Do all computations in double-precision unless otherwise asked. Answer what is asked in the question; the inclusion of irrelevant facts may result in a small point deduction. Hand in a single pdf file to MarkUS with your code uploaded in separate files but not your solutions file, and also hand in your code separately to MarkUs. If you handwrite your assignment, illegible solutions or parts of solutions will not be graded and may receive zero. If you typeset everything in latex, 5 bonus points will be awarded to your assignment.

1. [40 points] Given the following data points

x	y
2	1
2.5	$\sqrt{2}-1$
3	-1
3.5	$\sqrt{2}-1$
4	1

   (a) [10 points] Plot three interpolants, the cubic spline interpolant, the linear spline interpolant, and the polynomial interpolant.

   (b) [30 points] The data satisfies the equation

$$y = 2\sqrt{|x-3|} - 1 \tag{1}$$

   Given this information,

   i. [15 points] Sample additional points $(x_i + x_{i+1})/2$ (i.e. the midpoints of each subinterval – 9 points total). What is the error at $x = 2.125$ for both the original interpolant and the refined interpolant?

   ii. [15 points] Calculate the rate of convergence of this problem for the linear spline and the cubic spline. What do you observe?

2. [30 points] The standard normal distribution has the density function

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2) \tag{2}$$

   and the probability of a standard normal random variable Z being less than a given constant a is

$$\int_{-\infty}^{a} f(x)dx \tag{3}$$

   A closed-form integral for $f(x)$ is not known. Thus, numerical integration must be used. One of the simplest ways to address the problem of integrating over indefinite integrals is to use symmetry and split the integral into different domains:

$$\int_{-\infty}^{0} f(x)dx = \int_{0}^{\infty} f(x)dx = 1/2. \tag{4}$$

   Additionally,

$$\int_{a}^{c} f(x)dx = \int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx. \tag{5}$$

   (a) [10 points] What is $P(Z < 2)$? Use 2001 evenly spaced points and trapezoidal quadrature to solve for a numerical solution. Report the error. The value of $P(Z < 2)$ is 0.977249868051821 to 15 digits of accuracy.

   (b) [10 points] Repeat the exercise with Simpson's rule.

   (c) [10 points] Trapezoidal quadrature is $\mathcal{O}(h^2)$ order of accuracy and Simpson's rule is $\mathcal{O}(h^4)$ order of accuracy. If you repeat exercises (b) and (c) with 4001 evenly spaced points in the same interval, what error do you expect to get?

3. [30 points] Higher-order methods can be derived from multiple lower-order methods using Richardson extrapolation.

   (a) [10 points] Apply Simpson's rule to the function $f(x)$ from question 2 on the interval $[0, 1]$, with 4, 8, and 16 subintervals. Denote the computed results as R_1, R_2, R_3. What is the error of each R_i? The exact solution, accurate to 15 decimal places, is `0.341344746068543`

   (b) [10 points] Can you find a way to cancel out the lower-order error terms in Simpson's method, leaving only the higher-order terms? *Hint:* Write down the error with h and $2h$, then write down a linear combination of R_i and R_{i+1} which zeros the leading error term. Let Q_1 denote the result computed from R_1 and R_2, and let Q_2 denote the result computed from R_2 and R_3. Give a formula for Q_i in terms of R_i and R_{i+1}.

   (c) Report the error for Q_1 and Q_2. What order of accuracy do you see? Is it to be expected?