Do all computations in double-precision unless otherwise asked. Answer what is asked in the question; the inclusion of irrelevant facts may result in a small point deduction. Hand in a single pdf file to MarkUS with your code uploaded in separate files but not your solutions file, and also hand in your code separately to MarkUs. If you handwrite your assignment, illegible solutions or parts of solutions will not be graded and may receive zero. If you typeset everything in latex, 5 bonus points will be awarded to your assignment.

1. [15 points] Prove the following statements:
   (a) [5 points] Show that the condition number of a nonsingular matrix is equal to that of its inverse, that is; $\kappa(A) = \kappa(A^{-1})$.
   (b) [5 points] Show that for any unit vector u, the matrix $P = I - 2uu^T$ is orthogonal, that is, $P^{-1} = P^T$.
   (c) [5 points] Show that if a square matrix A is nonsingular, then $B = A^T A$ is symmetric positive definite.

2. [60 points] In the finite difference numerical solution of partial differential equations (PDEs), we convert a continuous problem into a discrete problem represented by a linear system, and we solve the resulting system of linear equations.

   Note that this problem is much easier to code in matlab than python, due to operations with sparse matrices

   Consider the PDE
   $$u''(x) = f(x), u(a) = \beta_1, u(b) = \beta_2 \tag{1}$$
   where u is the unknown function we are solving for, $[a, b]$ is the domain of interest, and $u(a)$, $u(b)$ specify Dirichlet boundary conditions on the endpoints.

   (a) [10 points] From Assignment 1, we/you derived finite difference formulas for the first derivative. Show that the finite difference formula
   $$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{2}$$
   is second order accurate, that is, the error behaves like $\mathcal{O}(h^2)$.
   (b) [10 points] Suppose we discretize our domain $[a, b]$ into n uniform partitions with the equidistant points $x_0 = a, x_1, x_2, \ldots, x_n = b$ and consider the solution on these points alone. Take $h = x_j - x_{j-1}$. What are the $n-1$ equations that the solution u must satisfy on the interior points $x_1, x_2, \ldots, x_{n-1}$? Write out the set of equations in matrix form for $n = 6$. Note that the system of equations is **not necessarily** a square matrix.
   (c) [5 points] Add in the two boundary conditions, $u(a) = \beta_1$ and $u(b) = \beta_2$ to make it a square system.
   (d) [5 points] The system of equations above is not symmetric. Re-write the system of equations to make it so. For an arbitrary n, what is the percentage of nonzero entries in the matrix?
   (e) [15 points] Let $f(x) = -\sin(x)$, $u(a = 0) = 0$, $u(b = \pi/2) = 1$. The exact solution is $u(x) = \sin(x)$. Write a program to solve the linear system for $n = 8, 16, 32, 64, 128, 256, \ldots, 65536$ (for the purposes of this assignment, you must run the computations to this resolution). Create a table with columns n, condition number of the matrix (use `condest` in matlab), error (defined as $\max |u(x_i) - \sin(x_i)|$), and the rate of convergence (defined as $\log_2(\text{error}_{2h}/\text{error}_h)$).
   Update Feb 1 2023: **Use sparse matrices, otherwise you may run into memory issues.**
   (f) [15 points] What do you observe? What is the order of convergence? Is it what we expect? What can you say about the order of convergence as n becomes larger? Can you explain the behavior of the order of convergence? (Hint: I didn't make you compute the condition number in part (e) for fun).

3. [25 points] You have obtained data (x_i, y_i) given in the table. We think that the data is from the model
   $$y = a_1 x^{a_2}. \tag{3}$$
   where a_i are unknown coefficients to be determined.

x_i	4.6416e-02	2.1544e-02	4.6416e-03	2.1544e-03	4.6416e-04	2.1544e-04	4.6416e-05	2.1544e-05
y_i	3.3069e-04	7.1251e-05	3.3073e-06	7.1253e-07	3.3073e-08	7.1252e-09	3.3064e-10	7.0618e-11

   (a) [10 points] Convert the data to a more suitable form for linear least squares.
   (b) [5 points] Give the system of overdetermined linear equations to solve.
   (c) [5 points] Calculate a_1 and a_2 by solving the normal equations. What is the interpretation of the value of a_2?
   (d) [5 points] Show how to use the QR decomposition to solve for a_1 and a_2.