P^{j,k} v^{j,k}]_i|)} \} < tol \text{ and } P_A^{j,k} = P_A^{j,k-1} \tag{20}$$

Typical values are $scale = 1$ and $tol = 10^{-6}$.

Second stopping criterion for policy iteration is

$$\max_i \{ \frac{|A(Q^{j,k}) v^{j,k} - A(Q^{j,k-1}) v^{j,k}|}{\max(scale, A(Q^{j,k}) v^{j,k})} \} < tol \text{ and } \mu^{j,k} = \mu^{j,k-1}. \tag{21}$$

Discussion on convergence

For convergence, there are two things to prove:

- the penalty iteration converges
- the discretization converges overall to the HJB solution.

We make certain assumptions that are sufficient to carry the proofs but not necessary to obtain the desired numerical behavior.

We give a brief overview on how we prove the statements:

The first is easy to prove following standard arguments such as [Chen and Christara, 2021] and [Forsyth and Vetzal, 2002].

The second we follow arguments made in [Barles, 1997] and [Pooley et al., 2003], where a stable, consistent, monotone scheme ensures convergence to the viscosity solution.

Outline

Stock Borrowing Fee problem (long position) with American exercise rights

Common Information					Penalty Iters		Policy Iters	
Nodes	Tstep	Value	Change	Rate	Total	Avg	Total	Avg
101	40	23.076824	—	—	71	1.77	68	1.70
201	82	23.082631	5.81e-03	—	142	1.73	139	1.70
401	166	23.083667	1.04e-03	2.49	277	1.67	273	1.64
801	332	23.083875	2.09e-04	2.31	561	1.69	558	1.68
1601	664	23.083922	4.68e-05	2.16	1127	1.70	1126	1.70
3201	1327	23.083932	1.05e-05	2.16	2237	1.69	2296	1.73

Table 1: Long position of Stock Borrowing Fees problem with straddle payoff, American exercise rights and variable timesteps; value computed at K; Penalty results by Algorithm 1, Policy results by Algorithm 2. Parameters: $\sigma = 0.30$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$, $T = 1.0$, $K = 100$, $S_{\max} = 1000$

Uncertain Volatility (best case)

Common Information					Penalty Iters		Policy Iters	
Nodes	Tstep	Value	Change	Rate	Total	Avg	Total	Avg
101	27	4.888611	—	—	36	1.33	36	1.33
201	52	4.883171	-5.44e-03	—	70	1.35	71	1.37
401	102	4.881935	-1.24e-03	2.14	140	1.37	140	1.37
801	202	4.881634	-3.01e-04	2.04	273	1.35	274	1.36
1601	402	4.881560	-7.44e-05	2.02	543	1.35	544	1.35
3201	802	4.881541	-1.82e-05	2.03	1084	1.35	1086	1.35

Table 2: Best Case of Uncertain Volatility problem with butterfly payoff and constant timesteps; value computed at K; Penalty results by Algorithm 1, Policy results by Algorithm 2. Parameters: $\sigma_{\max} = 0.25$, $\sigma_{\min} = 0.15$, $r = 0.1$, $T = 0.25$, $K_1 = 90$, $K = 100$, $K_2 = 110$, $S_{\max} = 500$.

Transaction Cost Model (European exercise rights, Put Payoff)

Common Information					Penalty Iters		Policy Iters	
Nodes	Tstep	Value	Change	Rate	Total	Avg	Total	Avg
101	102	15.843845	—	—	103	1.01	103	1.01
201	202	15.850002	6.16e-03	—	203	1.00	203	1.00
401	402	15.851542	1.54e-03	2.00	403	1.00	403	1.00
801	802	15.851927	3.85e-04	2.00	803	1.00	803	1.00
1601	1602	15.852023	9.63e-05	2.00	1603	1.00	1603	1.00
3201	3202	15.852047	2.41e-05	2.00	3203	1.00	3203	1.00

Table 3: European Transaction Model with Put payoff (linear problem) and constant timesteps; value computed at K; Penalty results by Algorithm 1, Policy results by Algorithm 2. Parameters: $\sigma = 0.65$, $r = 0.05$, $T = 1.0$, $\kappa = 0.1$, $K = 100$, $S_{\max} = 1000$. Exact solution is 15.852055.

Note that, as expected, we only take one iteration per timestep (due to linearity).

Transaction Cost Model (European exercise rights, Butterfly Spread Payoff)

Common Information					Penalty Iters		Policy Iters	
Nodes	Tstep	Value	Change	Rate	Total	Avg	Total	Avg
101	102	0.126405	—	—	121	1.19	121	1.19
201	202	0.125742	-6.63e-04	—	236	1.17	236	1.17
401	402	0.125485	-2.57e-04	1.37	474	1.18	474	1.18
801	802	0.125361	-1.24e-04	1.05	936	1.17	935	1.17
1601	1602	0.125323	-3.83e-05	1.70	1879	1.17	1874	1.17
3201	3202	0.125311	-1.20e-05	1.68	3736	1.17	3719	1.16

Table 4: European Transaction Cost model with Butterfly Spread payoff and constant timesteps; value computed at K; Penalty results by Algorithm 1, Policy results by Algorithm 2. Parameters: $\sigma = 0.65$, $r = 0.05$, $T = 1$, $\kappa = 0.1$, $K = 100$, $S_{\max} = 1000$.

Transaction Cost Model (American exercise rights, Put Payoff)

Common Information					Penalty Iters		Policy Iters	
Nodes	Tstep	Value	Change	Rate	Total	Avg	Total	Avg
101	42	14.671527	—	—	66	1.57	66	1.57
201	85	14.677064	5.54e-03	—	136	1.60	134	1.58
401	171	14.678432	1.37e-03	2.02	278	1.63	281	1.64
801	344	14.678768	3.36e-04	2.03	565	1.64	577	1.68
1601	687	14.678851	8.29e-05	2.02	1144	1.67	1146	1.67
3201	1374	14.678872	2.06e-05	2.01	2272	1.65	2287	1.66

Table 5: American Transaction Cost model with Put payoff and variable timesteps; value computed at K. Penalty results by Algorithm 1, Policy results by Algorithm 2. Parameters: $\sigma = 1.0$, $r = 0.1$, $T = 0.25$, $\kappa = 0.18$, $K = 100$, $S_{\max} = 1000$.

Transaction Cost Model (American exercise rights, Butterfly Spread Payoff)

Common Information					Penalty Iters		Policy Iters	
Nodes	Tstep	Value	Change	Rate	Total	Avg	Total	Avg
101	42	8.556308	—	—	54	1.20	54	1.20
201	87	8.558431	2.12e-03	—	110	1.24	109	1.22
401	176	8.558946	5.15e-04	2.04	220	1.24	219	1.24
801	353	8.559073	1.27e-04	2.02	433	1.23	427	1.21
1601	704	8.559073	3.12e-05	2.03	868	1.23	853	1.21
3201	1407	8.559112	7.81e-06	2.00	1731	1.23	1720	1.22

Table 6: American Transaction Cost model with Butterfly Spread payoff and variable timesteps; value computed at $1.1K$; Penalty results by Algorithm 1, Policy results by Algorithm 2. Parameters: $\sigma = 0.65$, $r = 0.05$, $T = 1$, $\kappa = 0.1$, $K = 100$, $S_{\max} = 1000$.

We do not compute the convergence at K, because the value at that point remains constant, as it is bound by the constraint arising from American exercise rights ($v^j \geq V^*$) and only has rounding and no discretization error.

Outline

Conclusion

- Double-penalty method is similar to [Chen and Christara, 2021] for pricing valuation adjustments. Here we extend the method to account for nonlinear terms involving max/min of first and second derivatives.
- The improved policy iteration method works well with American options and variable timestepping.
- We have proven the convergence of the individual iterations at a specified timestep and also the convergence of the discretization scheme to the viscosity solution. Please see accompanying paper for the proofs under certain assumptions.
- Penalty (PDE) and Policy (HJB) methods take approximately the same number of iterations.
- However, penalty methods avoid the enumeration of all possible cases, which makes them more efficient than the policy iteration methods.

References I

Avellaneda, M., Levy, A., and Paras, A. (1995).
Pricing and hedging derivative securities in markets with uncertain volatilities.
Applied Mathematical Finance, 2(2):73–88.

Barles, G. (1997).
Convergence of numerical schemes for degenerate parabolic equations arising in finance theory.
Numerical methods in finance, 13(1).

Black, F. and Scholes, M. (1973).
The pricing of options and corporate liabilities.
Journal of political economy, 81(3):637–654.

Chen, Y. and Christara, C. (2021).
Penalty methods for bilateral XVA pricing in European and American contingent claims by a PDE model.
Journal of Computational Finance, 24(4):41–70.

Duffie, D., Garleanu, N., and Pedersen, L. H. (2002).
Securities lending, shorting, and pricing.
Journal of Financial Economics, 66(2-3):307–339.

Forsyth, P. and Labahn, G. (2007).
Numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in finance.
Journal of Computational Finance, 11(2):1–43.

References III

Forsyth, P. and Vetzal, K. (2002).
Quadratic convergence for valuing American options using a penalty method.
SIAM J. Sci. Comput., 23(6):2095–2122.

Leland, H. E. (1985).
Option pricing and replication with transactions costs.
The Journal of Finance, 40(5):1283–1301.

Pooley, D. M., Forsyth, P. A., and Vetzal, K. R. (2003).
Numerical convergence properties of option pricing PDEs with uncertain volatility.
IMA Journal of Numerical Analysis, 23(2):241–267.