Do all computations in double-precision unless otherwise asked. Answer what is asked in the question; the inclusion of irrelevant facts may result in a small point deduction. Hand in a single pdf file to MarkUS with your code uploaded in separate files but not your solutions file, and also hand in your code separately to MarkUs. If you handwrite your assignment, illegible solutions or parts of solutions will not be graded and may receive zero. If you typeset everything in latex, 5 bonus points will be awarded to your assignment.

1. [20 points] Show that the energy norm is given by
$$\|x\|_A = \sqrt{x^T A x} \tag{1}$$
   (a) [10 points] Show that if A is positive definite then $\|x\|_A$ is a norm.
   (b) [10 points] Show (with a counterexample) that if A is not positive definite, then one of the norm properties does not hold.

2. [50 points] Consider the representative problem discussed in class:
$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = g(x, y) \tag{2}$$
   We extend it to three dimensions with
$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = g(x, y, z) \tag{3}$$
   (a) [10 points] Using a lexiographical ordering, derive the discrete equations that arise when the 2nd derivatives are approximated with a 3 point centered finite difference method.
   (b) [10 points] Describe the matrix that arises from the discretization of this problem.
   (c) [10 points] For $n = 8$, produce the following plots:
      i. The sparsity pattern of the matrix A.
      ii. The sparsity pattern of either the LU decomposition or the Cholesky decomposition of A.

      What is the order of the number of nonzero entries in each case?
   (d) [20 points] Consider a given vector g. Let $N = 8, 16, 32, 64, \ldots$. Using both backslash and conjugate gradient, compare the performances of the methods. Hint: Use tic() and toc() to record the time in matlab, and time.time() in python. Discuss the output and the runtime.

3. [30 points] Consider the iterative scheme
$$x_{k+1} = x_k + \alpha(b - Ax_k) \tag{4}$$
   This is the gradient descent method with constant stepsize.
   (a) [10 points] What is the iteration matrix T?
   (b) [10 points] Suppose A is SPD, with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. What α guarantees convergence for any initial guess?
   (c) [10 points] What α gives the best rate of convergence?