Do all computations in double-precision unless otherwise asked. Answer what is asked in the question; the inclusion of irrelevant facts may result in a small point deduction. Hand in a single pdf file to MarkUS with your code uploaded in separate files but not your solutions file, and also hand in your code separately to MarkUs. If you handwrite your assignment, illegible solutions or parts of solutions will not be graded and may receive zero. If you typeset everything in latex, 5 bonus points will be awarded to your assignment.

1. [30 points] Consider the minimization of the Rosenbrock function:

$$\min f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \tag{1}$$

   (a) [10 points] What is the gradient and the Hessian? What is the unique minimizer?

   (b) [10 points] Implement gradient descent and Newton's method. Determine the number of iterations required for convergence for the initial guess $x_0 = [0, 0]^T$. For gradient descent, use a tolerance of 10^{-4} and set the stepsize to 10^{-3} to ensure convergence if you do not want to implement backtracking linesearch.

   (c) [10 points] Comment on the difference in performance.

2. [20 points] Consider the network matrix

$$A = \begin{bmatrix} 0 & 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 & 1/2 & 0 \end{bmatrix} \tag{2}$$

   Write and run code for the following subproblems:

   (a) [10 points] Use the power method to determine the dominant eigenpair, and report the normalized (in the 1-norm) eigenvector. Use an initial guess of a normalized vector with all entries being equal.

   (b) [10 points] What happens if you use an inverse iteration to try to determine the dominant eigenpair? What value of α should be chosen? Is this practical in applications?

3. [50 points] Consider the following nonlinear PDE:

$$u'' + (u')^2 + u - \ln(x) = 0, \quad 1 < x < 2 \tag{3}$$

   with boundary conditions $u(1) = 0$, $u(2) = \ln(2)$. Similar to the previous assignment, we will solve this PDE with finite differences. You have already derived these formulas in Assignments 1 and 2.

   (a) [10 points] Show that $u = \ln(x)$ is a solution.

   (b) [10 points] Consider a uniform discretization of $n + 1$ points of the interval $[1, 2]$. What is the system of nonlinear equations to be solved?

   (c) [10 points] What is the Jacobian? Give a high-level description on how you would use Newton's Method to solve the discretized equations.

   (d) [20 points] Write code to determine the solution u. Use Newton's method with an initial guess of all zeros. When the problem is in the form of $f(x) = 0$, use the criterion $\|f(x)\| \leq 10^{-8}$ to determine convergence.

   Output a table giving the number of partitions (from $n = 8$ to 1024, doubling each time), number of iterations for convergence, the maximum error among all grid points, and the order of convergence.